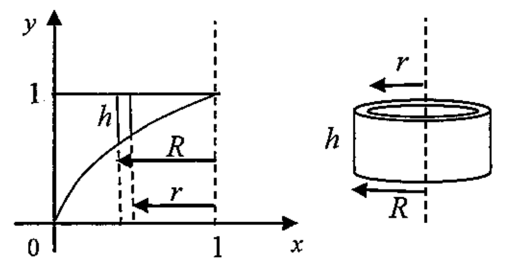
Independent Trial HSC 2014 Mathematics Extension 2 Marking Guidelines

Section 1 Questions 1-10 (1 mark each)

Question	Answer	Solution	Outcomes
1.	B	$\left.\begin{array}{l} a=1 \\ r=2\cos^2\theta \end{array}\right\}$ $\quad \dfrac{a}{1-r}=\dfrac{1}{1-(1+\cos 2\theta)}=\dfrac{1}{-\cos 2\theta}=-\sec 2\theta$	H5
2.	C	$f(x)=\sin^{-1}x+\tan^{-1}x$ is an increasing function with domain $-1\le x\le 1$. $\therefore \sin^{-1}(-1)+\tan^{-1}(-1)\le y\le \sin^{-1}1+\tan^{-1}1$ $(-\frac{\pi}{2})+(-\frac{\pi}{4})\le y\le \frac{\pi}{2}+\frac{\pi}{4}$ $\quad \therefore -\frac{3\pi}{4}\le y\le \frac{3\pi}{4}$	HE4
3.	A	$e^x+e^y=1$ $e^x+e^y\dfrac{dy}{dx}=0$ $\quad \therefore \dfrac{dy}{dx}=\dfrac{-e^x}{e^y}=-e^{x-y}$	E6
4.	B	$\lvert z-6\rvert=2\lvert z\rvert$ $\quad (x-6)^2+y^2=4(x^2+y^2)$ $36=3x^2+12x+3y^2$ $\quad (x+2)^2+y^2=16$	E3
5.	C	$\left.\begin{array}{l} e=\sqrt{2} \\ a=\sqrt{k} \end{array}\right\}$ $\quad S(\sqrt{2k},0) \quad \therefore \sqrt{2k}=4 \quad \therefore k=8$	E4
6.	D	$\int xe^{-x}\,dx=x(-e^{-x})-\int(-e^{-x})\,dx=-xe^{-x}-e^{-x}+c$	E8
7.	D	 $h=1-\sqrt{x}$ $R=1-x$ $r=1-x-\delta x$ $\delta V=\pi(R^2-r^2)h$ $=\pi(R+r)(R-r)h$ $\therefore \delta V=\pi\{2(1-x)-\delta x\}\delta x(1-\sqrt{x})$ $\delta V=2\pi(1-x)(1-\sqrt{x})\delta x$ (ignoring terms in $(\delta x)^2$) $\therefore V=\lim_{\delta x\to 0}\sum_{x=0}^{x=1}2\pi(1-x)(1-\sqrt{x})\delta x=2\pi\int_0^1(1-x)(1-\sqrt{x})\,dx$	E7
8.	C	$-\frac{1}{\alpha}$, $-\frac{1}{\beta}$ and $-\frac{1}{\gamma}$ satisfy $(-\frac{1}{x})^3-4(-\frac{1}{x})-2=0$ Rearranging gives $2x^3-4x^2+1=0$	E4
9.	A	Resolving forces horizontally and applying Newton's 2nd law gives $T\sin\theta=mr\omega^2$. Then $r=l\sin\theta$ gives $T=ml\omega^2$	E5
10.	D	$\lim_{n\to\infty}\dfrac{{}^nC_1\,.\,{}^nC_2}{{}^nC_3}=\lim_{n\to\infty}\left\{n.\dfrac{n(n-1)}{2!}.\dfrac{3!}{n(n-1)(n-2)}\right\}=3\lim_{n\to\infty}\dfrac{1}{1-\frac{2}{n}}=3$	HE3

Section II

Question 11

a. Outcomes assessed: E3

Marking Guidelines

Criteria	Marks
i • find the difference	1
ii • find the product	1

Answer

i. $\overline{z}-w=(1-3i)-(2-i)=-1-2i$ ii. $zw=(1+3i)(2-i)=5+5i$

b. Outcomes assessed: E3

Marking Guidelines

Criteria	Marks
i • find the modulus	1
• find the argument	1
ii • use deMoivre's theorem	1
• simplify into required form	1

Answer

i. $-1+\sqrt{3}\,i=2\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$

ii. $z^8=2^8\left(\cos\frac{16\pi}{3}+i\sin\frac{16\pi}{3}\right)=2^8\left(\cos\frac{-2\pi}{3}+i\sin\frac{-2\pi}{3}\right)=2^7\left(-1-\sqrt{3}\,i\right)$

$16z^4=16.2^4\left(\cos\frac{8\pi}{3}+i\sin\frac{8\pi}{3}\right)=2^8\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)=2^7\left(-1+\sqrt{3}\,i\right)$

$z^8-16z^4=-2^8\sqrt{3}\,i=-256\sqrt{3}\,i$

c. Outcomes assessed: E3

Marking Guidelines

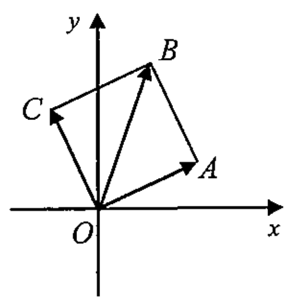
Criteria	Marks
i • find z represented by vector OC	1
• find z represented by vector OB	1
ii • write an expression for z using complex numbers in form $a+ib$	1
• evaluate z	1

Answer

i. $\overrightarrow{OC}$ represents iz (anticlockwise rotation of $\overrightarrow{OA}$ by $\frac{\pi}{2}$)

$\overrightarrow{OB}$ is the vector sum of $\overrightarrow{OA}$ and $\overrightarrow{OC}$.

Hence $\overrightarrow{OB}$ represents $z+iz=(1+i)z$



ii. $4+2i=(1+i)z$

$(4+2i)(1-i)=2z$

$6-2i=2z$

$z=3-i$

Q11 (cont)

d. Outcomes assessed: E4

Marking Guidelines

Criteria	Marks
i • use the relationships between roots and coefficients to evaluate the sum of squares	1
• evaluate the sum of fourth powers by writing it in terms of sums of lower powers	1
ii • use the negative value of the sum of the fourth powers to deduce at least one root is non-real	1
• deduce that there are either 4 non-real roots, or 2 real and 2 non-real roots	1
• show that one root is real by establishing the change of sign of the polynomial function	1

Answer

i. $\alpha^2+\beta^2+\gamma^2+\delta^2=(\alpha+\beta+\gamma+\delta)^2-2(\alpha\beta+\beta\gamma+\gamma\delta+\delta\alpha)=0-2(-2)=4$

Each of $\alpha,\beta,\gamma,\delta$ satisfies $x^4-2x^2-5x+3=0$. Hence

$$(\alpha^4+\beta^4+\gamma^4+\delta^4)-2(\alpha^2+\beta^2+\gamma^2+\delta^2)-5(\alpha+\beta+\gamma+\delta)+(3+3+3+3)=0$$

$$(\alpha^4+\beta^4+\gamma^4+\delta^4)-2\times4-5\times0+12=0$$

$$\therefore \alpha^4+\beta^4+\gamma^4+\delta^4=-4$$

ii. At least one of $\alpha,\beta,\gamma,\delta$ must be non-real (since the fourth powers are not all non-negative). However the non-real roots come in complex conjugate pairs (since the coefficients are real). Hence either there are 4 non-real roots, or there are 2 non-real and 2 real roots.
Considering the continuous polynomial function $P(x)=x^4-2x^2-5x+3$, $P(0)=3>0$ and $P(1)=-3<0$ and hence there is a real root of $P(x)=0$ lying between 0 and 1.
Hence the equation must have 2 real and 2 non-real roots.

Question 12

a. Outcomes assessed: E8

Marking Guidelines

Criteria	Marks
• complete the square	1
• write the primitive function	1

Answer

$$\int \frac{1}{\sqrt{3-(x^2-2x)}}\,dx=\int \frac{1}{\sqrt{4-(x-1)^2}}\,dx=\sin^{-1}\left(\frac{x-1}{2}\right)+c$$

b. Outcomes assessed: E8

Marking Guidelines

Criteria	Marks
• rearrange integrand into appropriate form	1
• write the primitive	1

Answer

$$\int \frac{e^{2x}}{e^x+1}\,dx=\int \frac{e^x(e^x+1)-e^x}{e^x+1}\,dx=\int\left\{e^x-\frac{e^x}{e^x+1}\right\}dx=e^x-\ln(e^x+1)+c$$

Q12 (cont)
c. Outcomes assessed: E8

Marking Guidelines

Criteria	Marks
• apply integration by parts	1
• complete the primitive function	1

Answer

$$\int 1.\tan^{-1}x\ dx = x\tan^{-1}x - \int \frac{x}{1+x^2}\,dx = x\tan^{-1}x - \tfrac{1}{2}\ln\left(1+x^2\right)+c$$

d. Outcomes assessed: E8

Marking Guidelines

Criteria	Marks
• express the integrand in terms of t	1
• write the definite integral in terms of t	1
• find the primitive function	1
• evaluate using t limits	1

Answer

$$t=\tan\tfrac{x}{2}$$
$$dt=\tfrac{1}{2}\sec^2\tfrac{x}{2}\,dx$$
$$dx=\frac{2}{1+t^2}dt$$

$$x=0 \Rightarrow t=0$$
$$x=\tfrac{\pi}{2} \Rightarrow t=1$$

$$5+4\sin x+3\cos x$$
$$=\frac{5(1+t^2)+8t+3(1-t^2)}{1+t^2}$$
$$=\frac{2(t^2+4t+4)}{1+t^2}$$
$$=\frac{2(t+2)^2}{1+t^2}$$

$$\int_0^{\frac{\pi}{2}}\frac{1}{5+4\sin x+3\cos x}\,dx$$
$$=\int_0^1 \frac{1+t^2}{2(t+2)^2}\cdot\frac{2}{1+t^2}\,dt$$
$$=-\left[\frac{1}{t+2}\right]_0^1$$
$$=\tfrac{1}{6}$$

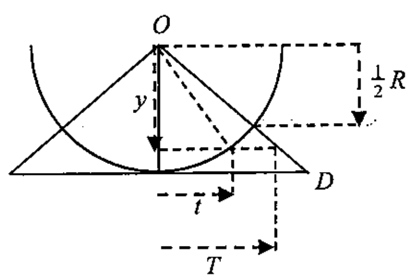
e. Outcomes assessed: E7, E8

Marking Guidelines

Criteria	Marks
i • find either inner or outer radius of the annular cross section	1
• find other radius of the annulus and then its area	1
• express V as a limiting sum of slice volumes and hence as a definite integral	1
ii • find the primitive function	1
• evaluate in terms of R.	1

Answer

i. Cross section y below O is an annulus with inner radius t and outer radius T, where $t^2=R^2-y^2$ and $T=y\tan\frac{\pi}{3}=y\sqrt{3}$ (since vertical through O makes angle $\frac{\pi}{3}$ with OD)



Hence area of cross section is $\pi\left\{3y^2-\left(R^2-y^2\right)\right\}$

$$\therefore\ V=\lim_{\delta y\to 0}\sum_{y=\frac{1}{2}R}^{R}\pi\left(4y^2-R^2\right)\delta y=\pi\int_{\frac{1}{2}R}^{R}\left(4y^2-R^2\right)dy$$

ii. $V=\pi\left[\tfrac{4}{3}y^3-R^2y\right]_{\frac{1}{2}R}^{R}=\pi\left\{\tfrac{4}{3}\left(R^3-(\tfrac{1}{2}R)^3\right)-R^2\left(R-\tfrac{1}{2}R\right)\right\}$

$$\therefore V=\pi R^3\left(\tfrac{4}{3}\times\tfrac{7}{8}-\tfrac{1}{2}\right)=\tfrac{2}{3}\pi R^3$$

Question 13

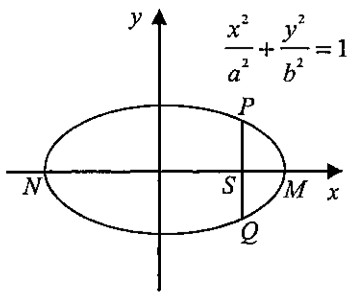
a. Outcomes assessed: E4

Marking Guidelines

Criteria	Marks
• write expressions for *MS*, *NS* in terms of *a* and *e*	1
• find *PS* in terms of *a* and *e*	1
• find *PQ* in terms of *a* and *e*	1
• find the sum of the reciprocals of *MS* and *NS* and rearrange to obtain result	1

Answer

$$\frac{1}{MS}+\frac{1}{NS}=\frac{1}{a(1-e)}+\frac{1}{a(1+e)}=\frac{(1+e)+(1-e)}{a(1-e^2)}=\frac{2}{a(1-e^2)}$$



Using the locus definition of the ellipse and the directrix $x=\frac{a}{e}$,

$PS=e\left(\frac{a}{e}-ae\right)=a\left(1-e^2\right)$ and hence $PQ=2PS=2a(1-e^2)$

$$\therefore \frac{1}{MS}+\frac{1}{NS}=\frac{4}{PQ}$$

b. Outcomes assessed: E4

Marking Guidelines

Criteria	Marks
i • find gradient of *PQ* and hence gradient of *MX*	1
• find coordinates of *M* and gradient of *OM*	1
• deduce *MX* and *OM* make equal acute angles with the *x*-axis so that ΔMOX is isosceles	1
ii • find the parameter at *T* in terms of *p* and *q*	1
• compare the gradient of the tangent at *T* with the gradient of *PQ*	1

Answer

i. $gradient\ MX = Gradient\ PQ=\frac{c(\frac{1}{p}-\frac{1}{q})}{c(p-q)}=-\frac{1}{pq}$. *M* has coordinates $\left(\frac{c(p+q)}{2},\frac{c(p+q)}{2pq}\right)$

$gradient\ OM=\frac{c(p+q)}{2pq}\div\frac{c(p+q)}{2}=\frac{1}{pq}$ $\therefore gradient\ MX=-\ gradient\ OM$

If *MX* and *OM* make angles α and β respectively with the positive *x* axis, then $\tan\alpha=-\tan\beta$. Hence $\beta=180°-\alpha$ and in ΔMOX, $\angle MOX=\angle MXO=\beta$. Then ΔMOX is isosceles with $MX=OM$.

ii. Let *T* have coordinates $\left(ct,\frac{c}{t}\right)$. Then $m_{OT}=\frac{c}{t}\div ct=\frac{1}{t^2}$. But $m_{OT}=m_{OM}$. $\therefore t^2=pq$.

At *T*, $\frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}=\frac{-c}{t^2}\div c=-\frac{1}{t^2}=-\frac{1}{pq}=m_{PQ}$. Hence the tangent at *T* is parallel to *PQ*.

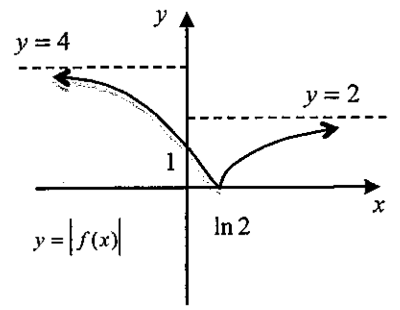
Q13 (cont)
c. Outcomes assessed: E6

Marking Guidelines

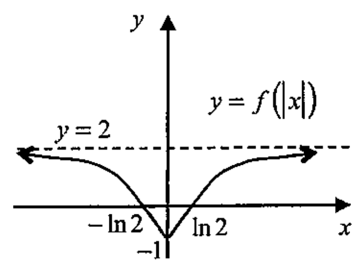
Criteria	Marks
i • reflect section of graph below x axis in x axis	1
ii • reflect section of graph to right of y axis in y axis to obtain graph for $x<0$	1
iii • sketch upper branch with asymptotes	1
• sketch lower branch with asymptotes and y intercept	1
iv • correct domain, shape and vertical asymptotes	1
• correct intercepts on axes	1

Answer

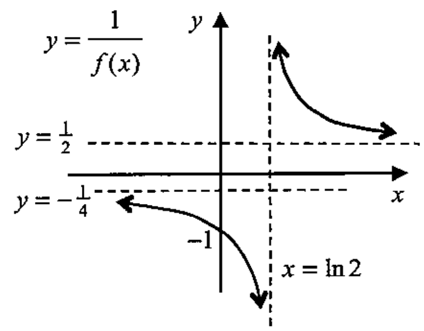
i.



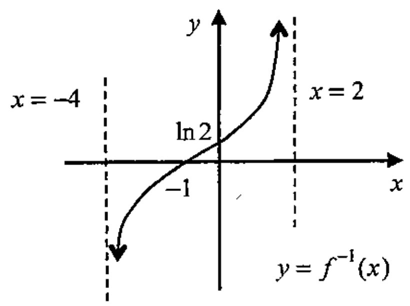
ii.



iii.



iv.



Question 14
a. Outcomes assessed: E4

Marking Guidelines

Criteria	Marks
• find A	1
• find B	1
• find C	1

Answer

$$x^2+3x+4 \equiv A\left(x^2+3\right)+\left(Bx+C\right)\left(x-2\right)$$

$x=2 \;\Rightarrow\; 14=7A \qquad \therefore A=2$

$x=0 \;\Rightarrow\; 4=3A-2C \qquad \therefore C=1$

Equate coeff. of x^2: $1=A+B \qquad \therefore B=-1$

$$\frac{x^2+3x+4}{\left(x-2\right)\left(x^2+3\right)}=\frac{2}{x-2}+\frac{-x+1}{x^2+3}$$

Q14 (cont)

b. Outcomes assessed: HE2

Marking Guidelines

Criteria	Marks
• define a sequence of statements and show the first two are true	1
• use the recurrence relation to write T_{k+1} in terms of values of T_k, T_{k-1} given $S(n)$ true, $n \le k$	1
• rearrange to establish conditional truth of $S(k+1)$ and complete the induction process	1

Answer

Let $S(n)$, $n = 1, 2, 3, \ldots$ be the sequence of statements defined by $S(n):\ T_n = (n+1)\,3^n$.

Consider $S(1)$ and $S(2)$: $\quad T_1 = 6 = (1+1)\times 3^1 \quad \therefore S(1)$ is true

$T_2 = 27 = (2+1)\times 3^2 \quad \therefore S(2)$ is true

If $S(n)$ is true for $n \le k$ (where $k \ge 2$): $\quad T_n = (n+1)\,3^n$, $\quad n = 1, 2, 3, \ldots, k \qquad *$

Consider $S(k+1)$, $k \ge 2$:

$$\begin{aligned} T_{k+1} &= 6T_k - 9T_{k-1} \\ &= 6(k+1)\,3^k - 9k\,.\,3^{k-1} \quad \text{if } S(n) \text{ is true for } n \le k \text{, using } * \\ &= \left\{2(k+1) - k\right\} 3^{k+1} \\ &= \left\{(k+1)+1\right\} 3^{k+1} \end{aligned}$$

Hence if $S(n)$ is true for $n \le k$ (where $k \ge 2$) then $S(k+1)$ is true. But $S(n)$ is true for $n \le 2$. Hence $S(3)$ is true, then $S(n)$ true for $n \le 3 \Rightarrow S(4)$ is true and so on. Hence by Mathematical Induction, $T_n = (n+1)\,3^n$ for all integers $n \ge 1$.

c. Outcomes assessed: E5

Marking Guidelines

Criteria	Marks
i • find $\frac{dx}{dv}$ in terms of v	1
• use initial conditions to find x in terms of v	1
• find expression for H by finding x when v is zero	1
ii • find equation of motion for downward journey	1
• find distance fallen in terms of v	1
• use expression for maximum height to establish required equation	1
iii • note continuity and establish change of sign	1
• apply Newton's method	1
iv • find terminal velocity and use value of λ to obtain required percentage	1

Answer

i.

$$\ddot{x} = -\tfrac{1}{10}(100+v)$$

$$v\frac{dv}{dx} = -\tfrac{1}{10}(100+v)$$

$$-\tfrac{1}{10}\frac{dx}{dv} = \frac{v}{100+v}$$

$$-\tfrac{1}{10}\frac{dx}{dv} = 1 - \frac{100}{100+v}$$

$$-\tfrac{1}{10}x = v - 100\ln(100+v) + c$$

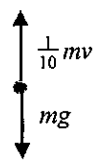
$$\left.\begin{aligned} t&=0 \\ x=0,\ v&=200 \end{aligned}\right\} \quad \begin{aligned} 0 &= 200 - 100\ln 300 + c \\ \tfrac{1}{10}x &= (200-v) - 100\ln\left(\tfrac{300}{100+v}\right) \end{aligned}$$

$$x = H,\ v = 0 \Rightarrow \quad \tfrac{1}{10}H = 200 - 100\ln 3$$

$$\therefore\ H = 1000\,(2 - \ln 3)$$

Q14 c (cont)

ii. For the downward journey, let x be the distance fallen below the position of maximum height, with initial conditions $x=0,\ v=0$.



By Newton's 2nd Law

$$m\ddot{x} = mg - \tfrac{1}{10}mv$$
$$\ddot{x} = \tfrac{1}{10}(100-v)$$
$$v\frac{dv}{dx} = \tfrac{1}{10}(100-v)$$
$$\tfrac{1}{10}\frac{dx}{dv} = \frac{v}{100-v}$$
$$-\tfrac{1}{10}\frac{dx}{dv} = 1 - \frac{100}{100-v}$$

$$-\tfrac{1}{10}x = v + 100\ln(100-v) + c$$
$$t=0, x=0, v=0 \Rightarrow \quad 0 = 100\ln 100 + c$$
$$-\tfrac{1}{10}x = v + 100\ln\left(\tfrac{100-v}{100}\right)$$
$$x = H \Rightarrow \quad -100(2-\ln 3) = v + 100\ln\left(1-\tfrac{v}{100}\right)$$
$$\tfrac{v}{100} + \ln\left(1-\tfrac{v}{100}\right) + \left(2-\ln 3\right) = 0$$

iii. Let $f(\lambda) = \lambda + \ln(1-\lambda) + (2-\ln 3)$.

$$f'(\lambda) = 1 - \tfrac{1}{1-\lambda} = \tfrac{-\lambda}{1-\lambda}$$

Then $f(\lambda)$ is continuous for $0<\lambda<1$ and

$f(0\cdot 8) \approx 0\cdot 09 > 0$, $f(0\cdot 9) \approx -0\cdot 50 < 0$

Hence $f(\lambda)=0$ for some $0\cdot 8 < \lambda < 0\cdot 9$.

Using $\lambda_0 = 0\cdot 82$, $\lambda_1 = 0\cdot 82 - \dfrac{0\cdot 82 + \ln(1-0\cdot 82) + 2 - \ln 3}{\left(\frac{-0\cdot 82}{1-0\cdot 82}\right)} \approx 0\cdot 82$

iv. Since Newton's method returned the same approximate root to 2 decimal places, $\dfrac{v}{100} \approx 0\cdot 82$ gives the speed v on return to projection point as 82 ms^{-1} (to nearest 1).

For the downward journey, $\ddot{x} \to 0$ as $v \to 100$. Hence the terminal velocity is 100 ms^{-1} .
Hence particle has attained 82% of its terminal velocity on return to its point of projection.

Question 15

a. Outcomes assessed: E8

Marking Guidelines

Criteria	Marks
i • rearrange integrand	1
• evaluate definite integral to obtain reduction formula	1
ii • evaluate I_0	1
• reduce and evaluate I_4	1

Answer

i. $$\int_0^1 \frac{x^n}{1+x^2}\,dx = \int_0^1 \frac{\left\{(1+x^2)-1\right\}x^{n-2}}{1+x^2}\,dx \ , \quad n=2,3,4,\ldots$$
$$I_n = \int_0^1 x^{n-2}\,dx - I_{n-2}$$
$$= \tfrac{1}{n-1}\left[x^{n-1}\right]_0^1 - I_{n-2}$$
$$= \tfrac{1}{n-1} - I_{n-2}$$

ii. $$I_0 = \int_0^1 \frac{1}{1+x^2}\,dx = \left[\tan^{-1}x\right]_0^1 = \tfrac{\pi}{4}$$
$$I_4 = \tfrac{1}{3} - I_2$$
$$= \tfrac{1}{3} - (1 - I_0)$$
$$= \tfrac{\pi}{4} - \tfrac{2}{3}$$

Q15(cont)

b. Outcomes assessed: PE3

Marking Guidelines

Criteria	Marks
i • show the sum of a positive real number and its reciprocal is at least 2	1
ii α • rearrange given expression into such sums	1
• apply result from (i) to establish required inequality	1
β • make appropriate replacements for a, b, c	1
• rearrange to establish required inequality	1

Answer

i. $\left(a+\frac{1}{a}\right)^2 = \left(a-\frac{1}{a}\right)^2 + 4 \ge 4$, since $\left(a-\frac{1}{a}\right)^2 \ge 0$ for real $a \ne 0$

$\therefore a+\frac{1}{a} \ge 2$ for real $a>0$

ii(α). $\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c} = \left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{b}{c}+\frac{c}{b}\right)+\left(\frac{c}{a}+\frac{a}{c}\right)$, where each of $\frac{a}{b}, \frac{b}{c}, \frac{c}{a}$ is real and positive.

$\ge 2 + 2 + 2$ using (i)

$$\therefore \frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c} \ge 6$$

ii (β). Replacing $a \to b+c,\ b \to c+a,\ c \to a+b$:

$$\frac{(c+a)+(a+b)}{b+c}+\frac{(a+b)+(b+c)}{c+a}+\frac{(b+c)+(c+a)}{a+b} \ge 6$$

$$1+\frac{2a}{b+c} + 1+\frac{2b}{c+a} + 1+\frac{2c}{a+b} \ge 6$$

$$2\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right) \ge 3$$

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \ge \frac{3}{2}$$

c. Outcomes assessed: H5

Marking Guidelines

Criteria	Marks
i • expand using compound angle trigonometric identities then simplify	1
ii • use identity from (i) to simplify sum	1
• use trigonometric identity converting difference to product	1
• simplify to obtain required result	1
iii • use appropriate trigonometric identity	1
• use result from (ii) to evaluate sum	1

Answer

i. $\sin(2k+1)\theta = \sin 2k\theta \cos\theta + \cos 2k\theta \sin\theta$ and $\sin(2k-1)\theta = \sin 2k\theta \cos\theta - \cos 2k\theta \sin\theta$

$\therefore \sin(2k+1)\theta - \sin(2k-1)\theta = 2\sin\theta\cos 2k\theta$

ii. $2\sin\theta \sum_{k=1}^{n} \cos 2k\theta = \sum_{k=1}^{n}\left\{\sin(2k+1)\theta - \sin(2k-1)\theta\right\} = \sin(2n+1)\theta - \sin\theta$

Using $\sin A - \sin B = 2\sin\frac{A-B}{2}\cos\frac{A+B}{2}$, $A = 2(n+1)\theta,\ B=\theta$: $\sin\theta\sum_{k=1}^{n}\cos 2k\theta = \sin n\theta \cos(n+1)\theta$

iii. $\sum_{k=1}^{10} \sin^2\frac{k\pi}{10} = \frac{1}{2}\sum_{k=1}^{10}\left(1-\cos\frac{2k\pi}{10}\right) = \frac{1}{2}\left\{10 - \frac{\sin\frac{10\pi}{10}\cos\frac{11\pi}{10}}{\sin\frac{\pi}{10}}\right\} = 5$

Question 16

a. Outcomes assessed: P5, H5, PE3

Marking Guidelines

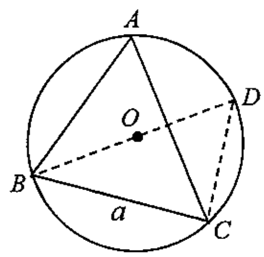
Criteria	Marks
i • provide a sequence of deductions leading to required result	1
• justify deductions using geometric properties and trigonometry	1
ii • use the sine rule and (i) to obtain result	1
iii • write *f* explicitly as function of α for fixed A and find derivative	1
• show derivative is negative throughout domain of f	1
iv • deduce $f(\alpha)$ takes its maximum value for $\alpha = 0$ and hence when ΔABC is equilateral	1
• find this maximum value	1

Answer

i. Construct diameter BD through centre O and construct DC.

$\angle BDC = \angle A$ ($\angle$'s *in same segment subtended by chord BC are equal*)

$\angle BCD = \frac{\pi}{2}$ ($\angle$ *in semi-circle is a right angle*)



In ΔBCD, $\sin\angle BDC = \dfrac{a}{BD} = \dfrac{a}{2R}$. $\quad \therefore R = \dfrac{a}{2\sin A}$

ii. $$\frac{Area\ \Delta ABC}{Area\ \ circle\ ABC} = \frac{\frac{1}{2}bc\sin A}{\pi R^2} = \frac{bc\sin A(2\sin A)^2}{2\pi a^2} = \frac{2}{\pi}abc\left(\frac{\sin A}{a}\right)^3$$

Using the sine rule in ΔABC, $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$.

$$\frac{Area\ \Delta ABC}{Area\ \ circle\ ABC} = \frac{2}{\pi}abc.\frac{\sin A}{a}.\frac{\sin B}{b}.\frac{\sin C}{c} = \frac{2}{\pi}\sin A\sin B\sin C$$

iii. $C = A-\alpha,\ \ B = \pi-(2A-\alpha)$ (*since* $\angle$ *sum of* ΔABC *is* π)

$f(\alpha) = \frac{2}{\pi}\sin A\sin B\sin C = \frac{2}{\pi}\sin A\sin(2A-\alpha)\sin(A-\alpha)$, where $0 \le \alpha < A$

Using $\cos(p-q)-\cos(p+q) = 2\sin p\sin q$ with $p = 2A-\alpha,\ \ q = A-\alpha$:

$f(\alpha) = \frac{1}{\pi}\sin A\{\cos A - \cos(3A-2\alpha)\}$

$f'(\alpha) = -\frac{2}{\pi}\sin A\sin(3A-2\alpha)$

But $3A-2\alpha = A+(A-\alpha)+(A-\alpha) = A+C+C \le A+B+C = \pi$ (*since* $A \ge B \ge C$)

Hence $\sin(3A-2\alpha) \ge 0$ and $f'(\alpha) \le 0$ for $0 \le \alpha < A$.

iv. $f(\alpha)$ is a decreasing function throughout its domain $0 \le \alpha < A$, hence it takes its maximum value when $\alpha = 0$ and $C = A$. But then $A \ge B \ge C \Rightarrow A = B = C = \frac{\pi}{3}$ and ΔABC is equilateral.

The maximum value of $\dfrac{Area\ \Delta ABC}{Area\ \ circle\ ABC}$ is $f(0) = \frac{2}{\pi}\left(\sin\frac{\pi}{3}\right)^3 = \frac{3\sqrt{3}}{4\pi}$.

Q16 (cont)

b. Outcomes assessed: PE3, HE3

Marking Guidelines

Criteria	Marks
i • expand the square and break up into separate sums	1
• manipulate sigma notation to obtain required result	1
ii • apply result to the sequence of $n+1$ binomial coefficients	1
• evaluate the sum of these binomial coefficients	1
• explain why strict inequality holds	1
iii • equate coefficients of x^n on both sides of identity using properties of binomial coefficients	1
iv • simplify factorial quotient then apply (ii) and (iii) to obtain appropriate inequality	1
• take logarithms to complete proof	1

Answer

i.

$$\sum_{k=1}^{n}\left(x_k-\tfrac{1}{n}S_n\right)^2=\sum_{k=1}^{n}\left\{x_k^{\,2}-2\left(\tfrac{1}{n}S_n\right)x_k+\left(\tfrac{1}{n}S_n\right)^2\right\}$$

$$=\sum_{k=1}^{n}x_k^{\,2}-2\left(\tfrac{1}{n}S_n\right)\sum_{k=1}^{n}x_k+\left(\tfrac{1}{n}S_n\right)^2\sum_{k=1}^{n}1$$

$$=\sum_{k=1}^{n}x_k^{\,2}-2\left(\tfrac{1}{n}S_n\right)S_n+n\left(\tfrac{1}{n}S_n\right)^2$$

$$=\sum_{k=1}^{n}x_k^{\,2}-\tfrac{1}{n}S_n^{\,2}$$

$$\sum_{k=1}^{n}x_k^{\,2}=\tfrac{1}{n}S_n^{\,2}+\sum_{k=1}^{n}\left(x_k-\tfrac{1}{n}S_n\right)^2$$

ii. $x_{k+1}={}^nC_k$, $k=0,1,2,\dots,n$. Then $(1+x)^n\equiv\sum_{k=0}^{n}{}^nC_kx^k$ gives $2^n=\sum_{k=0}^{n}{}^nC_k$. $\therefore S_{n+1}=2^n$

From (i), for real x_k , $\sum_{k=1}^{n+1}x_k^{\,2}\ge\frac{1}{n+1}S_{n+1}^{\,2}$, since each of $(x_k-\frac{1}{n+1}S_{n+1})^2\ge 0$.

Since equality only holds for $x_1=x_2=\dots=x_{n+1}$, $\sum_{k=0}^{n}\left({}^nC_k\right)^2>\frac{2^{2n}}{n+1}$ for $n=2,3,4,\dots$

iii. Considering the coefficient of x^n on both sides of the identity $\left\{(1+x)^n\right\}^2\equiv(1+x)^{2n}$,

$\sum_{k=0}^{n}{}^nC_k\,{}^nC_{n-k}={}^{2n}C_n$. But ${}^nC_{n-k}={}^nC_k$ and ${}^{2n}C_n=\frac{(2n)!}{n!\,n!}$. Hence $\sum_{k=0}^{n}\left({}^nC_k\right)^2=\frac{(2n)!}{(n!)^2}$.

iv. $$\frac{(2n)!}{(n!)^2}=\frac{2n(2n-1)\{2(n-1)\}(2n-3)\{2(n-2)\}\dots 2.1}{\{n(n-1)(n-2)\dots 1\}^2}=\frac{2^n(2n-1)(2n-3)\dots 3.1}{n(n-1)(n-2)\dots 1}$$

Using (ii) and (iii), $\frac{2^n(2n-1)(2n-3)\dots 3.1}{n(n-1)(n-2)\dots 1}>\frac{2^{2n}}{n+1}$, giving $\frac{1.3.5\dots(2n-3)(2n-1)}{2.4.6\dots\{2(n-1)\}\{2n\}}>\frac{1}{n+1}$

Taking logs of both sides gives $\ln 1-\ln 2+\ln 3-\ln 4+\dots+\ln(2n-1)-\ln(2n)>-\ln(n+1)$

$$\ln 2-\ln 3+\ln 4-\dots+\ln(2n)<\ln(n+1)$$

Also $\ln 2-\ln 3+\ln 4-\dots+\ln(2n)=\ln 2+\ln\frac{4}{3}+\ln\frac{6}{5}+\dots+\ln\frac{2n}{2n-1}>0$

$\therefore\ 0<\ln 2-\ln 3+\ln 4-\dots+\ln(2n)<\ln(n+1)$ for $n=2,3,4,\dots$

Independent HSC Trial Examinations 2014 Mathematics Extension 2 Mapping Grid

Question	Marks	Content	Syllabus Outcomes	Targeted Performance Bands
1	1	Further trigonometry	H5	E2-E3
2	1	Inverse functions	HE4	E2-E3
3	1	Graphs	E6	E2-E3
4	1	Complex numbers	E3	E2-E3
5	1	Conic sections	E4	E2-E3
6	1	Integration	E8	E3-E4
7	1	Volumes	E7	E3-E4
8	1	Polynomials	E4	E2-E3
9	1	Mechanics	E5	E3-E4
10	1	Binomial theorem	HE3	E3-E4
11 a i	1	Complex numbers	E3	E2-E3
ii	1	Complex numbers	E3	E2-E3
b i	2	Complex numbers	E3	E2-E3
ii	2	Complex numbers	E3	E2-E3
c i	2	Complex numbers	E3	E2-E3
ii	2	Complex numbers	E3	E2-E3
d i	2	Polynomials	E4	E2-E3
ii	3	Polynomials	E4	E2-E3
12 a	2	Integration	E8	E2-E3
b	2	Integration	E8	E2-E3
c	2	Integration	E8	E3-E4
d	4	Integration	E8	E2-E3
e i	3	Volumes	E7	E3-E4
ii	2	Integration	E8	E2-E3
13 a	4	Conic sections	E4	E2-E3
b i	3	Conic sections	E4	E2-E3
ii	2	Conic sections	E4	E2-E3
c i	1	Graphs	E6	E2-E3
ii	1	Graphs	E6	E2-E3
iii	2	Graphs	E6	E2-E3
iv	2	Graphs	E6	E2-E3
14 a	3	Polynomials	E4	E2-E3
b	3	Induction	HE2	E3-E4
c i	3	Mechanics	E5	E3-E4
ii	3	Mechanics	E5	E3-E4
iii	2	Mechanics	E5	E3-E4
iv	1	Mechanics	E5	E3-E4
15 a i	2	Integration	E8	E2-E3
ii	2	Integration	E8	E2-E3
b i	1	Inequalities	PE3	E2-E3
ii	4	Inequalities	PE3	E3-E4
c i	1	Further trigonometry	H5	E2-E3
ii	3	Further trigonometry	H5	E3-E4
iii	2	Further trigonometry	H5	E3-E4

Maths Extension 2 Mapping grid (continued)

Question	Marks	Content	Syllabus Outcomes	Targeted Performance Bands
16 a i	2	Circle geometry	PE3	E2-E3
ii	1	Trigonometry	H5	E2-E3
iii	2	Further trigonometry	H5	E3-E4
iv	2	Functions	P5	E3-E4
b i	2	Inequalities	PE3	E3-E4
ii	3	Inequalities	PE3	E3-E4
iii	1	Binomial Theorem	HE3	E3-E4
iv	2	Inequalities	PE3	E3-E4