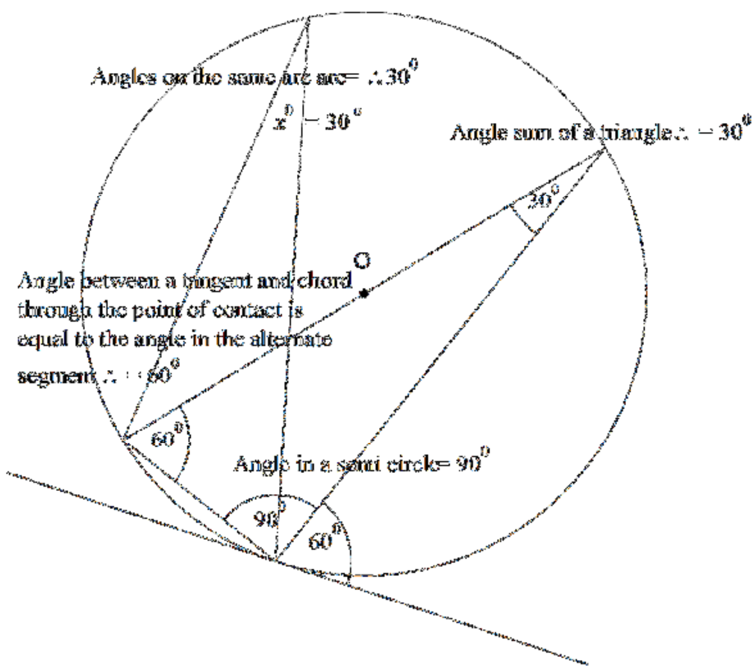
Frensham 2014
HSC Mathematics Extension 1 Trial HSC Examination
Worked solutions and marking guidelines

Section I		
	Solution	**Criteria**
1	Domain: $-1 \le \frac{x}{2} \le 1$ or $-2 \le x \le 2$. Range: $\frac{1}{2}\times 0 \le y \le \frac{1}{2}\times \pi$ or $0 \le y \le \frac{\pi}{2}$	1 Mark: C
2	$P(x) = x^3 + ax + 1$ $P(-2) = (-2)^3 + a\times -2 + 1 = 5$ $-2a = 12$ $a = -6$	1 Mark: A
3	$\int \frac{x}{(2-x^2)^3}dx = -\frac{1}{2}\int \frac{1}{u^3}du$ $u = 2 - x^2$ $\frac{du}{dx} = -2x$ $-\frac{1}{2}du = xdx$ $= -\frac{1}{2}\times -\frac{1}{2}u^{-2} + C$ $= \frac{1}{4(2-x^2)^2} + C$	1 Mark: B
4	$\int_0^1 \frac{1}{x^2+1}dx = \left[\tan^{-1} x\right]_0^1$ $= \frac{\pi}{4} - 0$ $= \frac{\pi}{4}$	1 Mark: A
5	Number of arrangements $= \frac{11!}{2!\times 2!}$ (2 I's and 2 B's) $= 9\ 979\ 200$	1 Mark: B
6	For $y = 2x$ then $m_1 = 2$ For $x + y - 5 = 0$ then $m_2 = -1$	1 Mark: D

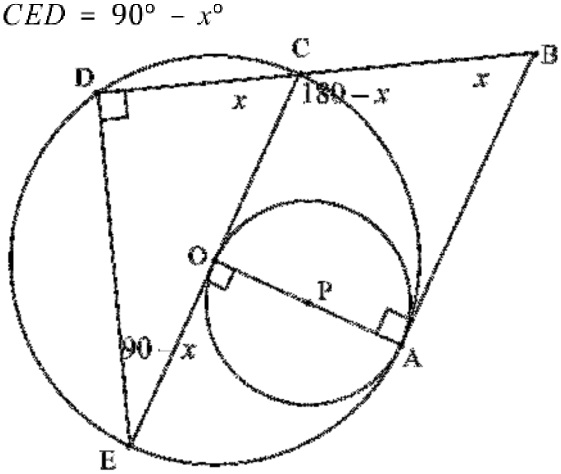
	$\tan\theta = \left\lvert\frac{m_1 - m_2}{1 + m_1 m_2}\right\rvert = \left\lvert\frac{2-(-1)}{1+2\times -1}\right\rvert$ $= 3$ $\theta = 71.56505118\ldots$ $\approx 72^\circ$	

7	. 	1 Mark: A
8	$A(-1,2)$ and $B(3,5)$ with $3:-1$ $x = \frac{mx_2 + nx_1}{m+n} = \frac{3\times 3 + -1\times -1}{3+-1} = 5$ $y = \frac{my_2 + ny_1}{m+n} = \frac{3\times 5 + -1\times 2}{3+-1} = 6.5$ Point is (5,6.5)	1 Mark: D
9	$(x-4)^2 \times \frac{2x-5}{x-4} \ge x\times(x-4)^2 \quad (x\ne 4)$ $(x-4)(2x-5) - x(x-4)^2 \ge 0$ $(x-4)[(2x-5) - x(x-4)] \ge 0$ $(x-4)(-x^2+6x-5) \ge 0$ $(x-4)(x-5)(1-x) \ge 0$ Critical points are 1,4 and 5 or use a sketch and where the	1 Mark: D

	polynomial is above the x-axis. Test values in each region $x \le 1$ and $4 < x \le 5$ Note Alternate method: since $x \ne 4$ answer must be B or D then test $x = 0$ in original inequality which gives $\frac{5}{4} > 0$ which is true so $x = 0$ must be included in the solution $\therefore$ D	
10	$v = 2x + 5$ $v^2 = 4x^2 + 20x + 25$ $\frac{1}{2}v^2 = 2x^2 + 10x + \frac{25}{2}$ $a = \frac{d}{dx}\left(2x^2 + 10x + \frac{25}{2}\right)$ $= 4x + 10$ When $x = 1$ then $a = 14$	1 Mark: C

<table>
<tr><td colspan="3">Section II</td></tr>
<tr><td>11(a)</td><td>
Let the roots be α, β and $\alpha - \beta$.
$4x^3 - 4x^2 - 29x + 15 = 0$
$\alpha + \beta + (\alpha - \beta) = -\frac{b}{a} = -\frac{-4}{4} = 1$
$2\alpha = 1$ or $\alpha = \frac{1}{2}$
$\alpha\beta(\alpha - \beta) = -\frac{d}{a}$
$\frac{1}{2}\beta(\frac{1}{2} - \beta) = -\frac{15}{4}$
$\beta(\frac{1}{2} - \beta) = -\frac{15}{2}$
$2\beta^2 - \beta - 15 = 0$
$(2\beta + 5)(\beta - 3) = 0$
$\beta = -\frac{5}{2}$ or $\beta = 3$
Roots are $x = -\frac{5}{2}$, $x = \frac{1}{2}$ and $x = 3$
</td><td>
3 Marks: Correct answer.

2 Marks: Makes significant progress towards the solution.

1 Mark: Finds the sum or product of the roots.
</td></tr>
<tr><td>11(b)</td><td>
$\angle PAB = 90°$ (Angle between a tangent and radius)
$\angle POC = 90°$ (Angle between a tangent and radius)
$\therefore OC \parallel AB$ (cointerior angles are supplementary)
$\therefore \angle OCB = 180 - x°$ (Cointerior angles on $\parallel$ lines)
$\therefore \angle DCE = x°$ (angles on straight line)
$\angle EDC = 90°$ (angle in a semicircle is a right angle)
$\angle CED = 180° - 90° - x°$ (angle sum ΔCED)
$\therefore \angle CED = 90° - x°$


</td><td>
2 Marks: Correct answer.
1 Marks: Makes some progress towards the solution.
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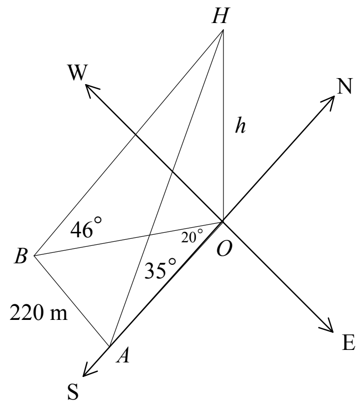
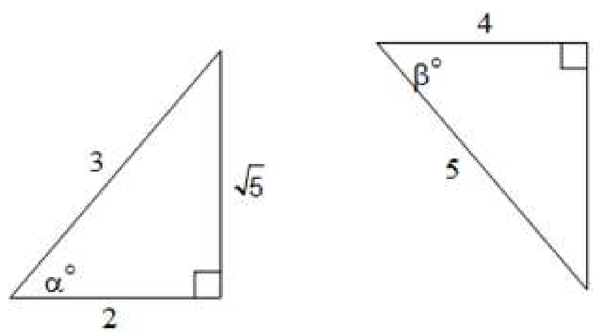
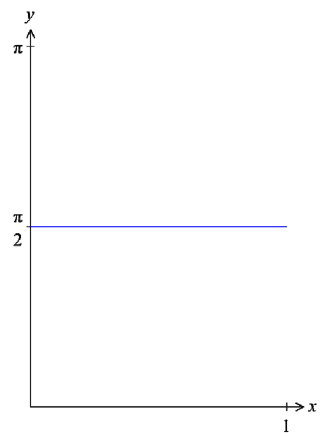
11(c)	$\text{LHS} = \dfrac{\sin\theta}{\sin\theta+\cos\theta}+\dfrac{\sin\theta}{\cos\theta-\sin\theta}$ $= \dfrac{\sin\theta(\cos\theta-\sin\theta)+\sin\theta(\cos\theta+\sin\theta)}{(\cos\theta+\sin\theta)(\cos\theta-\sin\theta)}$ $= \dfrac{2\sin\theta\cos\theta}{\cos^2\theta-\sin^2\theta}$ $= \dfrac{\sin 2\theta}{\cos 2\theta}$ $= \tan 2\theta = \text{RHS}$	2 Marks: Correct answer. 1 Marks: Uses a relevant trigonometric identity
11(d)	$\text{Number of ways} = {}^{10}C_3 \times {}^{12}C_2$ $= 120 \times 66$ $= 7920$ Class can be selected in 7920 ways.	2 Marks: Correct answer. 1 Marks: Shows some understanding.

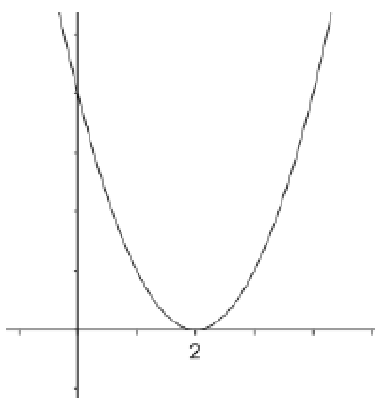
11(e) (i)	 In Δ HOA $\tan 35^\circ = \dfrac{h}{OA}$ $OA = \dfrac{h}{\tan 35^\circ}$ In Δ HOB $\tan 46^\circ = \dfrac{h}{OB}$ $OB = \dfrac{h}{\tan 46^\circ}$	2 Marks: Correct answer. 1 Mark: One correct expression or shows some understanding of the problem.
11(e) (ii)	$AB^2 = OA^2 + OB^2 - 2\times OA \times OB \times \cos 20^\circ$ $220^2 = \left(\dfrac{h}{\tan 35^\circ}\right)^2 + \left(\dfrac{h}{\tan 46^\circ}\right)^2 - 2\times\dfrac{h}{\tan 35^\circ}\times\dfrac{h}{\tan 46^\circ}\times\cos 20^\circ$ $= h^2\left(\dfrac{1}{\tan^2 35^\circ} + \dfrac{1}{\tan^2 46^\circ} - 2\times\dfrac{\cos 20^\circ}{\tan 35^\circ \times \tan 46^\circ}\right)$ $h^2 = 220^2 \div \left(\dfrac{1}{\tan^2 35^\circ} + \dfrac{1}{\tan^2 46^\circ} - 2\times\dfrac{\cos 20^\circ}{\tan 35^\circ \times \tan 46^\circ}\right)$ $= 127296.7453...$ $h = 356.7866944... \approx 357$ m	2 Marks: Correct answer. 1 Mark: Uses the cosine rule with at least one correct value.
11(f)	Factors of 5 are $\{\pm 1, \pm 5\}$ $P(1) = 1^3 + 3\times 1^2 - 9\times 1 + 5 = 0$ Therefore $(x-1)$ is a factor of $x^3 + 3x^2 - 9x + 5$ $x-1 \overline{)\,x^3 + 3x^2 - 9x + 5}$ with quotient $x^2 + 4x - 5$ $\underline{x^3 - x^2}$ $4x^2 - 9x$ $\underline{4x^2 - 4x}$ $-5x + 5$ $\underline{-5x + 5}$ $P(x) = x^3 + 3x^2 - 9x + 5 = (x-1)(x^2 + 4x - 5)$ $= (x-1)(x-1)(x+5) = (x-1)^2(x+5)$	2 Marks: Correct answer. 1 Mark: Finds one factor or shows some understanding.

12(a) (i)	To find the gradient of the tangent $y=\frac{1}{4a}x^2$ and $\frac{dy}{dx}=\frac{1}{2a}x$ At $P(2ap,ap^2)$ $\frac{dy}{dx}=\frac{1}{2a}\times 2ap=p$ Equation of the tangent at $P(2ap,ap^2)$ $y-y_1=m(x-x_1)$ $y-ap^2=p(x-2ap)$ $y=px-ap^2$ x-intercept ($y=0$) then $x=ap$. Hence $A(ap,0)$ y-intercept ($x=0$) then $y=-ap^2$. Hence $B(0,-ap^2)$ Midpoint of A and B. $M\left(\frac{ap+0}{2},\frac{0+-ap^2}{2}\right)=M\left(\frac{ap}{2},\frac{-ap^2}{2}\right)$	2 Marks: Correct answer. 1 Mark: Finds the gradient of the tangent or the coordinates of A and B.
12(a) (ii)	To find the locus of M eliminate p from coordinates of M Now $x=\frac{ap}{2}$ (1) and $y=\frac{-ap^2}{2}$ (2) From (1) $p=\frac{2x}{a}$ and sub into eqn (2) $y=\frac{-a(\frac{2x}{a})^2}{2}=\frac{-a}{2}\times\frac{4x^2}{a^2}=-\frac{2x^2}{a}$ or $x^2=-\frac{1}{2}ay$ (parabola)	1 Mark: Correct answer.
12(a) (iii)	$x^2=-\frac{1}{2}ay=4\times\left(-\frac{1}{8}a\right)\times y$ Focus is $\left(0,-\frac{1}{8}a\right)$ and equation of the directrix $y=\frac{1}{8}a$	1 Mark: Correct answer.
12(b)	Step 1: To prove the statement true for $n=1$ $\text{LHS}=1$ $\quad\text{RHS}=2^1-1=1$ Result is true for $n=1$ Step 2: Assume the result true for $n=k$ $1+2+4+...+2^{k-1}=2^k-1$ Step 3: To prove the result is true for $n=k+1$ i.e. prove $1+2+4+...+2^{k-1}+2^k=2^{k+1}-1$	3 Marks: Correct answer. 2 Marks: Proves the result true for $n=1$ and attempts to use the result of $n=k$ to prove the result for $n=k+1$.

	$\text{LHS} = 1+2+4+...+2^{k-1}+2^k$ $= 2^k - 1 + 2^k$ $= 2\times 2^k - 1$ using assumption $= 2^{k+1} - 1$ $=$ RHS Result is true for $n = k+1$ if true for $n = k$ Step 3: Proven true for n = 1, assuming true for n=k proven true for n=k+1, so true for n=1+1=2, 1+2=3, and Result true by principle of mathematical induction for all positive integers n.	1 Mark: Proves the result true for $n = 1$.
12(c)	$\sin\left[\cos^{-1}\frac{2}{3}+\tan^{-1}\left(\frac{-3}{4}\right)\right] = \sin\left[\cos^{-1}\frac{2}{3}-\tan^{-1}\frac{3}{4}\right]$ Let $\alpha = \cos^{-1}\frac{2}{3}$ and $\beta = \tan^{-1}\frac{3}{4}$  $\sin\left[\cos^{-1}\frac{2}{3}-\tan^{-1}\frac{3}{4}\right] = \sin\alpha\cos\beta - \cos\alpha\sin\beta$ $= \frac{\sqrt{5}}{3}\times\frac{4}{5}-\frac{2}{3}\times\frac{3}{5}$ $= \frac{4\sqrt{5}}{15}$	2 Marks: Correct answer. 1 Mark: Sets up the two triangles or shows some understanding of the problem.
12(d)	Let $\sin\theta+\cos\theta = R\sin(\theta+\alpha)$ $= R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$ $\therefore\ R\cos\alpha = 1$ and $R\sin\alpha = 1$ $R^2(\cos^2\alpha+\sin^2\alpha) = 2$ and $\tan\alpha = 1$ or $\alpha = \frac{\pi}{4}$ $R = \sqrt{2}$	3 Marks: Correct answer. 2 Marks: Finds two angles or makes significant progress towards the solution. 1 Mark: Sets up the sum of two

	$\sin\theta + \cos\theta = \sqrt{2}\sin(\theta + \frac{\pi}{4}) = 1$ $\sin(\theta + \frac{\pi}{4}) = \frac{1}{\sqrt{2}}$ $\theta + \frac{\pi}{4} = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \ldots$ $\theta = 0, \frac{\pi}{2}, 2\pi$	angles or shows some understanding of the problem.
12(e) (i)	$f(x) = \sin^{-1} x + \cos^{-1} x$ $f'(x) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0$	1 Mark: Correct answer.

12(e) (ii)	Since $f'(x)=0$, $f(x)$ is a constant (gradient of tangent is 0) Let $x=0$ then $f(0)=\sin^{-1}0+\cos^{-1}0=0+\frac{\pi}{2}=\frac{\pi}{2}$ Therefore $f(x)=\frac{\pi}{2}$ for $0\le x\le 1$ 	2 Marks: Correct answer. 1 Mark: Recognises that the graph is a horizontal line or shows some understanding of the problem.
13(a) (i)	$f(x)=xe^x-1$ $f(0)=0\times e^0-1=-1<0$ $f(1)=1\times e^1-1=e-1>0$ Since f(0) and f(1) have opposite signs and f(x) is a continuous function Therefore the root lies between $x=0$ and $x=1$.	1 Mark: Correct answer.
13(a) (ii)	$f(x)=xe^x-1$ $f'(x)=xe^x+e^x=e^x(x+1)$ $f(0.5)=0.5e^{0.5}-1$ $f'(0.5)=e^{0.5}(0.5+1)=1.5e^{0.5}$ $x_1=x_0-\frac{f(x_0)}{f'(x_0)}$ $=0.5-\left(\frac{0.5e^{0.5}-1}{1.5e^{0.5}}\right)=0.5710204398...\approx 0.57$	2 Marks: Correct answer. 1 Mark: Finds $f(0.5), f'(0.5)$ or shows some understanding of Newton's method.
13(b) (i)	Horizontal Motion $\ddot{x}=0$ $\dot{x}=c_1$ (when $t=0, \dot{x}=v\cos 40^\circ$) $\dot{x}=v\cos 40^\circ$ $x=v\cos 40^\circ t+c_2$ (when $t=0, x=0$) $x=v\cos 40^\circ t$ (1) Vertical Motion $\ddot{y}=-10$ $\dot{y}=-10t+c_1$ (when $t=0, \dot{y}=v\sin 40^\circ$) $\dot{y}=-10t+v\sin 40^\circ$ $y=-5t^2+v\sin 40^\circ t+c_2$ (when $t=0, y=0$) $y=-5t^2+v\sin 40^\circ t$ (2)	3 Marks: Correct answer. 2 Marks: Derives either the horizontal or vertical equations of motion. 1 Mark: States the expressions.

13(b) (ii)	From eqn (1) $t=\dfrac{x}{v\cos 40^\circ}$ sub into eqn (2) $y=-5\left(\dfrac{x}{v\cos 40^\circ}\right)^2+v\sin 40^\circ\left(\dfrac{x}{v\cos 40^\circ}\right)$ $=-\dfrac{5x^2}{v^2}\sec^2 40^\circ+x\tan 40^\circ$	2 Marks: Correct answer. 1 Mark: Eliminates t or shows some understanding.
13(b) (iii)	To find v for $x=20$ and $y=6$ $6=-\dfrac{5\times 20^2}{v^2}\sec^2 40^\circ+20\times\tan 40^\circ$ $v^2=\dfrac{5\times 20^2\times\sec^2 40^\circ}{20\tan 40^\circ-6}$ $v=17.77917137\ldots$ $\approx 17.8\text{ ms}^{-1}$	2 Marks: Correct answer. 1 Mark: Makes some progress towards the solution.
13(c) (i)	 Will have an inverse if strictly increasing or strictly decreasing only. Largest domain, containing $x=0$ where this occurs is $x\le 2$	1 Mark: Correct answer.
13(c) (ii)	Domain of $y=f^{-1}(x)$ is the range of $y=f(x)$. Range of $y=f(x)$ is $y\ge 0$. $\therefore$ domain of $y=f^{-1}(x)$ is $x\ge 0$.	1 Mark: Correct answer.
13(c) (iii)	Interchanging x and y, the inverse is $x=(y-2)^2$ $y-2=\pm\sqrt{x}$ $y=2\pm\sqrt{x}$ But as $x\le 2$ for the inverse to exist, $y=2-\sqrt{x}$.	1 Mark: Correct answer.
13(c) (iv)	$y=f(x)$ and $y=f^{-1}(x)$ **intersect on the line $y=x$.** $\therefore$ $y=(x-2)^2$ and $y=x$ can be solved simultaneously to give the points of intersection for $y=f(x)$ and $y=f^{-1}(x)$. They meet when $x=(x-2)^2$ i.e. when $x=x^2-4x+4$	2 Marks: correct explanation for why $x=(x-2)^2$ gives the point of intersection; and correctly solves equation and

	$x^2 - 5x + 4 = 0$ $(x-4)(x-1) = 0$ $x = 1$ or 4 But as $x \le 2$ for the inverse to exist, $y = f(x)$ and its inverse meet when $x=1$.	explains why one solution only. 1 Mark: one of above
14(a)	$\int \cos^2 2x dx = \int \frac{1}{2}(1+\cos 4x)dx$ $= \frac{1}{2}\left[x + \frac{1}{4}\sin 4x\right] + c$ $= \frac{x}{2} + \frac{1}{8}\sin 4x + c$	2 Marks: Correct answer. 1 Mark: Uses double angle formula.

14(b) (i)	Simple harmonic motion occurs when $\ddot{x} = -n^2 x$ Now $x = 3\cos 2t + 4\sin 2t$ $\dot{x} = -3\times 2\sin 2t + 4\times 2\cos 2t$ $\ddot{x} = -3\times 2^2\cos 2t - 4\times 2^2\sin 2t$ $= -2^2\left(3\cos 2t + 4\sin 2t\right)$ $\ddot{x} = -2^2 x$	2 Marks: Correct answer. 1 Mark: Recognises the condition for SHM.
14(b) (ii)	Maximum speed at $\ddot{x} = 0$ or $x = 0$ (centre of motion) $x = 3\cos 2t + 4\sin 2t = 0$ $4\sin 2t = -3\cos 2t$ $\tan 2t = -\frac{3}{4}$ $2t = \tan^{-1}\left(-0.75\right) + n\pi$, where n is an integer $2t = -0.6435011088\ldots + 0, \pi, 2\pi$ Smallest positive value of t for maximum speed $t = \frac{1}{2}(-0.6435011088\ldots + \pi) = 1.249045772\ldots$ $\dot{x} = -3\times 2\sin(2\times 1.24\ldots) + 4\times 2\cos(2\times 1.24...) = -10$ Maximum speed is 10 **Alternatively** using the auxillary angle method i.e. $v = -6\sin 2t + 8\cos 2t$ i.e. $v = 8\cos 2t - 6\sin 2t$ now writing this in the form $v = R\cos(2t + \alpha)$ $R = \sqrt{(-6)^2 + (8)^2} = 10$ $\alpha = \tan^{-1}(\frac{6}{8})$ $v = 10\cos(2t + \tan^{-1} 0.75)$ which has a maximum value of 10.	2 Marks: Correct answer. 1 Mark: Makes some progress towards the solution.
14(c) (i)	$T = Ae^{-kt} - 12$ or $Ae^{-kt} = T + 12$ $\frac{dT}{dt} = -kAe^{-kt}$ $= -k(T + 12)$	1 Mark: Correct answer.
14(c) (ii)	Initially $t = 0$ and $T = 24$, $T = Ae^{-kt} - 12$ $24 = Ae^{-k\times 0} - 12$ $A = 36$	1 Mark: Correct answer.
14(c) (iii)	Also $t = 15$ and $T = 9$ $9 = 36e^{-k\times 15} - 12$ $e^{-15k} = \frac{21}{36} = \frac{7}{12}$	3 Marks: Correct answer. 2 Marks: Determines the value of e^{-kt} or makes significant

	$-15k = \log_e \frac{7}{12}$ $k = -\frac{1}{15}\log_e \frac{7}{12}$ $= 0.03593310005...$ We need to find t when $T = 0$	progress. 1 Mark: Finds the exact value of k or shows some understanding.

	$0 = 36e^{-kt} - 12$ $e^{-kt} = \frac{12}{36} = \frac{1}{3}$ $-kt = \log_e \frac{1}{3}$ $t = -\frac{1}{k}\log_e \frac{1}{3}$ $= 30.5738243... \approx 31$ minutes It will take about 31 minutes for the water to cool to 0ºC	
14(d) (i)	Facing front: Number of ways $= 5\times 4\times 7!$ Facing back: Number of ways $= 4\times 3\times 7!$ Total number of ways $= (5\times 4 + 4\times 3)\times 7!$ $= 161\ 280$	2 Marks: Correct answer. 1 Mark: Makes some progress towards the solution.
14(d) (ii)	Alex facing front and Bella facing back Number of ways $= 5\times 4\times 7!$ Bella facing front and Alex facing back Number of ways $= 5\times 4\times 7!$ Total number of ways $= (5\times 4\times 7!)\times 2$ $= 201\ 600$	2 Marks: Correct answer. 1 Mark: Makes some progress towards the solution.