YEAR 11

CAMBRIDGE Mathematics 2 Unit

Second Edition

COLOUR VERSION WITH STUDENT CD-ROM

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Preface

This textbook has been written for students in Years 11 and 12 taking the 2 Unit calculus course 'Mathematics' for the NSW HSC. The book covers all the content of the course at the level required for the HSC examination. There are two volumes — the present volume is roughly intended for Year 11, and the next volume for Year 12. Schools will, however, differ in their choices of order of topics and in their rates of progress.

Although the Syllabus has not been rewritten for the new HSC, there has been a gradual shift of emphasis in recent examination papers.

- The interdependence of the course content has been emphasised.
- Graphs have been used much more freely in argument.
- Structured problem solving has been expanded.
- There has been more stress on explanation and proof.

This text addresses these new emphases, and the exercises contain a wide variety of different types of questions.

There is an abundance of questions and problems in each exercise — too many for any one student — carefully grouped in three graded sets, so that with proper selection, the book can be used at all levels of ability in the 2 Unit course.

This new second edition has been thoroughly rewritten to make it more accessible to all students. The exercises now have more early drill questions to reinforce each new skill, there are more worked exercises on each new algorithm, and some chapters and sections have been split into two so that ideas can be introduced more gradually. We have also added a review exercise to each chapter.

We would like to thank our colleagues at Sydney Grammar School and Newington College for their invaluable help in advising us and commenting on the successive drafts. We would also like to thank the Headmasters of our two schools for their encouragement of this project, and Peter Cribb, Chris Gray and the team at Cambridge University Press, Melbourne, for their support and help in discussions. Finally, our thanks go to our families for encouraging us, despite the distractions that the project has caused to family life.

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How to Use This Book

This book has been written so that it is suitable for the full range of 2 Unit students, whatever their abilities and ambitions.

The Exercises: **No-one should try to do all the questions!** We have written long exercises so that everyone will find enough questions of a suitable standard — each student will need to select from them, and there should be plenty left for revision. The book provides a great variety of questions, and representatives of all types should be attempted.

Each chapter is divided into a number of sections. Each of these sections has its own substantial exercise, subdivided into three groups of questions:

FOUNDATION: These questions are intended to drill the new content of the section at a reasonably straightforward level. There is little point in proceeding without mastery of this group.

DEVELOPMENT: This group is usually the longest. It contains more substantial questions, questions requiring proof or explanation, problems where the new content can be applied, and problems involving content from other sections and chapters to put the new ideas in a wider context.

CHALLENGE: Many questions in recent 2 Unit HSC examinations have been very demanding, and this section is intended to match the standard of those recent examinations. Some questions are algebraically challenging, some require more sophistication in logic, some establish more difficult connections between topics, and some complete proofs or give an alternative approach.

The Theory and the Worked Exercises: All the theory in the course has been properly developed, but students and their teachers should feel free to choose how thoroughly the theory is presented in any particular class. It can often be helpful to learn a method first and then return to the details of the proof and explanation when the point of it all has become clear.

The main formulae, methods, definitions and results have been boxed and numbered consecutively through each chapter. They provide a bare summary only, and students are advised to make their own short summary of each chapter using the numbered boxes as a basis.

The worked examples have been chosen to illustrate the new methods introduced in the section. They should provide sufficient preparation for the questions in the following exercise, but they cannot possibly cover the variety of questions that can be asked.

The Chapter Review Exercises: A Chapter Review Exercise has been added to each chapter of the second edition. These exercises are intended only as a basic review of the chapter — for harder questions, students are advised to work through more of the later questions in the exercises.

The Order of the Topics: We have presented the topics in the order that we have found most satisfactory in our own teaching. There are, however, many effective orderings of the topics, and apart from questions that provide links between topics, the book allows all the flexibility needed in the many different situations that apply in different schools.

We have left Euclidean geometry and probability until Chapter Seven of the Year 12 volume for two reasons. First, we believe that functions and calculus should be developed as early as possible because these are the fundamental ideas in the course. Secondly, the courses in Years 9 and 10 already develop most of the work in Euclidean geometry and probability, at least in an intuitive fashion, so that revisiting them in Year 12, with a greater emphasis now on proof in geometry, seems an ideal arrangement.

Many students, however, will want to study geometry in Year 11. The publishers have therefore made this chapter available free on their website at

www.cambridge.edu.au/education/2unitGeometry

The two geometry chapters from the 3 Unit volume are also on the website.

The Structure of the Course: Recent examination papers have made the interconnections amongst the various topics much clearer. Calculus is the backbone of the course, and the two processes of differentiation and integration, inverses of each other, are the basis of most of the topics. Both processes are introduced as geometrical ideas — differentiation is defined using tangents, and integration using areas — but the subsequent discussions, applications and exercises give many other ways of understanding them.

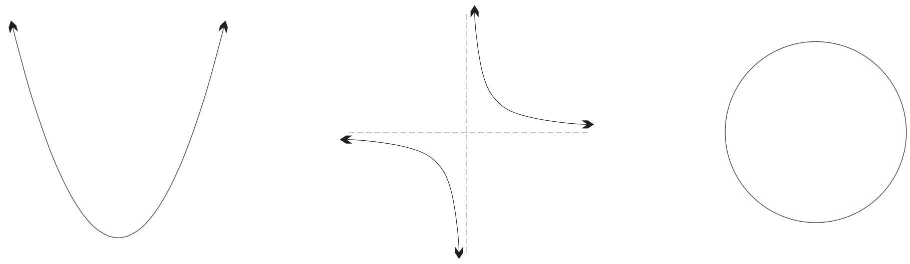
Besides linear functions, three groups of functions dominate the course:

THE QUADRATIC FUNCTIONS: (Covered in the Year 11 volume) These functions are known from earlier years. They are algebraic representations of the parabola, and arise naturally when areas are being considered or a constant acceleration is being applied. They can be studied without calculus, but calculus provides an alternative and sometimes quicker approach.

THE EXPONENTIAL AND LOGARITHMIC FUNCTIONS: (Covered in the Year 12 volume) Calculus is essential for the study of these functions. We have begun the topic with the exponential function. This has the great advantage of emphasising the fundamental property that the exponential function with base e is its own derivative — this is the reason why it is essential for the study of natural growth and decay, and therefore occurs in almost every application of mathematics. The logarithmic function, and its relationship with the rectangular hyperbola $y = 1/x$, has been covered in a separate chapter.

THE TRIGONOMETRIC FUNCTIONS: (Covered in the Year 12 volume) Calculus is also essential for the study of the trigonometric functions. Their definitions, like the associated definition of π, are based on the circle. The graphs of the sine and cosine functions are waves, and they are essential for the study of all periodic phenomena.

Thus the three basic functions in the course, x^2, e^x and $\sin x$, and the related numbers e and π, can all be developed from the three most basic degree-2 curves — the parabola, the rectangular hyperbola and the circle. In this way, everything in the course, whether in calculus, geometry, trigonometry, coordinate geometry or algebra, can easily be related to everything else.



Algebra and Graphs: One of the chief purposes of the course, stressed heavily in recent examinations, is to encourage arguments that relate a curve to its equation. Algebraic arguments are constantly used to investigate graphs of functions. Conversely, graphs are constantly used to solve algebraic problems. We have drawn as many sketches in the book as space allowed, but as a matter of routine, students should draw diagrams for most of the problems they attempt. It is because sketches can so easily be drawn that this type of mathematics is so satisfactory for study at school.

Theory and Applications: Although this course develops calculus in a purely mathematical way, using geometry and algebra, its content is fundamental to all the sciences. In particular, the applications of calculus to maximisation, motion, rates of change and finance are all parts of the syllabus. The course thus allows students to experience a double view of mathematics, as a system of pure logic on the one hand, and an essential part of modern technology on the other.

Limits, Continuity and the Real Numbers: This is a first course in calculus, and rigorous arguments about limits, continuity or the real numbers would be quite inappropriate. Any such ideas required in this course are not difficult to understand intuitively. Most arguments about limits need only the limit $\lim_{x \to \infty} 1/x = 0$ and occasionally the sandwich principle. Introducing the tangent as the limit of the secant is a dramatic new idea, clearly marking the beginning of calculus, and is quite accessible. The functions in the course are too well-behaved for continuity to be a real issue. The real numbers are defined geometrically as points on the number line, and any properties that are needed can be justified by appealing to intuitive ideas about lines and curves. Everything in the course apart from these subtle issues of 'foundations' can be proven completely.

Technology: There is much discussion about what role technology should play in the mathematics classroom and what calculators or software may be effective. This is a time for experimentation and diversity. We have therefore given only a few specific recommendations about technology, but we encourage such investigation, and to this new colour version we have added some optional technology resources which can be accessed via the student CD in the back of the book. The graphs of functions are at the centre of the course, and the more experience and intuitive understanding students have, the better able they are to interpret the mathematics correctly. A warning here is appropriate — any machine drawing of a curve should be accompanied by a clear understanding of why such a curve arises from the particular equation or situation.

About the Authors

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The mathematician's patterns, like the painter's or the poet's, must be beautiful. The ideas, like the colours or the words, must fit together in a harmonious way. Beauty is the first test.

— The English mathematician G. H. Hardy (1877–1947)

CHAPTER ONE

Methods in Algebra

Fluency in algebra, and particularly in factoring, is absolutely vital for everything in this course. Much of this chapter will be a review of earlier work, but several topics will probably be quite new, including:
- the sum and difference of cubes in Section 1E
- three simultaneous equations in three variables in Section 1H.

1 A Arithmetic with Pronumerals

A *pronumeral* is a symbol that stands for a number. The pronumeral may stand for a known number, or for an unknown number, or it may be a *variable*, standing for any one of a whole set of possible numbers. Pronumerals, being numbers, can therefore take part in all the operations that are possible with numbers, such as addition, subtraction, multiplication and division (except by zero).

Like and Unlike Terms: An *algebraic expression* consists of pronumerals, numbers and the operations of arithmetic. Here is an example:

$$x^2 + 2x + 3x^2 - 4x - 3$$

This particular algebraic expression can be *simplified* by combining *like terms*.
- The two like terms x^2 and $3x^2$ can be combined to give $4x^2$.
- Another pair of like terms $2x$ and $-4x$ can be combined to give $-2x$.
- This yields three *unlike terms*, $4x^2$, $-2x$ and -3, which cannot be combined.

WORKED EXERCISE:
Simplify each expression by combining like terms.
(a) $7a + 15 - 2a - 20$ (b) $x^2 + 2x + 3x^2 - 4x - 3$

SOLUTION:
(a) $7a + 15 - 2a - 20 = 5a - 5$ (b) $x^2 + 2x + 3x^2 - 4x - 3 = 4x^2 - 2x - 3$

Multiplying and Dividing: To simplify a product like $3y \times (-6y)$, or a quotient like $10x^2y \div 5y$, work systematically through the signs, then the numerals, and then each pronumeral in turn.

WORKED EXERCISE:
Simplify these products and quotients.
(a) $3y \times (-6y)$ (b) $4ab \times 7bc$ (c) $10x^2y \div 5y$

SOLUTION:
(a) $3y \times (-6y) = -18y^2$ (b) $4ab \times 7bc = 28ab^2c$ (c) $10x^2y \div 5y = 2x^2$

Index Laws: Here are the standard laws for dealing with indices. They will be covered in more detail in Chapter Seven.

1

THE INDEX LAWS:
- To multiply powers of the same base, add the indices: $a^x\, a^y = a^{x+y}$
- To divide powers of the same base, subtract the indices: $\dfrac{a^x}{a^y} = a^{x-y}$
- To raise a power to a power, multiply the indices: $(a^x)^n = a^{xn}$
- The power of a product is the product of the powers: $(ab)^x = a^x\, b^x$
- The power of a quotient is the quotient of the powers: $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

In expressions with several factors, work systematically through the signs, then the numerals, and then each pronumeral in turn.

WORKED EXERCISE:
Use the index laws above to simplify each expression.

(a) $3x^4 \times 4x^3$
(b) $(48x^7y^3) \div (16x^5y^3)$
(c) $(3a^4)^3$
(d) $(-5x^2)^3 \times (2xy)^4$
(e) $\left(\dfrac{2x}{3y}\right)^4$

SOLUTION:

(a) $3x^4 \times 4x^3 = 12x^7$ (multiplying powers of the same base)

(b) $(48x^7y^3) \div (16x^5y^3) = 3x^2$ (dividing powers of the same base)

(c) $(3a^4)^3 = 27a^{12}$ (raising a power to a power)

(d) $(-5x^2)^3 \times (2xy)^4 = -125x^6 \times 16x^4y^4$ (two powers of products)

$= -2000x^{10}y^4$ (multiplying powers of the same base)

(e) $\left(\dfrac{2x}{3y}\right)^4 = \dfrac{16x^4}{81y^4}$ (a power of a quotient)

Exercise 1A

1. Simplify:
(a) $5x + 3x$ (b) $5x - 3x$ (c) $-5x + 3x$ (d) $-5x - 3x$

2. Simplify:
(a) $-2a + 3a + 4a$ (b) $-2a - 3a + 4a$ (c) $-2a - 3a - 4a$ (d) $-2a + 3a - 4a$

3. Simplify:
(a) $-2x + x$
(b) $3y - y$
(c) $3a - 7a$
(d) $-8b + 5b$
(e) $4x - (-3x)$
(f) $-2ab - ba$
(g) $-3pq + 7pq$
(h) $-5abc - (-2abc)$

4. Simplify:
(a) $6x + 3 - 5x$
(b) $-2 + 2y - 1$
(c) $3a - 7 - a + 4$
(d) $3x - 2y + 5x + 6y$
(e) $-8t + 12 - 2t - 17$
(f) $2a^2 + 7a - 5a^2 - 3a$
(g) $9x^2 - 7x + 4 - 14x^2 - 5x - 7$
(h) $3a - 4b - 2c + 4a + 2b - c + 2a - b - 2c$

5. Simplify:

(a) $-3a \times 2$ (b) $-4a \times (-3a)$ (c) $a^2 \times a^3$ (d) $(a^2)^3$

6. Simplify:

(a) $-10a \div 5$ (b) $-24a \div (-8a)$ (c) $a^7 \div a^3$ (d) $7a^2 \div 7a$

7. Simplify:

(a) $\dfrac{5x}{x}$ (b) $\dfrac{-7x^3}{x}$ (c) $\dfrac{-12a^2b}{-ab}$ (d) $\dfrac{-27x^6y^7z^2}{9x^3y^3z}$

8. Simplify:

(a) $t^2 + t^2$ (b) $t^2 - t^2$ (c) $t^2 \times t^2$ (d) $t^2 \div t^2$

9. Simplify:

(a) $-6x + 3x$ (b) $-6x - 3x$ (c) $-6x \times 3x$ (d) $-6x \div 3x$

DEVELOPMENT

10. If $a = -2$, find the value of:

(a) $3a + 2$ (b) $a^3 - a^2$ (c) $3a^2 - a + 4$ (d) $a^4 + 3a^3 + 2a^2 - a$

11. If $x = 2$ and $y = -3$, find the value of:

(a) $3x + 2y$ (b) $y^2 - 5x$ (c) $8x^2 - y^3$ (d) $x^2 - 3xy + 2y^2$

12. Subtract:

(a) x from $3x$ (b) $-x$ from $3x$ (c) $2a$ from $-4a$ (d) $-b$ from $-5b$

13. Multiply:

(a) $5a$ by 2
(b) $6x$ by -3
(c) $-3a$ by a
(d) $-2a^2$ by $-3ab$
(e) $4x^2$ by $-2x^3$
(f) $-3p^2q$ by $2pq^3$

14. Divide:

(a) $-2x$ by x
(b) $3x^3$ by x^2
(c) x^3y^2 by x^2y
(d) a^6x^3 by $-a^2x^3$
(e) $14a^5b^4$ by $-2a^4b$
(f) $-50a^2b^5c^8$ by $-10ab^3c^2$

15. Find the sum of:

(a) $x + y + z$, $2x + 3y - 2z$ and $3x - 4y + z$
(b) $2a - 3b + c$, $15a - 21b - 8c$ and $24b + 7c + 3a$
(c) $5ab + bc - 3ca$, $ab - bc + ca$ and $-ab + 2ca + bc$
(d) $x^3 - 3x^2y + 3xy^2$, $-2x^2y - xy^2 - y^3$ and $x^3 + 4y^3$

16. From:

(a) $7x^2 - 5x + 6$ take $5x^2 - 3x + 2$
(b) $4a - 8b + c$ take $a - 3b + 5c$
(c) $3a + b - c - d$ take $6a - b + c - 3d$
(d) $ab - bc - cd$ take $-ab + bc - 3cd$

17. Subtract:

(a) $x^3 - x^2 + x + 1$ from $x^3 + x^2 - x + 1$
(b) $3xy^2 - 3x^2y + x^3 - y^3$ from $x^3 + 3x^2y + 3xy^2 + y^3$
(c) $b^3 + c^3 - 2abc$ from $a^3 + b^3 - 3abc$
(d) $x^4 + 5 + x - 3x^3$ from $5x^4 - 8x^3 - 2x^2 + 7$

18. Simplify:
(a) $2a^2b^4 \times 3a^3b^2$ (b) $-6ab^5 \times 4a^3b^3$ (c) $(-3a^3)^2$ (d) $(-2a^4b)^3$

CHALLENGE

19. What must be added to $4x^3 - 3x^2 + 2$ to give $3x^3 + 7x - 6$?

20. Simplify:
(a) $\dfrac{3a \times 3a \times 3a}{3a + 3a + 3a}$ (b) $\dfrac{3c \times 4c^2 \times 5c^3}{3c^2 + 4c^2 + 5c^2}$ (c) $\dfrac{ab^2 \times 2b^2c^3 \times 3c^3a^4}{a^3b^3 + 2a^3b^3 + 3a^3b^3}$

21. Simplify:
(a) $\dfrac{(-2x^2)^3}{-4x}$ (b) $\dfrac{(3xy^3)^3}{3x^2y^4}$ (c) $\dfrac{(-ab)^3 \times (-ab^2)^2}{-a^5b^3}$ (d) $\dfrac{(-2a^3b^2)^2 \times 16a^7b}{(2a^2b)^5}$

22. Divide the product of $(-3x^7y^5)^4$ and $(-2xy^6)^3$ by $(-6x^3y^8)^2$.

1 B Expanding Brackets

Expanding brackets is routine in arithmetic. For example, to calculate 7×61,

$$7 \times (60 + 1) = 7 \times 60 + 7 \times 1,$$

which quickly gives the result $7 \times 61 = 420 + 7 = 427$. The algebraic version of this procedure can be written as:

2 EXPANDING BRACKETS IN ALGEBRA:

$$a(x + y) = ax + ay$$

WORKED EXERCISE:
Expand and simplify each expression.
(a) $3x(4x - 7)$ (b) $5a(3 - b) - 3b(6 - 5a)$

SOLUTION:
(a) $3x(4x - 7) = 12x^2 - 21x$
(b) $5a(3 - b) - 3b(6 - 5a) = 15a - 5ab - 18b + 15ab$
$= 15a + 10ab - 18b$

Expanding the Product of Two Bracketed Terms: Each pair of brackets should be expanded in turn and the reulting expression should then be simplified.

WORKED EXERCISE:
Expand and simplify each expression.
(a) $(x + 3)(x - 5)$ (b) $(3 + x)(9 + 3x + x^2)$

SOLUTION:
(a) $(x + 3)(x - 5)$
$= x(x - 5) + 3(x - 5)$
$= x^2 - 5x + 3x - 15$
$= x^2 - 2x - 15$

(b) $(3 + x)(9 + 3x + x^2)$
$= 3(9 + 3x + x^2) + x(9 + 3x + x^2)$
$= 27 + 9x + 3x^2 + 9x + 3x^2 + x^3$
$= 27 + 18x + 6x^2 + x^3$

Special Expansions: These three identities are important and must be memorised. Examples of these expansions occur very frequently, and knowing the formulae greatly simplifies the working. They are proven in the exercises.

3
SQUARE OF A SUM: $(A+B)^2 = A^2 + 2AB + B^2$
SQUARE OF A DIFFERENCE: $(A-B)^2 = A^2 - 2AB + B^2$
DIFFERENCE OF SQUARES: $(A+B)(A-B) = A^2 - B^2$

WORKED EXERCISE: Use the three special expansions above to simplify:
(a) $(4x+1)^2$ (b) $(s-3t)^2$ (c) $(x+3y)(x-3y)$

SOLUTION:
(a) $(4x+1)^2 = 16x^2 + 8x + 1$ (the square of a sum)
(b) $(s-3t)^2 = s^2 - 6st + 9t^2$ (the square of a difference)
(c) $(x+3y)(x-3y) = x^2 - 9y^2$ (the difference of squares)

Exercise 1B

1. Expand:
(a) $3(x-2)$ (d) $-2(x-3)$ (g) $-(x-2)$
(b) $2(x-3)$ (e) $-3(x+2)$ (h) $-(2-x)$
(c) $-3(x-2)$ (f) $-2(x+3)$ (i) $-(x+3)$

2. Expand:
(a) $3(x+y)$ (d) $x(x-7)$ (g) $5(a+3b-2c)$
(b) $-2(p-q)$ (e) $-x(x-3)$ (h) $-3(2x-3y+5z)$
(c) $4(a+2b)$ (f) $-a(a+4)$ (i) $xy(2x-3y)$

3. Expand and simplify:
(a) $2(x+1)-x$ (e) $3-(x+1)$ (i) $-4(a-b)-3(a+2b)$
(b) $3a+5+4(a-2)$ (f) $b+c-(b-c)$ (j) $4(s-t)-5(s+t)$
(c) $2+2(x-3)$ (g) $(2x-3y)-(3x-2y)$ (k) $2x(x+6y)-x(x-5y)$
(d) $-3(a+2)+10$ (h) $3(x-2)-2(x-5)$ (l) $-7(2a-3b+c)-6(-a+4b-2c)$

4. Expand and simplify:
(a) $(x+2)(x+3)$ (f) $(2a+3)(a+5)$ (k) $(6-c)(c-3)$
(b) $(y+4)(y+7)$ (g) $(u-4)(3u+2)$ (l) $(2d-3)(4+d)$
(c) $(t+6)(t-3)$ (h) $(4p+5)(2p-3)$ (m) $(2x+3)(y-2)$
(d) $(x-4)(x+2)$ (i) $(2b-7)(b-3)$ (n) $(a-2)(5b+4)$
(e) $(t-1)(t-3)$ (j) $(5a-2)(3a+1)$ (o) $(3-2m)(4-3n)$

DEVELOPMENT

5. (a) By expanding $(A+B)(A+B)$, prove the special expansion $(A+B)^2 = A^2+2AB+B^2$.
(b) Similarly, prove the special expansions:
(i) $(A-B)^2 = A^2 - 2AB + B^2$ (ii) $(A-B)(A+B) = A^2 - B^2$

6. Use the special expansions to expand:

(a) $(x+y)^2$
(b) $(x-y)^2$
(c) $(x-y)(x+y)$
(d) $(a+3)^2$
(e) $(b-4)^2$
(f) $(c+5)^2$
(g) $(d-6)(d+6)$
(h) $(7+e)(7-e)$
(i) $(8+f)^2$
(j) $(9-g)^2$
(k) $(h+10)(h-10)$
(l) $(i+11)^2$
(m) $(2a+1)^2$
(n) $(2b-3)^2$
(o) $(3c+2)^2$
(p) $(2d+3e)^2$
(q) $(2f+3g)(2f-3g)$
(r) $(3h-2i)(3h+2i)$
(s) $(5j+4)^2$
(t) $(4k-5\ell)^2$
(u) $(4+5m)(4-5m)$
(v) $(5-3n)^2$
(w) $(7p+4q)^2$
(x) $(8-3r)^2$

7. Expand:

(a) $-a(a^2-a-1)$
(b) $-2x(x^3-2x^2-3x+1)$
(c) $3xy(2x^2y-5x^3)$
(d) $-2a^2b(a^2b^3-2a^3b)$

8. Simplify:

(a) $14-\big(10-(3x-7)-8x\big)$
(b) $4\Big(a-2(b-c)-\big(a-(b-2)\big)\Big)$

9. Expand and simplify:

(a) $\left(t+\dfrac{1}{t}\right)^2$
(b) $\left(t-\dfrac{1}{t}\right)^2$
(c) $\left(t+\dfrac{1}{t}\right)\left(t-\dfrac{1}{t}\right)$

CHALLENGE

10. Subtract $a(b+c-a)$ from the sum of $b(c+a-b)$ and $c(a+b-c)$.

11. Multiply:

(a) $a-2b$ by $a+2b$
(b) $2-5x$ by $5+4x$
(c) $4x+7$ by itself
(d) x^2+3y by x^2-4y
(e) $a+b-c$ by $a-b$
(f) $9x^2-3x+1$ by $3x+1$

12. Expand and simplify:

(a) $(a-b)(a+b)-a(a-2b)$
(b) $(x+2)^2-(x+1)^2$
(c) $(a-3)^2-(a-3)(a+3)$
(d) $(p+q)^2-(p-q)^2$
(e) $(2x+3)(x-1)-(x-2)(x+1)$
(f) $3(a-4)(a-2)-2(a-3)(a-5)$

13. Use the special expansions to find the value of:

(a) 102^2
(b) 999^2
(c) 203×197

1 C Factoring

Factoring is the reverse process of expanding brackets, and will be needed on a routine basis throughout the course. There are four basic methods, but in every situation, common factors should always be taken out first.

4

THE FOUR BASIC METHODS OF FACTORING:

- HIGHEST COMMON FACTOR: *Always try this first.*
- DIFFERENCE OF SQUARES: This involves two terms.
- QUADRATICS: This involves three terms.
- GROUPING: This involves four or more terms.

Factoring should continue until each factor is *irreducible*, meaning that it cannot be factored further.

Factoring by Taking Out the Highest Common Factor: Always look first for any common factors of all the terms, and then take out the highest common factor.

WORKED EXERCISE:

Factor each expression by taking out the highest common factor.

(a) $4x^3 + 3x^2$ (b) $18a^2b^3 - 30b^3$

SOLUTION:

(a) The highest common factor of $4x^3$ and $3x^2$ is x^2,
so $4x^3 + 3x^2 = x^2(4x + 3)$.

(b) The highest common factor of $18a^2b^3$ and $30b^3$ is $6b^3$,
so $18a^2b^3 - 30b^3 = 6b^3(3a^2 - 5)$.

Factoring by Difference of Squares: The expression must have two terms, both of which are squares. Sometimes a common factor must be taken out first.

WORKED EXERCISE:

Use the difference of squares to factor each expression.

(a) $a^2 - 36$ (b) $80x^2 - 5y^2$

SOLUTION:

(a) $a^2 - 36 = (a + 6)(a - 6)$

(b) $80x^2 - 5y^2 = 5(16x^2 - y^2)$ (Take out the highest common factor.)
$= 5(4x - y)(4x + y)$ (Use the difference of squares.)

Factoring Monic Quadratics: A quadratic is called *monic* if the coefficient of x^2 is 1. Suppose that we want to factor the monic quadratic expression $x^2 - 13x + 36$. We look for two numbers:

- whose sum is -13 (the coefficient of x), and
- whose product is $+36$ (the constant term).

WORKED EXERCISE:

Factor these monic quadratics.

(a) $x^2 - 13x + 36$ (b) $a^2 + 12a - 28$

SOLUTION:

(a) The numbers with sum -13 and product $+36$ are -9 and -4,
so $x^2 - 13x + 36 = (x - 9)(x - 4)$.

(b) The numbers with sum $+12$ and product -28 are $+14$ and -2,
so $a^2 + 12a - 28 = (a + 14)(a - 2)$.

Factoring Non-monic Quadratics: In a *non-monic* quadratic like $2x^2 + 11x + 12$, where the coefficient of x^2 is not 1, we look for two numbers:

- whose sum is 11 (the coefficient of x), and
- whose product is $12 \times 2 = 24$ (the constant times the coefficient of x^2).

WORKED EXERCISE:

Factor these non-monic quadratics.

(a) $2x^2 + 11x + 12$ (b) $6s^2 - 11s - 10$

SOLUTION:

(a) The numbers with sum 11 and product $12 \times 2 = 24$ are 8 and 3,

so $2x^2 + 11x + 12 = (2x^2 + 8x) + (3x + 12)$ (Split $11x$ into $8x + 3x$.)

$= 2x(x + 4) + 3(x + 4)$ (Take out the HCF of each group.)

$= (2x + 3)(x + 4)$. ($x + 4$ is a common factor.)

(b) The numbers with sum -11 and product $-10 \times 6 = -60$ are -15 and 4,

so $6s^2 - 11s - 10 = (6s^2 - 15s) + (4s - 10)$ (Split $-11s$ into $-15s + 4s$.)

$= 3s(2s - 5) + 2(2s - 5)$ (Take out the HCF of each group.)

$= (3s + 2)(2s - 5)$. ($2s - 5$ is a common factor.)

Factoring by Grouping: When there are four or more terms, it is sometimes possible to split the expression into groups, factor each group in turn, and then factor the whole expression by taking out a common factor or by some other method.

WORKED EXERCISE:

Factor each expression by grouping.

(a) $12xy - 9x - 16y + 12$ (b) $s^2 - t^2 + s - t$

SOLUTION:

(a) $12xy - 9x - 16y + 12 = 3x(4y - 3) - 4(4y - 3)$ (Take out the HCF of each pair.)

$= (3x - 4)(4y - 3)$ ($4y - 3$ is a common factor.)

(b) $s^2 - t^2 + s - t = (s + t)(s - t) + (s - t)$ (Factor $s^2 - t^2$ using difference of squares.)

$= (s - t)(s + t + 1)$ ($s - t$ is a common factor.)

Exercise 1C

1. Factor, by taking out any common factors:

(a) $2x + 8$ (b) $6a - 15$ (c) $ax - ay$ (d) $20ab - 15ac$ (e) $x^2 + 3x$ (f) $p^2 + 2pq$ (g) $3a^2 - 6ab$ (h) $12x^2 + 18x$ (i) $20cd - 32c$ (j) $a^2b + b^2a$ (k) $6a^2 + 2a^3$ (l) $7x^3y - 14x^2y^2$

2. Factor, by grouping in pairs:

(a) $mp + mq + np + nq$ (b) $ax - ay + bx - by$ (c) $ax + 3a + 2x + 6$ (d) $a^2 + ab + ac + bc$ (e) $z^3 - z^2 + z - 1$ (f) $ac + bc - ad - bd$ (g) $pu - qu - pv + qv$ (h) $x^2 - 3x - xy + 3y$ (i) $5p - 5q - px + qx$ (j) $2ax - bx - 2ay + by$ (k) $ab + ac - b - c$ (l) $x^3 + 4x^2 - 3x - 12$ (m) $a^3 - 3a^2 - 2a + 6$ (n) $2t^3 + 5t^2 - 10t - 25$ (o) $2x^3 - 6x^2 - ax + 3a$

3. Factor, using the difference of squares:

(a) $a^2 - 1$ (b) $b^2 - 4$ (c) $c^2 - 9$ (d) $d^2 - 100$ (e) $25 - y^2$ (f) $1 - n^2$ (g) $49 - x^2$ (h) $144 - p^2$ (i) $4c^2 - 9$ (j) $9u^2 - 1$ (k) $25x^2 - 16$ (l) $1 - 49k^2$ (m) $x^2 - 4y^2$ (n) $9a^2 - b^2$ (o) $25m^2 - 36n^2$ (p) $81a^2b^2 - 64$

4. Factor each quadratic expression. They are all monic quadratics.

(a) $a^2 + 3a + 2$
(b) $k^2 + 5k + 6$
(c) $m^2 + 7m + 6$
(d) $x^2 + 8x + 15$
(e) $y^2 + 9y + 20$
(f) $t^2 + 12t + 20$
(g) $x^2 - 4x + 3$
(h) $c^2 - 7c + 10$
(i) $a^2 - 7a + 12$
(j) $b^2 - 8b + 12$
(k) $t^2 + t - 2$
(l) $u^2 - u - 2$
(m) $w^2 - 2w - 8$
(n) $a^2 + 2a - 8$
(o) $p^2 - 2p - 15$
(p) $y^2 + 3y - 28$
(q) $c^2 - 12c + 27$
(r) $u^2 - 13u + 42$
(s) $x^2 - x - 90$
(t) $x^2 + 3x - 40$
(u) $t^2 - 4t - 32$
(v) $p^2 + 9p - 36$
(w) $u^2 - 16u - 80$
(x) $t^2 + 23t - 50$

DEVELOPMENT

5. Factor each quadratic expression. They are all non-monic quadratics.

(a) $3x^2 + 4x + 1$
(b) $2x^2 + 5x + 2$
(c) $3x^2 + 16x + 5$
(d) $3x^2 + 8x + 4$
(e) $2x^2 - 3x + 1$
(f) $5x^2 - 13x + 6$
(g) $5x^2 - 11x + 6$
(h) $6x^2 - 11x + 3$
(i) $2x^2 - x - 3$
(j) $2x^2 + 3x - 5$
(k) $3x^2 + 2x - 5$
(l) $3x^2 + 14x - 5$
(m) $2x^2 - 7x - 15$
(n) $2x^2 + x - 15$
(o) $6x^2 + 17x - 3$
(p) $6x^2 - 7x - 3$
(q) $6x^2 + 5x - 6$
(r) $5x^2 + 23x + 12$
(s) $5x^2 + 4x - 12$
(t) $5x^2 - 19x + 12$
(u) $5x^2 - 11x - 12$
(v) $5x^2 + 28x - 12$
(w) $9x^2 - 6x - 8$
(x) $3x^2 + 13x - 30$

6. Use the techniques of the previous questions to factor each expression.

(a) $a^2 - 25$
(b) $b^2 - 25b$
(c) $c^2 - 25c + 100$
(d) $2d^2 + 25d + 50$
(e) $e^3 + 5e^2 + 5e + 25$
(f) $16 - f^2$
(g) $16g^2 - g^3$
(h) $h^2 + 16h + 64$
(i) $i^2 - 16i - 36$
(j) $5j^2 + 16j - 16$
(k) $4k^2 - 16k - 9$
(l) $2k^3 - 16k^2 - 3k + 24$
(m) $2a^2 + ab - 4a - 2b$
(n) $6m^3n^4 + 9m^2n^5$
(o) $49p^2 - 121q^2$
(p) $t^2 - 14t + 40$
(q) $3t^2 + 2t - 40$
(r) $5t^2 + 54t + 40$
(s) $5t^2 + 33t + 40$
(t) $5t^3 + 10t^2 + 15t$
(u) $u^2 + 15u - 54$
(v) $3x^3 - 2x^2y - 15x + 10y$
(w) $1 - 36a^2$
(x) $4a^2 - 12a + 9$

CHALLENGE

7. Write each expression as a product of three factors. (Take out any common factors first.)

(a) $3a^2 - 12$
(b) $x^4 - y^4$
(c) $x^3 - x$
(d) $5x^2 - 5x - 30$
(e) $25y - y^3$
(f) $16 - a^4$
(g) $4x^2 + 14x - 30$
(h) $a^4 + a^3 + a^2 + a$
(i) $c^3 + 9c^2 - c - 9$
(j) $x^3 - 8x^2 + 7x$
(k) $x^4 - 3x^2 - 4$
(l) $ax^2 - a - 2x^2 + 2$

1 D Algebraic Fractions

An *algebraic fraction* is a fraction containing pronumerals. They are manipulated in the same way as arithmetic fractions, and factoring may play a major role.

Adding and Subtracting Algebraic Fractions: A common denominator is needed. Finding the lowest common denominator may involve factoring each denominator.

5 ADDING AND SUBTRACTING ALGEBRAIC FRACTIONS:

- First factor every denominator.
- Then work with the lowest common denominator.

WORKED EXERCISE:

Use a common denominator to simplify each algebraic fraction.

(a) $\dfrac{5x}{6} + \dfrac{11x}{4}$ (b) $\dfrac{2}{3x} - \dfrac{3}{5x}$ (c) $\dfrac{1}{x-4} - \dfrac{1}{x}$ (d) $\dfrac{2+x}{x^2-x} - \dfrac{5}{x-1}$

SOLUTION:

(a) $$\frac{5x}{6} + \frac{11x}{4} = \frac{10x}{12} + \frac{33x}{12} = \frac{43x}{12}$$

(b) $$\frac{2}{3x} - \frac{3}{5x} = \frac{10}{15x} - \frac{9}{15x} = \frac{1}{15x}$$

(c) $$\frac{1}{x-4} - \frac{1}{x} = \frac{x-(x-4)}{x(x-4)} = \frac{4}{x(x-4)}$$

(d) $$\frac{2+x}{x^2-x} - \frac{5}{x-1} = \frac{2+x}{x(x-1)} - \frac{5}{x-1} = \frac{2+x-5x}{x(x-1)} = \frac{2-4x}{x(x-1)}$$

Cancelling Algebraic Fractions: The key step here is to factor the numerator and denominator completely before cancelling factors.

6 CANCELLING ALGEBRAIC FRACTIONS:

- First factor the numerator and denominator.
- Then cancel out all common factors.

WORKED EXERCISE:

Simplify each algebraic fraction.

(a) $\dfrac{6x+8}{6}$ (b) $\dfrac{x^2-x}{x^2-1}$

SOLUTION:

(a) $$\frac{6x+8}{6} = \frac{2(3x+4)}{6} = \frac{3x+4}{3}$$

(which could be written as $x + \frac{4}{3}$).

(b) $$\frac{x^2-x}{x^2-1} = \frac{x(x-1)}{(x+1)(x-1)} = \frac{x}{x+1}$$

Multiplying and Dividing Algebraic Fractions: These processes are done exactly as for arithmetic fractions.

7 MULTIPLYING AND DIVIDING ALGEBRAIC FRACTIONS:

- First factor all numerators and denominators completely.
- Then cancel common factors.

To divide by an algebraic fraction, multiply by its reciprocal.

WORKED EXERCISE:

Simplify these products and quotients of algebraic fractions.

(a) $\dfrac{2a}{a^2-9} \times \dfrac{a-3}{5a}$ (b) $\dfrac{12x}{x+1} \div \dfrac{6x}{x^2+2x+1}$

SOLUTION:

(a) $\dfrac{2a}{a^2-9} \times \dfrac{a-3}{5a} = \dfrac{2a}{(a-3)(a+3)} \times \dfrac{a-3}{5a}$ (Factor a^2-9.)

$= \dfrac{2}{5(a+3)}$ (Cancel $a-3$ and a.)

(b) $\dfrac{12x}{x+1} \div \dfrac{6x}{x^2+2x+1} = \dfrac{12x}{x+1} \times \dfrac{x^2+2x+1}{6x}$ (Multiply by the reciprocal.)

$= \dfrac{12x}{x+1} \times \dfrac{(x+1)^2}{6x}$ (Factor x^2+2x+1.)

$= 2(x+1)$ (Cancel $x+1$ and $6x$.)

Simplifying Compound Fractions: A *compound fraction* is a fraction in which either the numerator or the denominator is itself a fraction.

8 SIMPLIFYING COMPOUND FRACTIONS:

- Find the lowest common multiple of the denominators on the top and bottom.
- Multiply top and bottom by this lowest common multiple.

This will clear all the fractions from the top and bottom together.

WORKED EXERCISE:

Simplify each compound fraction.

(a) $\dfrac{\frac{1}{2}-\frac{1}{3}}{\frac{1}{4}+\frac{1}{6}}$ (b) $\dfrac{\frac{1}{t}+1}{\frac{1}{t}-1}$

SOLUTION:

(a) $\dfrac{\frac{1}{2}-\frac{1}{3}}{\frac{1}{4}+\frac{1}{6}} = \dfrac{\frac{1}{2}-\frac{1}{3}}{\frac{1}{4}+\frac{1}{6}} \times \dfrac{12}{12}$

$= \dfrac{6-4}{3+2}$

$= \frac{2}{5}$

(b) $\dfrac{\frac{1}{t}+1}{\frac{1}{t}-1} = \dfrac{\frac{1}{t}+1}{\frac{1}{t}-1} \times \dfrac{t}{t}$

$= \dfrac{1+t}{1-t}$

Exercise 1D

1. Simplify:

(a) $\dfrac{x}{x}$ (b) $\dfrac{2x}{x}$ (c) $\dfrac{x}{2x}$ (d) $\dfrac{a}{a^2}$ (e) $\dfrac{3x^2}{9xy}$ (f) $\dfrac{12ab}{4a^2b}$

2. Simplify:

(a) $\dfrac{x}{3} \times \dfrac{3}{x}$ (c) $x^2 \times \dfrac{3}{x}$ (e) $\dfrac{3x}{4} \times \dfrac{2}{x^2}$ (g) $\dfrac{2ab}{3} \times \dfrac{6}{ab^2}$

(b) $\dfrac{a}{4} \div \dfrac{a}{2}$ (d) $\dfrac{1}{2b} \times b^2$ (f) $\dfrac{5}{a} \div 10$ (h) $\dfrac{8ab}{5} \div \dfrac{4ab}{15}$

3. Write as a single fraction:

(a) $\frac{x}{2}+\frac{x}{5}$ (d) $\frac{2a}{3}+\frac{3a}{2}$ (g) $\frac{1}{x}+\frac{1}{2x}$ (j) $x+\frac{1}{x}$

(b) $\frac{a}{3}-\frac{a}{6}$ (e) $\frac{7b}{10}-\frac{19b}{30}$ (h) $\frac{3}{4x}+\frac{4}{3x}$ (k) $a+\frac{b}{a}$

(c) $\frac{x}{8}-\frac{y}{12}$ (f) $\frac{xy}{30}-\frac{xy}{12}$ (i) $\frac{1}{a}-\frac{1}{b}$ (l) $\frac{1}{x}-\frac{1}{x^2}$

DEVELOPMENT

4. Simplify:

(a) $\frac{x+1}{2}+\frac{x+2}{3}$ (d) $\frac{x+2}{2}-\frac{x+3}{3}$ (g) $\frac{2x+1}{3}-\frac{x-5}{6}+\frac{x+4}{4}$

(b) $\frac{2x-1}{5}+\frac{2x+3}{4}$ (e) $\frac{2x+1}{4}-\frac{2x-3}{5}$ (h) $\frac{3x-7}{5}+\frac{4x+3}{2}-\frac{2x-5}{10}$

(c) $\frac{x+3}{6}+\frac{x-3}{12}$ (f) $\frac{2x-1}{3}-\frac{2x+1}{6}$ (i) $\frac{x-5}{3x}-\frac{x-3}{5x}$

5. Factor where possible and then simplify:

(a) $\frac{2p+2q}{p+q}$ (e) $\frac{3a^2-6ab}{2a^2b-4ab^2}$ (i) $\frac{x^2+10x+25}{x^2+9x+20}$

(b) $\frac{3t-12}{2t-8}$ (f) $\frac{x^2+2x}{x^2-4}$ (j) $\frac{ac+ad+bc+bd}{a^2+ab}$

(c) $\frac{x^2+3x}{3x+9}$ (g) $\frac{a^2-9}{a^2+a-12}$ (k) $\frac{y^2-8y+15}{2y^2-5y-3}$

(d) $\frac{a}{ax+ay}$ (h) $\frac{x^2+2x+1}{x^2-1}$ (l) $\frac{9ax+6bx-6ay-4by}{9x^2-4y^2}$

6. Simplify:

(a) $\frac{1}{x}+\frac{1}{x+1}$ (d) $\frac{2}{x-3}+\frac{3}{x-2}$ (g) $\frac{x}{x+y}+\frac{y}{x-y}$

(b) $\frac{1}{x}-\frac{1}{x+1}$ (e) $\frac{3}{x+1}-\frac{2}{x-1}$ (h) $\frac{a}{x+a}-\frac{b}{x+b}$

(c) $\frac{1}{x+1}+\frac{1}{x-1}$ (f) $\frac{2}{x-2}-\frac{2}{x+3}$ (i) $\frac{x}{x-1}-\frac{x}{x+1}$

7. Simplify:

(a) $\frac{8a^3b}{5}\div\frac{4ab}{15}$ (c) $\frac{12x^2yz}{8xy^3}\times\frac{24xy^2}{36yz^2}$

(b) $\frac{2a}{3b}\times\frac{5c^2}{2a^2b}\times\frac{3b^2}{2c}$ (d) $\frac{3a^2b}{4b^3c}\times\frac{2c^2}{8a^3}\div\frac{6ac}{16b^2}$

CHALLENGE

8. Simplify:

(a) $\frac{3x+3}{2x}\times\frac{x^2}{x^2-1}$ (d) $\frac{x^2-x-20}{x^2-25}\times\frac{x^2-x-2}{x^2+2x-8}\div\frac{x+1}{x^2+5x}$

(b) $\frac{a^2+a-2}{a+2}\times\frac{a^2-3a}{a^2-4a+3}$ (e) $\frac{ax+bx-2a-2b}{3x^2-5x-2}\times\frac{9x^2-1}{a^2+2ab+b^2}$

(c) $\frac{c^2+5c+6}{c^2-16}\div\frac{c+3}{c-4}$ (f) $\frac{2x^2+x-15}{x^2+3x-28}\div\frac{x^2+6x+9}{x^2-4x}\div\frac{6x^2-15x}{x^2-49}$

9. Simplify:

(a) $\dfrac{1}{x^2+x}+\dfrac{1}{x^2-x}$

(b) $\dfrac{1}{x^2-4}+\dfrac{1}{x^2-4x+4}$

(c) $\dfrac{1}{x-y}+\dfrac{2x-y}{x^2-y^2}$

(d) $\dfrac{3}{x^2+2x-8}-\dfrac{2}{x^2+x-6}$

(e) $\dfrac{x}{a^2-b^2}-\dfrac{x}{a^2+ab}$

(f) $\dfrac{1}{x^2-4x+3}+\dfrac{1}{x^2-5x+6}-\dfrac{1}{x^2-3x+2}$

10. Simplify:

(a) $\dfrac{b-a}{a-b}$

(b) $\dfrac{v^2-u^2}{u-v}$

(c) $\dfrac{x^2-5x+6}{2-x}$

(d) $\dfrac{1}{a-b}-\dfrac{1}{b-a}$

(e) $\dfrac{m}{m-n}+\dfrac{n}{n-m}$

(f) $\dfrac{x-y}{y^2+xy-2x^2}$

11. Study the worked exercise on compound fractions and then simplify:

(a) $\dfrac{1-\frac{1}{2}}{1+\frac{1}{2}}$

(b) $\dfrac{2+\frac{1}{3}}{5-\frac{2}{3}}$

(c) $\dfrac{\frac{1}{2}-\frac{1}{5}}{1+\frac{1}{10}}$

(d) $\dfrac{\frac{17}{20}-\frac{3}{4}}{\frac{4}{5}-\frac{3}{10}}$

(e) $\dfrac{\frac{1}{x}}{1+\frac{2}{x}}$

(f) $\dfrac{t-\frac{1}{t}}{t+\frac{1}{t}}$

(g) $\dfrac{1}{\frac{1}{b}+\frac{1}{a}}$

(h) $\dfrac{\frac{x}{y}+\frac{y}{x}}{\frac{x}{y}-\frac{y}{x}}$

(i) $\dfrac{1-\frac{1}{x+1}}{\frac{1}{x}+\frac{1}{x+1}}$

(j) $\dfrac{\frac{3}{x+2}-\frac{2}{x+1}}{\frac{5}{x+2}-\frac{4}{x+1}}$

1 E Factoring the Sum and Difference of Cubes

The factoring of the *difference of cubes* is similar to the factoring of the difference of squares. The *sum of cubes* can also be factored, whereas the sum of squares cannot be factored.

9

DIFFERENCE OF CUBES: $A^3-B^3=(A-B)(A^2+AB+B^2)$

SUM OF CUBES: $A^3+B^3=(A+B)(A^2-AB+B^2)$

The proofs of these identities are left to the first question in the following exercise.

WORKED EXERCISE:

Factor each expression.

(a) x^3-8 (b) $27a^3+1$

SOLUTION:

(a) $x^3-8=(x-2)(x^2+2x+4)$ (Use the difference of cubes x^3-2^3.)

(b) $27a^3+1=(3a+1)(9a^2-3a+1)$ (Use the sum of cubes $(3a)^3+1^3$.)

WORKED EXERCISE:

(a) Simplify $\dfrac{a^3+1}{a+1}$.   (b) Factor a^3-b^3+a-b.

SOLUTION:

(a) $\dfrac{a^3+1}{a+1}=\dfrac{(a+1)(a^2-a+1)}{a+1}$

$=a^2-a+1$

(b) a^3-b^3+a-b

$=(a-b)(a^2+ab+b^2)+(a-b)$

$=(a-b)(a^2+ab+b^2+1)$

Exercise 1E

1. (a) Prove the identity $A^3 - B^3 = (A - B)(A^2 + AB + B^2)$ by expanding the RHS.
(b) Similarly, prove the identity $A^3 + B^3 = (A + B)(A^2 - AB + B^2)$.

2. Factor each expression.
(a) $x^3 + y^3$
(b) $a^3 - b^3$
(c) $y^3 + 1$
(d) $k^3 - 1$
(e) $a^3 + 8$
(f) $b^3 - 8$
(g) $27 - t^3$
(h) $27 + u^3$
(i) $x^3 + 64$
(j) $y^3 - 64$
(k) $125 + a^3$
(l) $125 - b^3$

DEVELOPMENT

3. Factor each expression.
(a) $8p^3 + 1$
(b) $8q^3 - 1$
(c) $u^3 - 64v^3$
(d) $t^3 + 64u^3$
(e) $27c^3 + 8$
(f) $27d^3 - 8$
(g) $64m^3 - 125n^3$
(h) $64p^3 + 125q^3$
(i) $216e^3 - 343f^3$
(j) $216g^3 + 343h^3$
(k) $a^3b^3c^3 + 1000$
(l) $729x^3 - 1331y^3$

4. Factor each expression fully. (Remember to take out any common factors first.)
(a) $5x^3 - 5$
(b) $2x^3 + 16$
(c) $a^4 - ab^3$
(d) $24t^3 + 81$
(e) $x^3y - 125y$
(f) $250p^3 - 432q^3$
(g) $27x^4 + 1000xy^3$
(h) $5x^3y^3 - 5$
(i) $x^6 + x^3y^3$

CHALLENGE

5. Simplify each expression.
(a) $\dfrac{x^3 - 1}{x^2 - 1}$
(b) $\dfrac{a^2 - 3a - 10}{a^3 + 8}$
(c) $\dfrac{a^3 + 1}{6a^2} \times \dfrac{3a}{a^2 + a}$
(d) $\dfrac{x^2 - 9}{x^4 - 27x} \div \dfrac{x + 3}{x^2 + 3x + 9}$

6. Simplify:
(a) $\dfrac{3}{a - 2} - \dfrac{3a}{a^2 + 2a + 4}$
(b) $\dfrac{1}{x^3 - 1} + \dfrac{x + 1}{x^2 + x + 1}$
(c) $\dfrac{1}{x^2 - 2x - 8} - \dfrac{1}{x^3 + 8}$
(d) $\dfrac{a^2}{a^3 + b^3} + \dfrac{a - b}{a^2 - ab + b^2} + \dfrac{1}{a + b}$

1 F Solving Linear Equations

The first principle in solving any equation is to simplify the equation by doing the same things to both sides. Linear equations can be solved completely by following this principle

10 SOLVING LINEAR EQUATIONS:
- Any number can be added to or subtracted from both sides.
- Both sides can be multiplied or divided by any non-zero number.

An equation involving algebraic fractions can often be reduced to a linear equation by following these steps.

WORKED EXERCISE: Solve each equation.

(a) $6x + 5 = 4x - 9$ (b) $\dfrac{4 - 7x}{4x - 7} = 1$

SOLUTION:

(a)
$$6x + 5 = 4x - 9$$
$\boxed{-4x}$ $\quad 2x + 5 = -9$

$\boxed{-5}$ $\quad 2x = -14$

$\boxed{\div 2}$ $\quad x = -7$

(b)
$$\frac{4 - 7x}{4x - 7} = 1$$
$\boxed{\times (4x - 7)}$ $\quad 4 - 7x = 4x - 7$

$\boxed{+7x}$ $\quad 4 = 11x - 7$

$\boxed{+7}$ $\quad 11 = 11x$

$\boxed{\div 11}$ $\quad x = 1$

Changing the Subject of a Formula: Similar sequences of operations allow the subject of a formula to be changed from one pronumeral to another.

WORKED EXERCISE:

Change the subject of each formula to x.

(a) $y = 4x - 3$ (b) $y = \dfrac{x + 1}{x + 2}$

SOLUTION:

(a)
$$y = 4x - 3$$
$\boxed{+3}$ $\quad y + 3 = 4x$

$\boxed{\div 4}$ $\quad \dfrac{y + 3}{4} = x$

$$x = \frac{y + 3}{4}$$

(b)
$$y = \frac{x + 1}{x + 2}$$
$\boxed{\times (x + 2)}$ $\quad xy + 2y = x + 1$

Rearranging, $\quad xy - x = 1 - 2y$

Factoring, $\quad x(y - 1) = 1 - 2y$

$\boxed{\div (y - 1)}$ $\quad x = \dfrac{1 - 2y}{y - 1}$

Exercise 1F

1. Solve:

(a) $a - 10 = 5$
(b) $t + 3 = 1$
(c) $5c = -35$
(d) $\dfrac{n}{6} = -3$
(e) $-2x = -20$
(f) $3x = 2$
(g) $-a = 5$
(h) $\dfrac{x}{-4} = -1$
(i) $-1 - x = 0$
(j) $0{\cdot}1y = 5$
(k) $2t = t$
(l) $-\frac{1}{2}x = 8$

2. Solve:

(a) $2x + 1 = 7$
(b) $5p - 2 = -2$
(c) $\dfrac{a}{2} - 1 = 3$
(d) $3 - w = 4$
(e) $3x - 5 = 22$
(f) $4x + 7 = -13$
(g) $1 - 2x = 9$
(h) $6x = 3x - 21$
(i) $-2 = 4 + \dfrac{t}{5}$
(j) $-13 = 5a - 6$
(k) $19 = 3 - 7y$
(l) $23 - \dfrac{u}{3} = 7$

3. Solve:

(a) $3n - 1 = 2n + 3$
(b) $4b + 3 = 2b + 1$
(c) $5x - 2 = 2x + 10$
(d) $5 - x = 27 + x$
(e) $16 + 9a = 10 - 3a$
(f) $13y - 21 = 20y - 35$
(g) $13 - 12x = 6 - 3x$
(h) $3(x + 7) = -2(x - 9)$
(i) $8 + 4(2 - x) = 3 - 2(5 - x)$
(j) $7x - (3x + 11) = 6 - (15 - 9x)$
(k) $4(x + 2) = 4x + 9$
(l) $3(x - 1) = 2(x + 1) + x - 5$

DEVELOPMENT

4. Solve:

(a) $\dfrac{x}{8}=\dfrac{1}{2}$
(b) $\dfrac{a}{12}=\dfrac{2}{3}$
(c) $\dfrac{y}{20}=\dfrac{4}{5}$
(d) $\dfrac{1}{x}=3$
(e) $\dfrac{2}{a}=5$
(f) $3=\dfrac{9}{2y}$
(g) $\dfrac{2x+1}{5}=-3$
(h) $\dfrac{5a}{3}=a+1$
(i) $\dfrac{7-4x}{6}=1$
(j) $\dfrac{5+a}{a}=-3$
(k) $\dfrac{9-2t}{t}=13$
(l) $6-\dfrac{c}{3}=c$
(m) $\dfrac{1}{a}+4=1-\dfrac{2}{a}$
(n) $\dfrac{4}{x-1}=-5$
(o) $\dfrac{3x}{1-2x}=7$
(p) $\dfrac{11t}{8t+13}=-2$

5. Solve:

(a) $(x-3)(x+6)=(x-4)(x-5)$
(b) $(1+2x)(4+3x)=(2-x)(5-6x)$
(c) $(x+3)^2=(x-1)^2$
(d) $(2x-5)(2x+5)=(2x-3)^2$

6. (a) If $v=u+at$, find a when $t=4$, $v=20$ and $u=8$.
(b) Given that $v^2=u^2+2as$, find the value of s when $u=6$, $v=10$ and $a=2$.
(c) Suppose that $\dfrac{1}{u}+\dfrac{1}{v}=\dfrac{1}{t}$. Find v, given that $u=-1$ and $t=2$.
(d) If $S=-15$, $n=10$ and $a=-24$, find ℓ, given that $S=\frac{n}{2}(a+\ell)$.
(e) The formula $F=\frac{9}{5}C+32$ relates temperatures in degrees Fahrenheit and Celsius. Find the value of C that corresponds to $F=95$.
(f) Suppose that c and d are related by the formula $\dfrac{3}{c+1}=\dfrac{5}{d-1}$. Find c when $d=-2$.

7. Solve each problem by forming, and then solving, a linear equation.
(a) Three less than four times a certain number is equal to 21. Find the number.
(b) Five more than twice a certain number is one more than the number itself. What is the number?
(c) Bill and Derek collect Batman cards. Bill has three times as many cards as Derek, and altogether they have 68 cards. How many cards does Derek have?
(d) If I paid \$1.45 for an apple and an orange, and the apple cost 15 cents more than the orange, how much did the orange cost?

8. Solve:

(a) $\dfrac{x}{3}-\dfrac{x}{5}=2$
(b) $y+\dfrac{y}{2}=1$
(c) $\dfrac{a}{10}-\dfrac{a}{6}=1$
(d) $\dfrac{x}{6}+\dfrac{2}{3}=\dfrac{x}{2}-\dfrac{5}{6}$
(e) $\dfrac{x}{3}-2=\dfrac{x}{2}-3$
(f) $\dfrac{1}{x}-3=\dfrac{1}{2x}$
(g) $\dfrac{1}{2x}-\dfrac{2}{3}=1-\dfrac{1}{3x}$
(h) $\dfrac{x-2}{3}=\dfrac{x+4}{4}$
(i) $\dfrac{3}{x-2}=\dfrac{4}{2x+5}$
(j) $\dfrac{x+1}{x+2}=\dfrac{x-3}{x+1}$
(k) $\dfrac{(3x-2)(3x+2)}{(3x-1)^2}=1$
(l) $\dfrac{a+5}{2}-\dfrac{a-1}{3}=1$

9. Rearrange each formula so that the pronumeral written in square brackets is the subject.

(a) $a=bc-d$ $[b]$
(b) $t=a+(n-1)d$ $[n]$
(c) $\dfrac{p}{q+r}=t$ $[r]$
(d) $u=1+\dfrac{3}{v}$ $[v]$

CHALLENGE

10. Solve:

(a) $\frac{3}{4} - \frac{x+1}{12} = \frac{2}{3} - \frac{x-1}{6}$

(b) $\frac{x+1}{2} - \frac{x-1}{3} = \frac{x+1}{3} - \frac{x-1}{2}$

(c) $\frac{2x}{5} + \frac{2-3x}{4} = \frac{3}{10} - \frac{3-5x}{2}$

(d) $\frac{3}{4}(x-1) - \frac{1}{2}(3x+2) = 0$

(e) $\frac{4x+1}{6} - \frac{2x-1}{15} = \frac{3x-5}{5} - \frac{6x+1}{10}$

(f) $\frac{7(1-x)}{12} - \frac{3+2x}{9} = \frac{5(2+x)}{6} - \frac{4-5x}{18}$

11. Solve each problem by forming, and then solving, a linear equation.

(a) My father is 40 years older than me and he is three times my age. How old am I?

(b) The fuel tank in my new car was 40% full. I added 28 litres and then found that it was 75% full. How much fuel does the tank hold?

(c) A basketballer has scored 312 points in 15 games. How many points must he average per game in his next 3 games to take his overall average to 20 points per game?

(d) A cyclist rides for 5 hours at a certain speed and then for 4 hours at a speed 6 km/h greater than his original speed. If he rides 294 km altogether, what was his first speed?

12. Rearrange each formula so that the pronumeral written in square brackets is the subject.

(a) $\frac{a}{2} - \frac{b}{3} = a$ $[a]$

(b) $\frac{1}{f} + \frac{2}{g} = \frac{5}{h}$ $[g]$

(c) $x = \frac{y}{y+2}$ $[y]$

(d) $a = \frac{b+5}{b-4}$ $[b]$

(e) $c = \frac{7+2d}{5-3d}$ $[d]$

(f) $u = \frac{v+w-1}{v-w+1}$ $[v]$

1 G Solving Quadratic Equations

There are three approaches to solving a quadratic equation:

- factoring
- completing the square
- using the quadratic formula.

This section reviews factoring and the quadratic formula. Completing the square will be reviewed in Section 1I.

Solving a Quadratic by Factoring: This method is the simplest, but does not always work.

11 SOLVING A QUADRATIC BY FACTORING:

1. Get all the terms on the left, then factor the left-hand side.
2. Use the principle that if $AB = 0$, then $A = 0$ or $B = 0$.

WORKED EXERCISE:

Solve the quadratic equation $5x^2 + 34x - 7 = 0$ by factoring.

SOLUTION:

$$5x^2 + 34x - 7 = 0$$

$$5x^2 + 35x - x - 7 = 0 \quad \text{(35 and } -1 \text{ have sum 34 and product } -7 \times 5 = -35.)$$

$$5x(x+7) - (x+7) = 0$$

$$(5x-1)(x+7) = 0 \quad \text{(The LHS is now factored.)}$$

$$5x - 1 = 0 \text{ or } x + 7 = 0 \quad \text{(One of the factors must be zero.)}$$

$$x = \tfrac{1}{5} \text{ or } x = -7 \quad \text{(There are two solutions.)}$$

Solving a Quadratic by the Formula: This method works whether the solutions are rational numbers or involve surds. It will be proven in Chapter Ten.

12 THE QUADRATIC FORMULA:
- The solutions of $ax^2 + bx + c = 0$ are:
$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$
- Always calculate $b^2 - 4ac$ first.
(Later, this quantity will be called the *discriminant* and given the symbol Δ.)

WORKED EXERCISE:
Solve each quadratic equation using the quadratic formula.
(a) $5x^2 + 2x - 7 = 0$
(b) $3x^2 + 4x - 1 = 0$

SOLUTION:
(a) For $5x^2 + 2x - 7 = 0$,
$a = 5$, $b = 2$ and $c = -7$.
Hence
$$\begin{aligned} b^2 - 4ac &= 2^2 + 140 \\ &= 144 \\ &= 12^2, \end{aligned}$$
so $x = \dfrac{-2+12}{10}$ or $\dfrac{-2-12}{10}$
$= 1$ or $-1\frac{2}{5}$.

(b) For $3x^2 + 4x - 1 = 0$,
$a = 3$, $b = 4$ and $c = -1$.
Hence
$$\begin{aligned} b^2 - 4ac &= 4^2 + 12 \\ &= 28 \\ &= 4 \times 7, \end{aligned}$$
so $x = \dfrac{-4 + 2\sqrt{7}}{6}$ or $\dfrac{-4 - 2\sqrt{7}}{6}$
$= \dfrac{-2 + \sqrt{7}}{3}$ or $\dfrac{-2 - \sqrt{7}}{3}$.

Exercise 1G

1. Solve:
(a) $x^2 = 9$
(b) $y^2 = 25$
(c) $a^2 - 4 = 0$
(d) $c^2 - 36 = 0$
(e) $1 - t^2 = 0$
(f) $x^2 = \frac{9}{4}$
(g) $4x^2 - 1 = 0$
(h) $9a^2 - 64 = 0$
(i) $25y^2 = 16$

2. Solve, by factoring:
(a) $x^2 - 5x = 0$
(b) $y^2 + y = 0$
(c) $c^2 + 2c = 0$
(d) $k^2 - 7k = 0$
(e) $t^2 = t$
(f) $3a = a^2$
(g) $2b^2 - b = 0$
(h) $3u^2 + u = 0$
(i) $4x^2 + 3x = 0$
(j) $2a^2 = 5a$
(k) $3y^2 = 2y$
(l) $12h + 5h^2 = 0$

3. Solve, by factoring:
(a) $x^2 + 4x + 3 = 0$
(b) $x^2 - 3x + 2 = 0$
(c) $x^2 + 6x + 8 = 0$
(d) $a^2 - 7a + 10 = 0$
(e) $t^2 - 4t - 12 = 0$
(f) $c^2 - 10c + 25 = 0$
(g) $n^2 - 9n + 8 = 0$
(h) $p^2 + 2p - 15 = 0$
(i) $a^2 - 10a - 24 = 0$
(j) $y^2 + 4y = 5$
(k) $p^2 = p + 6$
(l) $a^2 = a + 132$
(m) $c^2 + 18 = 9c$
(n) $8t + 20 = t^2$
(o) $u^2 + u = 56$
(p) $k^2 = 24 + 2k$
(q) $50 + 27h + h^2 = 0$
(r) $\alpha^2 + 20\alpha = 44$

DEVELOPMENT

4. Solve, by factoring:

(a) $2x^2 + 3x + 1 = 0$
(b) $3a^2 - 7a + 2 = 0$
(c) $4y^2 - 5y + 1 = 0$
(d) $2x^2 + 11x + 5 = 0$
(e) $2x^2 + x - 3 = 0$
(f) $3n^2 - 2n - 5 = 0$
(g) $3b^2 - 4b - 4 = 0$
(h) $2a^2 + 7a - 15 = 0$
(i) $2y^2 - y - 15 = 0$
(j) $3y^2 + 10y = 8$
(k) $5x^2 - 26x + 5 = 0$
(l) $4t^2 + 9 = 15t$
(m) $13t + 6 = 5t^2$
(n) $10u^2 + 3u - 4 = 0$
(o) $25x^2 + 1 = 10x$
(p) $6x^2 + 13x + 6 = 0$
(q) $12b^2 + 3 + 20b = 0$
(r) $6k^2 + 13k = 8$

5. Solve each equation, using the quadratic formula. Give exact answers, followed by approximations to four significant figures where appropriate.

(a) $x^2 - x - 1 = 0$
(b) $y^2 + y = 3$
(c) $a^2 + 12 = 7a$
(d) $u^2 + 2u - 2 = 0$
(e) $c^2 - 6c + 2 = 0$
(f) $4x^2 + 4x + 1 = 0$
(g) $2a^2 + 1 = 4a$
(h) $5x^2 + 13x - 6 = 0$
(i) $2b^2 + 3b = 1$
(j) $3c^2 = 4c + 3$
(k) $4t^2 = 2t + 1$
(l) $x^2 + x + 1 = 0$

6. Solve, by factoring:

(a) $x = \dfrac{x+2}{x}$
(b) $a + \dfrac{10}{a} = 7$
(c) $y + \dfrac{2}{y} = \dfrac{9}{2}$
(d) $(5b - 3)(3b + 1) = 1$
(e) $\dfrac{5k+7}{k-1} = 3k + 2$
(f) $\dfrac{u+3}{2u-7} = \dfrac{2u-1}{u-3}$

7. Find the exact solutions of:

(a) $x = \dfrac{1}{x} + 2$
(b) $\dfrac{4x-1}{x} = x$
(c) $a = \dfrac{a+4}{a-1}$
(d) $\dfrac{5m}{2} = 2 + \dfrac{1}{m}$
(e) $\dfrac{y+1}{y+2} = \dfrac{3-y}{y-4}$
(f) $2(k-1) = \dfrac{4-5k}{k+1}$

8. (a) If $y = px - ap^2$, find p, given that $a = 2$, $x = 3$ and $y = 1$.

(b) Given that $(x-a)(x-b) = c$, find x when $a = -2$, $b = 4$ and $c = 7$.

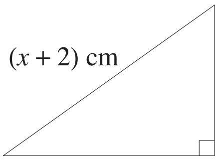
(c) Suppose that $S = \dfrac{n}{2}\big(2a + (n-1)d\big)$. Find the positive value of n that gives $S = 80$ when $a = 4$ and $d = 6$.

9. Solve each problem by forming and then solving a suitable quadratic equation.

(a) Find a positive integer that, when increased by 30, is 12 less than its square.

(b) Two positive numbers differ by 3 and the sum of their squares is 117. Find the numbers.

(c) Find the value of x in the diagram opposite.



CHALLENGE

10. Solve each equation.

(a) $\dfrac{2}{a+3} + \dfrac{a+3}{2} = \dfrac{10}{3}$
(b) $\dfrac{k+10}{k-5} - \dfrac{10}{k} = \dfrac{11}{6}$
(c) $\dfrac{3t}{t^2 - 6} = \sqrt{3}$
(d) $\dfrac{3m+1}{3m-1} - \dfrac{3m-1}{3m+1} = 2$

11. Solve each problem by constructing and then solving a quadratic equation.
(a) A rectangular area can be completely tiled with 200 square tiles. If the side length of each tile was increased by 1 cm, it would take only 128 tiles to tile the area. Find the side length of each tile.
(b) The numerator of a certain fraction is 3 less than its denominator. If 6 is added to the numerator and 5 to the denominator, the value of the fraction is doubled. Find the fraction.
(c) A photograph is 18 cm by 12 cm. It is to be surrounded by a frame of uniform width whose area is equal to that of the photograph. Find the width of the frame.
(d) A certain tank can be filled by two pipes in 80 minutes. The larger pipe by itself can fill the tank in 2 hours less than the smaller pipe by itself. How long does each pipe take to fill the tank on its own?
(e) Two trains each make a journey of 330 km. One of the trains travels 5 km/h faster than the other and takes 30 minutes less time. Find the speeds of the trains.

1 H Solving Simultaneous Equations

This section reviews the two algebraic approaches to solving simultaneous equations — substitution and elimination. Both linear and non-linear simultaneous equations are reviewed. The methods are extended to systems of three equations in three unknowns, which will be new for most readers.

Solution by Substitution: This method can be applied whenever one of the equations can be solved for one of the variables.

13 SOLVING SIMULTANEOUS EQUATIONS BY SUBSTITUTION:
- Solve one of the equations for one of the variables.
- Then substitute it into the other equation.

WORKED EXERCISE:

Solve each pair of simultaneous equations by substitution.

(a) $3x - 2y = 29$ (1)
$4x + y = 24$ (2)

(b) $y = x^2$ (1)
$y = x + 2$ (2)

SOLUTION:

(a) Solving (2) for y, $y = 24 - 4x$. (2A)
Substituting (2A) into (1), $3x - 2(24 - 4x) = 29$
$x = 7.$
Substituting $x = 7$ into (1), $21 - 2y = 29$
$y = -4.$
Hence $x = 7$ and $y = -4$. (This should be checked in the original equations.)

(b) Substituting (1) into (2), $x^2 = x + 2$
$x^2 - x - 2 = 0$
$(x - 2)(x + 1) = 0$
$x = 2$ or -1.
From (1), when $x = 2$, $y = 4$, and when $x = -1$, $y = 1$.
Hence $x = 2$ and $y = 4$, or $x = -1$ and $y = 1$. (Check in the original equations.)

Solution by Elimination: This method, when it can be used, is more elegant, and can involve less algebraic manipulation.

14 SOLVING SIMULTANEOUS EQUATIONS BY ELIMINATION:
Take suitable multiples of the equations so that one variable is eliminated when the equations are added or subtracted.

WORKED EXERCISE:

Solve each pair of simultaneous equations by elimination.

(a) $3x - 2y = 29$ (1)
$4x + 5y = 8$ (2)

(b) $x^2 + y^2 = 53$ (1)
$x^2 - y^2 = 45$ (2)

SOLUTION:

(a) Taking $4 \times$ (1) and $3 \times$ (2),

$$12x - 8y = 116 \qquad (1A)$$
$$12x + 15y = 24. \qquad (2A)$$

Subtracting (1A) from (2A),

$$23y = -92$$

$\boxed{\div 23}$ $\quad y = -4.$

Substituting into (1),

$$3x + 8 = 29$$
$$x = 7.$$

Hence $x = 7$ and $y = -4$.

(b) Adding (1) and (2),

$$2x^2 = 98$$
$$x^2 = 49.$$

Subtracting (2) from (1),

$$2y^2 = 8$$
$$y^2 = 4.$$

Hence $x = 7$ and $y = 2$,
or $x = 7$ and $y = -2$,
or $x = -7$ and $y = 2$,
or $x = -7$ and $y = -2$.

Systems of Three Equations in Three Variables: The key step here is to reduce the system to two equations in two variables.

15 SOLVING THREE SIMULTANEOUS EQUATIONS:
Using either substitution or elimination, produce two simultaneous equations in two of the variables.

WORKED EXERCISE:

Solve simultaneously:
$$3x - 2y - z = -8 \qquad (1)$$
$$5x + y + 3z = 23 \qquad (2)$$
$$4x + y - 5z = -18 \qquad (3)$$

SOLUTION:

Subtracting (3) from (2), $\quad x + 8z = 41. \qquad (4)$

Doubling (3), $\quad 8x + 2y - 10z = -36 \qquad (3A)$

and adding (1) and (3A), $\quad 11x - 11z = -44$

$$x - z = -4. \qquad (5)$$

Equations (4) and (5) are now two equations in two unknowns.

Subtracting (5) from (4), $\quad 9z = 45$

$$z = 5.$$

Substituting $z = 5$ into (5), $\quad x = 1$

and substituting into (2), $\quad y = 3.$

Hence $x = 1$, $y = 3$ and $z = 5$. (This should be checked in the original equations.)

Exercise 1H

1. Solve, by substituting the first equation into the second:

(a) $y = x$ and $2x + y = 9$
(b) $y = 2x$ and $3x - y = 2$
(c) $y = x - 1$ and $2x + y = 5$
(d) $a = 2b + 1$ and $a - 3b = 3$
(e) $p = 2 - q$ and $p - q = 4$
(f) $v = 1 - 3u$ and $2u + v = 0$

2. Solve, by either adding or subtracting the two equations:

(a) $x + y = 5$ and $x - y = 1$
(b) $3x - 2y = 7$ and $x + 2y = -3$
(c) $2x + y = 9$ and $x + y = 5$
(d) $a + 3b = 8$ and $a + 2b = 5$
(e) $4c - d = 6$ and $2c - d = 2$
(f) $p - 2q = 4$ and $3p - 2q = 0$

3. Solve, by substitution:

(a) $y = 2x$ and $3x + 2y = 14$
(b) $y = -3x$ and $2x + 5y = 13$
(c) $y = 4 - x$ and $x + 3y = 8$
(d) $x = 5y + 4$ and $3x - y = 26$
(e) $2x + y = 10$ and $7x + 8y = 53$
(f) $2x - y = 9$ and $3x - 7y = 19$
(g) $4x - 5y = 2$ and $x + 10y = 41$
(h) $2x + 3y = 47$ and $4x - y = 45$

4. Solve, by elimination:

(a) $2x + y = 1$ and $x - y = -4$
(b) $2x + 3y = 16$ and $2x + 7y = 24$
(c) $3x + 2y = -6$ and $x - 2y = -10$
(d) $5x - 3y = 28$ and $2x - 3y = 22$
(e) $3x + 2y = 7$ and $5x + y = 7$
(f) $3x + 2y = 0$ and $2x - y = 56$
(g) $15x + 2y = 27$ and $3x + 7y = 45$
(h) $7x - 3y = 41$ and $3x - y = 17$
(i) $2x + 3y = 28$ and $3x + 2y = 27$
(j) $3x - 2y = 11$ and $4x + 3y = 43$
(k) $4x + 6y = 11$ and $17x - 5y = 1$
(l) $8x = 5y$ and $13x = 8y + 1$

DEVELOPMENT

5. Solve, by substitution:

(a) $y = 2 - x$ and $y = x^2$
(b) $y = 2x - 3$ and $y = x^2 - 4x + 5$
(c) $y = 3x^2$ and $y = 4x - x^2$
(d) $x - y = 5$ and $y = x^2 - 11$
(e) $x - y = 2$ and $xy = 15$
(f) $3x + y = 9$ and $xy = 6$
(g) $x^2 - y^2 = 16$ and $x^2 + y^2 = 34$
(h) $x^2 + y^2 = 117$ and $2x^2 - 3y^2 = 54$

6. Solve each problem by constructing and then solving a pair of simultaneous equations.

(a) Find two numbers that differ by 16 and have a sum of 90.
(b) I paid 75 cents for a pen and a pencil. If the pen cost four times as much as the pencil, find the cost of each item.
(c) If 7 apples and 2 oranges cost \$4, while 5 apples and 4 oranges cost \$4·40, find the cost of each apple and orange.
(d) Twice as many adults as children attended a certain concert. If adult tickets cost \$8 each, child tickets cost \$3 each and the total takings were \$418, find the numbers of adults and children who attended.
(e) A man is 3 times as old as his son. In 12 years time he will be twice as old as his son. How old is each of them now?
(f) At a meeting of the members of a certain club, a proposal was voted on. If 357 members voted and the proposal was carried by a majority of 21, how many voted for and how many voted against?

7. Solve simultaneously:

(a) $\frac{y}{4} - \frac{x}{3} = 1$ and $\frac{x}{2} + \frac{y}{5} = 10$

(b) $4x + \frac{y-2}{3} = 12$ and $3y - \frac{x-3}{5} = 6$

CHALLENGE

8. Solve simultaneously:

(a) $x = 2y$, $y = 3z$, $x + y + z = 10$

(b) $x + 2y - z = -3$, $3x - 4y + z = 13$, $2x + 5y = -1$

(c) $2a - b + c = 10$, $a - b + 2c = 9$, $3a - 4c = 1$

(d) $p + q + r = 6$, $2p - q + r = 1$, $p + q - 2r = -9$

(e) $2x - y - z = 17$, $x + 3y + 4z = -20$, $5x - 2y + 3z = 19$

(f) $3u + v - 4w = -4$, $u - 2v + 7w = -7$, $4u + 3v - w = 9$

9. Solve simultaneously:

(a) $x + y = 15$ and $x^2 + y^2 = 125$

(b) $x - y = 3$ and $x^2 + y^2 = 185$

(c) $2x + y = 5$ and $4x^2 + y^2 = 17$

(d) $x + y = 9$ and $x^2 + xy + y^2 = 61$

(e) $x + 2y = 5$ and $2xy - x^2 = 3$

(f) $3x + 2y = 16$ and $xy = 10$

10. Set up a pair of simultaneous equations to solve each problem.

(a) The value of a certain fraction becomes $\frac{1}{5}$ if one is added to its numerator. If one is taken from its denominator, its value becomes $\frac{1}{7}$. Find the fraction.

(b) Kathy paid \$320 in cash for a CD player. If she paid in \$20 notes and \$10 notes and there were 23 notes altogether, how many of each type were there?

(c) A certain integer is between 10 and 100. Its value is 8 times the sum of its digits, and if it is reduced by 45, its digits are reversed. Find the integer.

(d) Two people are 16 km apart on a straight road. They start walking at the same time. If they walk towards each other, they will meet in 2 hours, but if they walk in the same direction (so that the distance between them is decreasing), they will meet in 8 hours. Find their walking speeds.

1 I Completing the Square

Completing the square can be done in all situations, whereas factoring is not always possible. In particular, the quadratic formula that was reviewed in Section 1G will proven in Chapter Ten by completing the square.

The review in this section is mostly restricted to monic quadratics, in which the coefficient of x^2 is 1. Chapter Ten will deal with non-monic quadratics.

Perfect Squares: The expansion of the quadratic $(x + \alpha)^2$ is

$$(x + \alpha)^2 = x^2 + 2\alpha x + \alpha^2.$$

Notice that the coefficient of x is twice α, and the constant is the square of α.

Reversing the process, the constant term in a perfect square can be found by taking half the coefficient of x and squaring the result.

16 COMPLETING THE SQUARE IN AN EXPRESSION $x^2 + bx + \cdots$:
Halve the coefficient b of x and square the result.

WORKED EXERCISE:

Complete the square in each expression.

(a) $x^2 + 16x + \cdots$ (b) $x^2 - 3x + \cdots$

SOLUTION:

(a) The coefficient of x is 16, half of 16 is 8, and $8^2 = 64$,
so $x^2 + 16x + 64 = (x + 8)^2$.

(b) The coefficient of x is -3, half of -3 is $-1\frac{1}{2}$, and $(-1\frac{1}{2})^2 = 2\frac{1}{4}$,
so $x^2 - 3x + 2\frac{1}{4} = (x - 1\frac{1}{2})^2$.

Solving Quadratic Equations by Completing the Square: This process always works.

17 SOLVING QUADRATIC EQUATIONS BY COMPLETING THE SQUARE:
Complete the square in the quadratic by adding the same to both sides.

WORKED EXERCISE:

Solve each quadratic equation by completing the square.

(a) $t^2 + 8t = 20$ (b) $x^2 - x - 1 = 0$

SOLUTION:

(a)
$$t^2 + 8t = 20$$
$$\boxed{+16} \quad t^2 + 8t + 16 = 36$$
$$(t + 4)^2 = 36$$
$$t + 4 = 6 \text{ or } t + 4 = -6$$
$$t = 2 \text{ or } -10$$

(b)
$$x^2 - x - 1 = 0$$
$$\boxed{+1} \quad x^2 - x = 1$$
$$\boxed{+\tfrac{1}{4}} \quad x^2 - x + \tfrac{1}{4} = 1\tfrac{1}{4}$$
$$(x - \tfrac{1}{2})^2 = \tfrac{5}{4}$$
$$x - \tfrac{1}{2} = \tfrac{1}{2}\sqrt{5} \text{ or } x - \tfrac{1}{2} = -\tfrac{1}{2}\sqrt{5}$$
$$x = \tfrac{1}{2} + \tfrac{1}{2}\sqrt{5} \text{ or } \tfrac{1}{2} - \tfrac{1}{2}\sqrt{5}$$

The Word 'Algebra': Al-Khwarizmi was a famous and influential mathematician who worked in Baghdad during the early ninth century when the Arabs excelled in science and mathematics. The Arabic word 'algebra' comes from the title of his most important work and means 'the restoration of broken parts' — a reference to the balancing of terms on both sides of an equation. Al-Khwarizmi's own name came into modern European languages as 'algorithm'.

Exercise 1I

1. What constant must be added to each expression in order to create a perfect square?

(a) $x^2 + 2x$ (b) $y^2 - 6y$ (c) $a^2 + 10a$ (d) $m^2 - 18m$ (e) $c^2 + 3c$ (f) $x^2 - x$ (g) $b^2 + 5b$ (h) $t^2 - 9t$

2. Factor:

(a) $x^2 + 4x + 4$ (b) $y^2 + 2y + 1$ (c) $p^2 + 14p + 49$ (d) $m^2 - 12m + 36$ (e) $t^2 - 16t + 64$ (f) $x^2 + 20x + 100$ (g) $u^2 - 40u + 400$ (h) $a^2 - 24a + 144$

3. Copy and complete:

(a) $x^2 + 6x + \cdots = (x + \cdots)^2$
(b) $y^2 + 8y + \cdots = (y + \cdots)^2$
(c) $a^2 - 20a + \cdots = (a - \cdots)^2$
(d) $b^2 - 100b + \cdots = (b - \cdots)^2$
(e) $u^2 + u + \cdots = (u + \cdots)^2$
(f) $t^2 - 7t + \cdots = (t - \cdots)^2$
(g) $m^2 + 50m + \cdots = (m + \cdots)^2$
(h) $c^2 - 13c + \cdots = (c - \cdots)^2$

DEVELOPMENT

4. Solve each quadratic equation by completing the square.

(a) $x^2 - 2x = 3$ (d) $y^2 + 3y = 10$ (g) $x^2 - 10x + 20 = 0$

(b) $x^2 - 6x = 0$ (e) $b^2 - 5b - 14 = 0$ (h) $y^2 - y + 2 = 0$

(c) $a^2 + 6a + 8 = 0$ (f) $x^2 + 4x + 1 = 0$ (i) $a^2 + 7a + 7 = 0$

CHALLENGE

5. Solve, by dividing both sides by the coefficient of x^2 and then completing the square:

(a) $3x^2 - 15x + 18 = 0$ (d) $2x^2 + 8x + 3 = 0$ (g) $3x^2 - 8x - 3 = 0$

(b) $2x^2 - 4x - 1 = 0$ (e) $4x^2 + 4x - 3 = 0$ (h) $2x^2 + x - 15 = 0$

(c) $3x^2 + 6x + 5 = 0$ (f) $4x^2 - 2x - 1 = 0$ (i) $2x^2 - 10x + 7 = 0$

6. (a) If $x^2 + y^2 + 4x - 2y + 1 = 0$, show that $(x+2)^2 + (y-1)^2 = 4$.

(b) Show that the equation $x^2 + y^2 - 6x - 8y = 0$ can be written in the form

$$(x-a)^2 + (y-b)^2 = c,$$

where a, b and c are constants. Hence find a, b and c.

(c) If $x^2 + 1 = 10x + 12y$, show that $(x-5)^2 = 12(y+2)$.

(d) Find values for A, B and C if $y^2 - 6x + 16y + 94 = (y+C)^2 - B(x+A)$.

1J Chapter Review Exercise

1. Simplify:

(a) $-8y + 2y$ (b) $-8y - 2y$ (c) $-8y \times 2y$ (d) $-8y \div 2y$

2. Simplify:

(a) $-2a^2 - a^2$ (b) $-2a^2 - (-a^2)$ (c) $-2a^2 \times (-a^2)$ (d) $-2a^2 \div (-a^2)$

3. Simplify:

(a) $3t - 1 - t$ (c) $7x - 4y - 6x + 2y$

(b) $-6p + 3q + 10p$ (d) $2a^2 + 8a - 13 + 3a^2 - 11a - 5$

4. Simplify:

(a) $-6k^6 \times 3k^3$ (b) $-6k^6 \div 3k^3$ (c) $(-6k^6)^2$ (d) $(3k^3)^3$

5. Expand and simplify:

(a) $4(x+3) + 5(2x-3)$ (e) $(n+7)(2n-3)$ (i) $(t-8)^2$

(b) $8(a-2b) - 6(2a-3b)$ (f) $(r+3)^2$ (j) $(2c+7)(2c-7)$

(c) $-(a-b) - (a+b)$ (g) $(y-5)(y+5)$ (k) $(4p+1)^2$

(d) $-4x^2(x+3) - 2x^2(x-1)$ (h) $(3x-5)(2x-3)$ (l) $(3u-2)^2$

6. Factor:

(a) $18a + 36$ (g) $36 - 25g^2$ (m) $4m^2 + 4m - 15$

(b) $20b - 36$ (h) $h^2 - 9h - 36$ (n) $n^3 + 8$

(c) $9c^2 + 36c$ (i) $i^2 + 5i - 36$ (o) $p^3 - 27$

(d) $d^2 - 36$ (j) $2j^2 + 11j + 12$ (p) $p^3 + 9p^2 + 4p + 36$

(e) $e^2 + 13e + 36$ (k) $3k^2 - 7k - 6$ (q) $qt - rt - 5q + 5r$

(f) $f^2 - 12f + 36$ (l) $5\ell^2 - 14\ell + 8$ (r) $u^2w + vw - u^2x - vx$

7. Simplify:

(a) $\frac{x}{2}+\frac{x}{4}$
(b) $\frac{x}{2}-\frac{x}{4}$
(c) $\frac{x}{2}\times\frac{x}{4}$
(d) $\frac{x}{2}\div\frac{x}{4}$
(e) $\frac{3a}{2b}+\frac{2a}{3b}$
(f) $\frac{3a}{2b}-\frac{2a}{3b}$
(g) $\frac{3a}{2b}\times\frac{2a}{3b}$
(h) $\frac{3a}{2b}\div\frac{2a}{3b}$
(i) $\frac{x}{y}+\frac{y}{x}$
(j) $\frac{x}{y}-\frac{y}{x}$
(k) $\frac{x}{y}\times\frac{y}{x}$
(l) $\frac{x}{y}\div\frac{y}{x}$

8. Simplify:

(a) $\frac{x+4}{5}+\frac{x-5}{3}$
(b) $\frac{5}{x+4}+\frac{3}{x-5}$
(c) $\frac{x+1}{2}-\frac{x-4}{5}$
(d) $\frac{2}{x+1}-\frac{5}{x-4}$
(e) $\frac{x}{2}-\frac{x+3}{4}$
(f) $\frac{2}{x}-\frac{4}{x+3}$

9. Factor where possible, then simplify:

(a) $\frac{6a+3b}{10a+5b}$
(b) $\frac{2x-2y}{x^2-y^2}$
(c) $\frac{x^2+2x-3}{x^2-5x+4}$
(d) $\frac{2x^2+3x+1}{2x^3+x^2+2x+1}$
(e) $\frac{a+b}{a^2+2ab+b^2}$
(f) $\frac{3x^2-19x-14}{9x^2-4}$
(g) $\frac{x^3-8}{x^2-4}$
(h) $\frac{a^2-2a-3}{a^3+1}$

10. Solve each linear equation.

(a) $3x+5=17$
(b) $3(x+5)=17$
(c) $\frac{x+5}{3}=17$
(d) $\frac{x}{3}+5=17$
(e) $7a-4=2a+11$
(f) $7(a-4)=2(a+11)$
(g) $\frac{a-4}{7}=\frac{a+11}{2}$
(h) $\frac{a}{7}-4=\frac{a}{2}+11$

11. Solve each quadratic equation by factoring the left-hand side.

(a) $a^2-49=0$
(b) $b^2+7b=0$
(c) $c^2+7c+6=0$
(d) $d^2+6d-7=0$
(e) $e^2-5e+6=0$
(f) $2f^2-f-6=0$
(g) $2g^2-13g+6=0$
(h) $3h^2+2h-8=0$

12. Solve, using the quadratic formula. Write the solutions in simplest exact form.

(a) $x^2-4x+1=0$
(b) $y^2+3y-3=0$
(c) $t^2+6t+4=0$
(d) $3x^2-2x-2=0$
(e) $2a^2+5a-5=0$
(f) $4k^2-6k-1=0$

13. Solve each quadratic by completing the square on the left-hand side.

(a) $x^2+4x=6$
(b) $y^2-6y+3=0$
(c) $x^2-2x=12$
(d) $y^2+10y+7=0$

CHAPTER **TWO**

Numbers and Surds

Arithmetic is the study of numbers and operations on them. This chapter reviews the arithmetic of integers, rational numbers and real numbers, with particular attention to surds. Most of this material will be familiar from earlier years, but Section 2B on recurring decimals may be new.

2 A Integers and Rational Numbers

Our ideas about numbers arise from the two quite distinct sources:

- The *integers* and the *rational numbers* are developed from counting.
- The *real numbers* are developed from geometry and the number line.

Sections 2A and 2B deal with the integers and the rational numbers.

The Integers: *Counting* is the first operation in arithmetic. Counting things like people in a room requires *zero* (if the room is empty) and all the *positive integers*:

$$0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ \ldots$$

The number zero is the first number on this list, but there is no last number, because every number is followed by another number. The list is called *infinite*, which means that it never 'finishes'.

Any two of these numbers can be *added* or *multiplied*, and the result is another number on the list. *Subtraction*, however, requires the *negative integers* as well:

$$\ldots,\ -6,\ -5,\ -4,\ -3,\ -2,\ -1$$

so that calculations like $5 - 7 = -2$ can be completed.

1 THE INTEGERS: The *integers*, or *whole numbers*, are the numbers

$$\ldots,\ -6,\ -5,\ -4,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ \ldots$$

- Two integers can be added, multiplied or subtracted to give another integer.
- There are infinitely many positive and infinitely many negative integers.
- The number zero is neither positive nor negative.

Multiples and the LCM: The *multiples* of a positive integer a are all the products $a,\ 2a,\ 3a,\ 4a,\ 5a,\ 6a,\ \ldots$ For example, here are the multiples of 6 and 8.

The multiples of 6 are: 6, 12, 18, 24, 30, 36, 42, 48, 54, 60, 66, 72, ...

The multiples of 8 are: 8, 16, 24, 32, 40, 48, 56, 64, 72, 80, 88, 96, ...

The *lowest common multiple* or *LCM* of two positive integers is the smallest number that appears on both lists. In this example, the LCM of 6 and 8 is 24.

The key to adding and subtracting fractions is finding the LCM of their denominators, called the *lowest common denominator*:

$$\frac{1}{6}+\frac{5}{8}=\frac{1\times 4}{24}+\frac{5\times 3}{24} = \frac{19}{24} \qquad \frac{1}{6}-\frac{5}{8}=\frac{1\times 4}{24}-\frac{5\times 3}{24} = -\frac{11}{24}$$

Divisors and the HCF: *Division* of positive integers yields a quotient and a remainder:

$$27 \div 10 = 2,\ \text{remainder } 7 \qquad 42 \div 6 = 7,\ \text{remainder } 0$$

Because the remainder is zero in the second case, 6 is called a *divisor* or *factor* of 42. Here are the lists of all divisors of 42 and 63.

The divisors of 42 are: 1, 2, 3, 6, 7, 14, 21, 42

The divisors of 63 are: 1, 3, 7, 9, 21, 63

The *highest common factor* or *HCF* of two positive integers is the largest number that appears on both lists. In this example, the HCF of 42 and 63 is 21.

The key to cancelling a fraction down to its *lowest terms*, and to multiplying fractions, is dividing the numerator and denominator by their HCF:

$$\frac{42}{63}=\frac{2\times 21}{3\times 21}=\frac{2}{3} \qquad \frac{5}{16}\times\frac{4}{35}=\frac{1}{4}\times\frac{1}{7}=\frac{1}{28}$$

Prime Numbers: A *prime number* is an integer greater than 1 whose only divisors are itself and 1. The primes form a sequence whose distinctive pattern has confused every mathematician since Greek times:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, \ldots$$

An integer that is greater than 1 and is not prime is called a *composite number*. Without giving its proof, we shall assume the *unique factorisation theorem*:

THE UNIQUE FACTORISATION THEOREM:

2 Every positive integer can be written as a product of prime numbers in one and only one way, apart from the order of the factors. For example,

$$24 = 2^3 \times 3 \qquad \text{and} \qquad 30 = 2\times 3\times 5.$$

The Rational Numbers: Problems like ‘Divide 7 cakes into 3 equal parts’ lead naturally to *fractions*, where the whole is ‘fractured’ or ‘broken’ into pieces. This gives the system of *rational numbers*, which can be written as the ‘ratio’ of two integers.

DEFINITION OF RATIONAL NUMBERS:

3

- A *rational number* is a number that can be written as a *fraction* $\frac{a}{b}$, where a and b are integers and $b \neq 0$. Here are some examples:

$$2\tfrac{1}{3}=\frac{7}{3} \qquad -\frac{1}{3}=\frac{-1}{3} \qquad 30\div 24=\frac{5}{4} \qquad 3{\cdot}72=\frac{372}{100} \qquad 4=\frac{4}{1}$$

- Every integer a can be written as a fraction $\frac{a}{1}$.
Hence every integer is a rational number.

The Four Operations on the Rational Numbers: Addition, multiplication, subtraction and division (except by 0) can all be carried out within the rational numbers. All except division have already been discussed.

4 DIVISION OF RATIONAL NUMBERS:

- The *reciprocal* of a fraction $\frac{a}{b}$ is $\frac{b}{a}$.
- To divide by a fraction, multiply by the reciprocal:

$$\frac{2}{7} \div \frac{3}{4} = \frac{2}{7} \times \frac{4}{3} = \frac{8}{21} \qquad \left(\text{The reciprocal of } \frac{3}{4} \text{ is } \frac{4}{3}.\right)$$

Exercise 2A

1. Write down all the prime numbers: (a) less than 20, (b) between 20 and 50.

2. Write each number as a product of its prime factors.
(a) 10 (c) 12 (e) 28 (g) 27
(b) 21 (d) 18 (f) 45 (h) 40

3. Find the HCF (highest common factor) of:
(a) 6 and 8 (c) 14 and 21 (e) 18 and 27 (g) 24 and 32
(b) 6 and 15 (d) 12 and 20 (f) 12 and 42 (h) 36 and 60

4. Find the LCM (lowest common multiple) of:
(a) 2 and 5 (c) 4 and 6 (e) 6 and 8 (g) 10 and 25
(b) 3 and 6 (d) 4 and 7 (f) 9 and 12 (h) 6 and 15

DEVELOPMENT

5. Write each number as a product of its prime factors.
(a) 30 (c) 39 (e) 108 (g) 154
(b) 36 (d) 48 (f) 128 (h) 136

6. Find the HCF of the numerator and denominator, then use it to cancel each fraction down to lowest terms.
(a) $\frac{4}{12}$ (c) $\frac{10}{15}$ (e) $\frac{16}{40}$ (g) $\frac{24}{42}$ (i) $\frac{36}{60}$
(b) $\frac{8}{10}$ (d) $\frac{21}{28}$ (f) $\frac{21}{45}$ (h) $\frac{45}{54}$ (j) $\frac{54}{72}$

7. Find the lowest common denominator, then simplify:
(a) $\frac{1}{2} + \frac{1}{4}$ (c) $\frac{1}{2} + \frac{1}{3}$ (e) $\frac{1}{6} + \frac{1}{9}$ (g) $\frac{7}{10} + \frac{2}{15}$
(b) $\frac{3}{10} + \frac{2}{5}$ (d) $\frac{2}{3} - \frac{2}{5}$ (f) $\frac{5}{12} - \frac{3}{8}$ (h) $\frac{2}{25} - \frac{1}{15}$

8. Find the value of:
(a) $\frac{1}{4} \times 20$ (c) $\frac{1}{2} \times \frac{1}{5}$ (e) $\frac{2}{5} \times \frac{5}{8}$ (g) $\frac{3}{4} \div 3$ (i) $1\frac{1}{2} \div \frac{3}{8}$
(b) $\frac{2}{3} \times 12$ (d) $\frac{1}{3} \times \frac{3}{7}$ (f) $2 \div \frac{1}{3}$ (h) $\frac{1}{3} \div \frac{1}{2}$ (j) $\frac{5}{12} \div 1\frac{2}{3}$

CHALLENGE

9. Express in lowest terms without using a calculator:
(a) $\frac{588}{630}$ (b) $\frac{455}{1001}$ (c) $\frac{500}{1\,000\,000}$

10. Without using a calculator, find the value of:
(a) $\frac{1}{12}+\frac{1}{15}+\frac{1}{20}$ (b) $\frac{11}{18}-\frac{9}{16}$ (c) $\frac{1}{2}+\frac{1}{4}+\frac{1}{5}+\frac{1}{25}+\frac{1}{100}$

11. Write each number as a product of its prime factors. Then find its square root by halving the indices.
(a) 576 (b) 1225 (c) 1 000 000

2 B Terminating and Recurring Decimals

Decimal notation extends 'place value' to negative powers of 10. For example:

$$123{\cdot}456 = 1\times 10^2 + 2\times 10^1 + 3\times 10^0 + 4\times 10^{-1} + 5\times 10^{-2} + 6\times 10^{-3}$$

Such a number can be written as a fraction, and so is a rational number.

Writing a Fraction as a Terminating or Recurring Decimal:

If a rational number can be written as a fraction whose denominator is a power of 10, then it can easily be written as a *terminating decimal*:

$$\tfrac{3}{25} = \tfrac{12}{100} = 0{\cdot}12 \qquad \text{and} \qquad 578\tfrac{3}{50} = 578 + \tfrac{6}{100} = 578{\cdot}06$$

If a rational number cannot be written with a power of 10 as its denominator, then repeated division will yield an infinite string of digits in its decimal representation. This string will cycle once the same remainder recurs, giving a *recurring decimal*.

$$\tfrac{2}{3} = 0{\cdot}666\,666\,666\,666\ldots = 0{\cdot}\dot{6} \qquad \text{(which has cycle length 1)}$$
$$6\tfrac{3}{7} = 6{\cdot}428\,571\,428\,571\ldots = 6{\cdot}\dot{4}2857\dot{1} \qquad \text{(which has cycle length 6)}$$
$$24\tfrac{35}{44} = 24{\cdot}795\,454\,545\,45\ldots = 24{\cdot}79\dot{5}\dot{4} \qquad \text{(which has cycle length 2)}$$

Writing a Recurring Decimal as a Fraction:

Conversely, every recurring decimal can be written as a fraction. The following worked exercise shows the method.

WORKED EXERCISE:
Write as a fraction in lowest terms:
(a) $0{\cdot}\dot{5}\dot{1}$ (which has cycle length 2), (b) $7{\cdot}3\dot{2}8\dot{4}$ (which has cycle length 3).

SOLUTION:

(a) Let $x = 0{\cdot}\dot{5}\dot{1}$.
Then $x = 0{\cdot}515\,151\ldots$
[× 100] $100x = 51{\cdot}515\,151\ldots$
Subtracting the last two lines,
$99x = 51$
[÷ 99] $x = \frac{17}{33}$,
so $0{\cdot}\dot{5}\dot{1} = \frac{17}{33}$.

(b) Let $x = 7{\cdot}3\dot{2}8\dot{4}$.
Then $x = 7{\cdot}328\,428\,4\ldots$
[× 1000] $1000x = 7328{\cdot}428\,428\,4\ldots$
Subtracting the last two lines,
$999x = 7321{\cdot}1$
[÷ 999] $x = \frac{7321{\cdot}1}{999}$,
so $7{\cdot}3\dot{2}8\dot{4} = \frac{73\,211}{9990}$.

5 WRITING A RECURRING DECIMAL AS A FRACTION:
If the cycle length is n, multiply by 10^n and subtract.

Percentages: Many practical situations involving fractions, decimals and ratios are commonly expressed in terms of percentages.

6 PERCENTAGES:

- To convert a fraction to a percentage, multiply by $\frac{100}{1}\%$:

$$\frac{3}{20} = \frac{3}{20} \times \frac{100}{1}\% = 15\%$$

- To convert a percentage to a fraction, replace % by $\frac{1}{100}$:

$$15\% = 15 \times \frac{1}{100} = \frac{3}{20}$$

Many problems are best solved by the *unitary method*, illustrated below.

WORKED EXERCISE:

(a) A table marked \$1400 has been discounted by 30%. How much does it now cost?
(b) A table discounted by 30% now costs \$1400. What was the original cost?

SOLUTION:

(a)

	100%	is	\$1400
÷ 10	10%	is	\$140
× 7	70%	is	\$980

so the discounted price is \$980.

(b)

	70%	is	\$1400
÷ 7	10%	is	\$200
× 10	100%	is	\$2000

so the original price was \$2000.

Exercise 2B

1. Write as a fraction in lowest terms:
(a) 30% (b) 80% (c) 75% (d) 5%

2. Write as a decimal:
(a) 60% (b) 27% (c) 9% (d) 16·5%

3. Write as a percentage:
(a) $\frac{1}{4}$ (b) $\frac{2}{5}$ (c) $\frac{6}{25}$ (d) $\frac{13}{20}$

4. Write as a percentage:
(a) 0·32 (b) 0·09 (c) 0·225 (d) 1·5

5. Express each fraction as a terminating decimal by first rewriting it as a fraction with denominator 10, 100 or 1000.
(a) $\frac{1}{2}$ (c) $\frac{3}{5}$ (e) $\frac{1}{25}$ (g) $\frac{1}{8}$
(b) $\frac{1}{5}$ (d) $\frac{3}{4}$ (f) $\frac{7}{20}$ (h) $\frac{5}{8}$

6. Express each fraction as a recurring decimal by dividing the numerator by the denominator.
(a) $\frac{1}{3}$ (c) $\frac{1}{9}$ (e) $\frac{3}{11}$ (g) $\frac{1}{6}$
(b) $\frac{2}{3}$ (d) $\frac{5}{9}$ (f) $\frac{1}{11}$ (h) $\frac{5}{6}$

7. Express each terminating decimal as a fraction in lowest terms.

(a) $0{\cdot}4$ (b) $0{\cdot}25$ (c) $0{\cdot}15$ (d) $0{\cdot}16$ (e) $0{\cdot}78$ (f) $0{\cdot}005$ (g) $0{\cdot}375$ (h) $0{\cdot}264$

8. Express each recurring decimal as a fraction in lowest terms.

(a) $0{\cdot}\dot{2}$ (b) $0{\cdot}\dot{7}$ (c) $0{\cdot}\dot{4}$ (d) $0{\cdot}\dot{6}\dot{5}$ (e) $0{\cdot}\dot{5}\dot{4}$ (f) $0{\cdot}\dot{8}\dot{4}$ (g) $0{\cdot}\dot{0}\dot{6}$ (h) $0{\cdot}\dot{1}3\dot{5}$ (i) $0{\cdot}\dot{7}6\dot{2}$ (j) $0{\cdot}\dot{0}3\dot{3}$

DEVELOPMENT

9. (a) Find 12% of \$5.

(b) Find $7{\cdot}5\%$ of 200 kg.

(c) Increase \$6000 by 30%.

(d) Decrease $1\frac{1}{2}$ hours by 20%.

10. Express each fraction as a decimal.

(a) $\dfrac{33}{250}$ (b) $\dfrac{1}{40}$ (c) $\dfrac{5}{16}$ (d) $\dfrac{27}{80}$ (e) $\dfrac{7}{12}$ (f) $1\dfrac{9}{11}$ (g) $\dfrac{2}{15}$ (h) $\dfrac{13}{55}$

11. Express each decimal as a fraction.

(a) $1{\cdot}\dot{6}$ (b) $3{\cdot}\dot{2}\dot{1}$ (c) $2{\cdot}\dot{4}2\dot{3}$ (d) $1{\cdot}\dot{0}7\dot{4}$ (e) $0{\cdot}2\dot{3}$ (f) $0{\cdot}1\dot{5}$ (g) $0{\cdot}6\dot{3}\dot{8}$ (h) $0{\cdot}3\dot{4}\dot{5}$

12. (a) Steve's council rates increased by 5% this year to \$840. What were his council rates last year?

(b) Joanne received a 10% discount on a pair of shoes. If she paid \$144, what was the original price?

(c) Marko spent \$135 this year at the Easter Show, a $12{\cdot}5\%$ increase on last year. How much did he spend last year?

13. Work out the recurring decimals for $\frac{1}{7}$, $\frac{2}{7}$, $\frac{3}{7}$, $\frac{4}{7}$, $\frac{5}{7}$ and $\frac{6}{7}$. Is there a pattern?

CHALLENGE

14. (a) Work out the recurring decimals for $\frac{1}{11}$, $\frac{2}{11}$, $\frac{3}{11}$, $\frac{4}{11}$, ..., $\frac{10}{11}$. Is there a pattern?

(b) Work out the recurring decimals for $\frac{1}{13}$, $\frac{2}{13}$, $\frac{3}{13}$, $\frac{4}{13}$, ..., $\frac{12}{13}$. Is there a pattern?

15. Use the method for converting a recurring decimal into a fraction to prove that:

(a) $0{\cdot}\dot{9} = 1$ (b) $2{\cdot}4\dot{9} = 2{\cdot}5$

16. (The numbers you obtain in this question may vary depending on the calculator used.)

(a) Use your calculator to express $\frac{1}{3}$ as a decimal by entering $1 \div 3$.

(b) Subtract $0{\cdot}333\,333\,33$ from this, multiply the result by 10^8 and then take the reciprocal.

(c) Show arithmetically that the final answer in part (b) is 3. Is the answer on your calculator also equal to 3? What does this tell you about the way fractions are stored on a calculator?

2 C Real Numbers and Approximations

There are two good reasons why decimals are used so often:

- Any two decimals can easily be compared with each other.
- Any quantity can be approximated 'as closely as we like' by a decimal.

A measurement is only approximate, no matter how good the instrument, and rounding using decimals is a useful way of showing how accurate it is.

Rounding to a Certain Number of Decimal Places: The rules for rounding a decimal are:

RULES FOR ROUNDING A DECIMAL NUMBER:
To round a decimal to, say, two decimal places, look at the third digit.

7 • If the third digit is 0, 1, 2, 3 or 4, leave the second digit alone.

• If the third digit is 5, 6, 7, 8 or 9, increase the second digit by 1.

Always use $\doteqdot$ rather than $=$ when a quantity has been rounded or approximated.

For example,

$3{\cdot}8472 \doteqdot 3{\cdot}85$, correct to two decimal places. (Look at 7, the third digit.)

$3{\cdot}8472 \doteqdot 3{\cdot}8$, correct to one decimal place. (Look at 4, the second digit.)

Scientific Notation and Rounding to a Certain Number of Significant Figures: The very large and very small numbers common in astronomy and atomic physics are easier to comprehend when they are written in scientific notation:

$1\,234\,000 = 1{\cdot}234 \times 10^6$ (There are four significant figures.)

$0{\cdot}000\,065\,432 = 6{\cdot}5432 \times 10^{-5}$ (There are five significant figures.)

The digits in the first factor are called the *significant figures* of the number. It is often more sensible to round a quantity correct to a given number of significant figures rather than to a given number of decimal places.

To round to, say, three significant figures, look at the fourth digit. If it is 5, 6, 7, 8 or 9, increase the third digit by 1. Otherwise, leave the third digit alone.

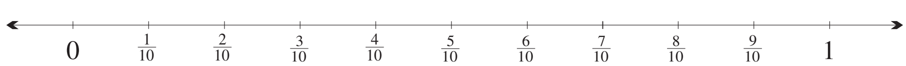
$3{\cdot}085 \times 10^9 \doteqdot 3{\cdot}09 \times 10^9$, correct to three significant figures.

$2{\cdot}789\,654 \times 10^{-29} \doteqdot 2{\cdot}790 \times 10^{-29}$, correct to four significant figures.

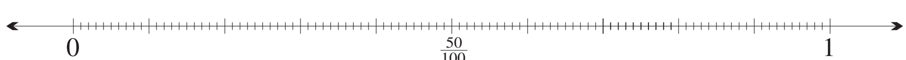
The number can be in normal notation and still be rounded this way:

$31{\cdot}203 \doteqdot 31{\cdot}20$, correct to four significant figures.

There are Numbers that are Not Rational: At first glance, it would seem reasonable to believe that all the numbers on the number line are rational, because the rational numbers are clearly spread 'as finely as we like' along the whole number line. Between 0 and 1 there are 9 rational numbers with denominator 10:

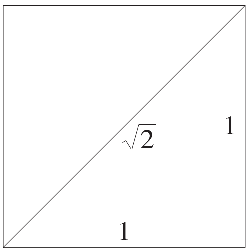


Between 0 and 1 there are 99 rational numbers with denominator 100:



Most points on the number line, however, represent numbers that cannot be written as fractions, and are called *irrational numbers*. Some of the most important numbers in this course are irrational, like $\sqrt{2}$ and π and the number e that will be introduced much later in the course.

The Square Root of 2 is Irrational: The number $\sqrt{2}$ is particularly important, because by Pythagoras' theorem, $\sqrt{2}$ is the diagonal of a unit square. Here is a proof by contradiction that $\sqrt{2}$ is an irrational number.



Suppose that $\sqrt{2}$ were a rational number.

Then $\sqrt{2}$ could be written as a fraction in lowest terms.

That is, $\sqrt{2} = \dfrac{a}{b}$, where $b > 1$ since $\sqrt{2}$ is not an integer.

Squaring, $2 = \dfrac{a^2}{b^2}$, where $b^2 > 1$ because $b > 1$.

Since $\dfrac{a}{b}$ is in lowest terms, $\dfrac{a^2}{b^2}$ is also in lowest terms,

which is impossible, since $\dfrac{a^2}{b^2} = 2$, but $b^2 > 1$.

This is a contradiction, so $\sqrt{2}$ cannot be a rational number.

The Greek mathematicians were greatly troubled by the existence of irrational numbers. Their concerns can still be seen in modern English, where the word 'irrational' means both 'not a fraction' and 'not reasonable'.

The Real Numbers and the Number Line: The integers and the rational numbers were based on *counting*. The existence of irrational numbers, however, means that this approach to arithmetic is inadequate, and a more general idea of number is needed. We have to turn away from counting and make use of *geometry*.

8 DEFINITION OF THE REAL NUMBERS:
- The *real numbers* are defined to be the points on the number line.
- All rational numbers are real, but real numbers like $\sqrt{2}$ and π are irrational.

At this point, geometry replaces counting as the basis of arithmetic.

Exercise 2C

1. Write each number correct to one decimal place.
(a) 0·32 (b) 5·68 (c) 12·75 (d) 0·05 (e) 3·03 (f) 9·96

2. Write each number correct to two significant figures.
(a) 0·429 (b) 5·429 (c) 5·029 (d) 0·0429 (e) 429 (f) 4290

3. Use a calculator to find each number correct to three decimal places.
(a) $\sqrt{10}$ (b) $\sqrt{47}$ (c) $\frac{9}{16}$ (d) $\frac{37}{48}$ (e) π (f) π^2

4. Use a calculator to find each number correct to three significant figures.
(a) $\sqrt{58}$ (b) $\sqrt[3]{133}$ (c) 62^2 (d) 14^5 (e) $\sqrt[4]{0{\cdot}3}$ (f) 124^{-1}

5. To how many significant figures are each of these numbers accurate?
(a) 0·04 (b) 0·40 (c) 0·404 (d) 0·044 (e) 4·004 (f) 400

6. Classify these real numbers as rational or irrational. Express those that are rational in the form $\frac{a}{b}$, where a and b are integers.

(a) -3 (d) $\sqrt{4}$ (g) $\sqrt{\frac{4}{9}}$ (j) $0{\cdot}333$ (m) π

(b) $1\frac{1}{2}$ (e) $\sqrt[3]{27}$ (h) $0{\cdot}45$ (k) $0{\cdot}\dot{3}$ (n) $3{\cdot}14$

(c) $\sqrt{3}$ (f) $\sqrt[4]{8}$ (i) 12% (l) $3\frac{1}{7}$ (o) 0

DEVELOPMENT

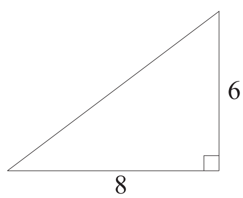
7. Use a calculator to evaluate each expression correct to three decimal places.

(a) $\dfrac{67 \times 29}{43}$ (c) $\dfrac{67}{43 \times 29}$ (e) $\dfrac{67 + 29}{43 + 71}$

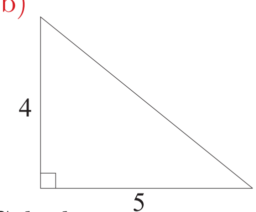
(b) $\dfrac{67 + 29}{43}$ (d) $\dfrac{67}{43 + 29}$ (f) $\dfrac{67 + 71}{43 \times 29}$

8. Use Pythagoras' theorem to find the length of the unknown side in each triangle, and state whether it is rational or irrational.

(a)



(b)



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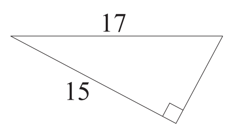

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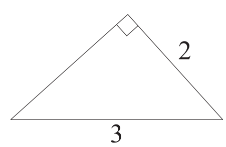
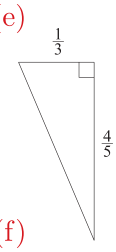


(e)



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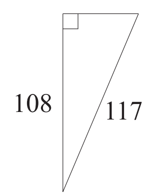


9. Calculate, correct to four significant figures:

(a) $10^{-0{\cdot}4}$

(b) $\dfrac{1}{240 - 13 \times 17}$

(c) $\dfrac{\sqrt{6{\cdot}5 + 8{\cdot}3}}{2{\cdot}7}$

(d) $\sqrt[3]{10{\cdot}57 \times 12{\cdot}83}$

(e) $\dfrac{3{\cdot}5 \times 10^4}{2{\cdot}3 \times 10^5}$

(f) $20\,000(1{\cdot}01)^{25}$

(g) $\dfrac{11{\cdot}3}{\sqrt{19{\cdot}5 - 14{\cdot}7}}$

(h) $\dfrac{3\frac{2}{3} + 5\frac{1}{4}}{4\frac{1}{2} + 6\frac{4}{5}}$

(i) $(87{\cdot}3 \times 10^4) \div (0{\cdot}629 \times 10^{-8})$

(j) $\dfrac{\sqrt{3} + \sqrt[3]{4}}{\sqrt[4]{5} + \sqrt[5]{6}}$

(k) $\dfrac{\left(\frac{2}{5}\right)^4 \times \left(\frac{3}{4}\right)^5}{\left(\frac{6}{7}\right)^2 + \left(\frac{2}{3}\right)^3}$

(l) $\sqrt{\dfrac{36{\cdot}41 - 19{\cdot}57}{23{\cdot}62 - 11{\cdot}39}}$

CHALLENGE

Use a calculator to answer the remaining questions. Write each answer in scientific notation.

10. The speed of light is approximately $2{\cdot}997\,929 \times 10^8$ m/s.

(a) How many metres are there in a light year? Assume that there are $365\frac{1}{4}$ days in a year and write your answer in metres, correct to three significant figures.

(b) The nearest large galaxy is Andromeda, which is estimated to be 2 200 000 light years away. How far is that in metres, correct to two significant figures?

(c) The time since the Big Bang is estimated to be 13·6 billion years. How long is that in seconds, correct to three significant figures?

(d) How far would light have travelled since the Big Bang? Give your answer in metres, correct to two significant figures.

11. The mass of a proton is $1{\cdot}6726 \times 10^{-27}$ kg and the mass of an electron is $9{\cdot}1095 \times 10^{-31}$ kg.
(a) Calculate, correct to four significant figures, the ratio of the mass of a proton to the mass of an electron.
(b) How many protons, correct to one significant figure, does it take to make 1 kg?

12. The diameter of a proton is approximately 10^{-15} metres.
(a) Assuming that a proton is a sphere, calculate the volume of a proton, correct to two significant figures. (The volume of a sphere is $\frac{4}{3}\pi r^3$.)
(b) Given that density is mass divided by volume, calculate, correct to one significant figure, the density of a proton. (This, very roughly, is the density of a neutron star.)
(c) Water has a density of $1\,\text{g/cm}^3$. How many times denser is a proton?

13. Prove that $\sqrt{3}$ is irrational. (Adapt the given proof that $\sqrt{2}$ is irrational.)

2 D Surds and their Arithmetic

Numbers like $\sqrt{2}$ and $\sqrt{3}$ occur constantly in this course because they occur in the solutions of quadratic equations. The last three sections of this chapter review various methods of dealing with them.

Square Roots and Positive Square Roots: The square of any real number is positive, except that $0^2 = 0$. Hence a negative number cannot have a square root, and the only square root of 0 is 0 itself.

A positive number, however, has two square roots, which are the opposites of each other. For example, the square roots of 9 are 3 and -3.

Note that the well-known symbol $\sqrt{x}$ does not mean 'the square root of x'. It is defined to mean the *positive* square root of x (or zero, if $x = 0$).

9 DEFINITION OF THE SYMBOL $\sqrt{x}$:
- For $x > 0$, $\sqrt{x}$ means the *positive* square root of x.
- For $x = 0$, $\sqrt{0} = 0$.
- For $x < 0$, $\sqrt{x}$ is undefined.

For example, $\sqrt{25} = 5$, even though 25 has two square roots, -5 and 5. The symbol for the negative square root of 25 is $-\sqrt{25}$.

Cube Roots: Cube roots are less complicated. Every number has exactly one cube root, and so the symbol $\sqrt[3]{x}$ simply means 'the cube root of x'. For example,

$$\sqrt[3]{8} = 2 \qquad \text{and} \qquad \sqrt[3]{-8} = -2 \qquad \text{and} \qquad \sqrt[3]{0} = 0$$

What is a Surd: The word *surd* is often used to refer to any expression involving a square or higher root. More precisely, however, surds do not include expressions like $\sqrt{\frac{4}{9}}$ and $\sqrt[3]{8}$, which can be simplified to rational numbers.

10 SURDS:
An expresson $\sqrt[n]{x}$, where x is a rational number and $n \geq 2$ is an integer, is called a *surd* if it is not itself a rational number.

The word 'surd' is related to 'absurd', meaning 'irrational'.

Simplifying Expressions Involving Surds: Here are some laws from earlier years for simplifying expressions involving square roots. The first pair restate the definition of the square root, and the second pair are easily proven by squaring.

11 LAWS CONCERNING SURDS: Let a and b be positive real numbers. Then:

(a) $\sqrt{a^2} = a$ (c) $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$

(b) $\left(\sqrt{a}\right)^2 = a$ (d) $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$

Taking Out Square Divisors: A surd like $\sqrt{500}$ is not regarded as being simplified, because 500 is divisible by the square 100, and so $\sqrt{500}$ can be written as $10\sqrt{5}$:

$$\sqrt{500} = \sqrt{100 \times 5} = \sqrt{100} \times \sqrt{5} = 10\sqrt{5}.$$

12 SIMPLIFYING A SURD:

Check the number inside the square root for divisibility by one of the squares

$$4,\ 9,\ 16,\ 25,\ 36,\ 49,\ 64,\ 81,\ 100,\ 121,\ 144,\ \ldots$$

It is quickest to divide by the largest possible square divisor.

WORKED EXERCISE:

Simplify these two expressions involving surds.

(a) $\sqrt{108}$ (b) $5\sqrt{27}$

SOLUTION:

(a) $\sqrt{108} = \sqrt{36 \times 3}$
$= \sqrt{36} \times \sqrt{3}$
$= 6\sqrt{3}$

(b) $5\sqrt{27} = 5\sqrt{9 \times 3}$
$= 5 \times \sqrt{9} \times \sqrt{3}$
$= 15\sqrt{3}$

WORKED EXERCISE:

Simplify the surds in these expressions, then collect like terms.

(a) $\sqrt{44} + \sqrt{99}$ (b) $\sqrt{72} - \sqrt{50} + \sqrt{12}$

SOLUTION:

(a) $\sqrt{44} + \sqrt{99} = 2\sqrt{11} + 3\sqrt{11}$
$= 5\sqrt{11}$

(b) $\sqrt{72} - \sqrt{50} + \sqrt{12}$
$= 6\sqrt{2} - 5\sqrt{2} + 2\sqrt{3}$
$= \sqrt{2} + 2\sqrt{3}$

Exercise 2D

1. Write down the value of:

(a) $\sqrt{16}$ (c) $\sqrt{81}$ (e) $\sqrt{144}$ (g) $\sqrt{2500}$

(b) $\sqrt{36}$ (d) $\sqrt{121}$ (f) $\sqrt{400}$ (h) $\sqrt{10\,000}$

2. Simplify:

(a) $\sqrt{12}$ (e) $\sqrt{28}$ (i) $\sqrt{54}$ (m) $\sqrt{80}$

(b) $\sqrt{18}$ (f) $\sqrt{40}$ (j) $\sqrt{200}$ (n) $\sqrt{98}$

(c) $\sqrt{20}$ (g) $\sqrt{32}$ (k) $\sqrt{60}$ (o) $\sqrt{800}$

(d) $\sqrt{27}$ (h) $\sqrt{99}$ (l) $\sqrt{75}$ (p) $\sqrt{1000}$

3. Simplify:

(a) $\sqrt{3}+\sqrt{3}$
(b) $5\sqrt{7}-3\sqrt{7}$
(c) $2\sqrt{5}-\sqrt{5}$
(d) $-3\sqrt{2}+\sqrt{2}$
(e) $4\sqrt{3}+3\sqrt{2}-2\sqrt{3}$
(f) $-5\sqrt{5}-2\sqrt{7}+6\sqrt{5}$
(g) $7\sqrt{6}+5\sqrt{3}-4\sqrt{6}-7\sqrt{3}$
(h) $-6\sqrt{2}-4\sqrt{5}+3\sqrt{2}-2\sqrt{5}$
(i) $3\sqrt{10}-8\sqrt{5}-7\sqrt{10}+10\sqrt{5}$

DEVELOPMENT

4. Simplify:

(a) $3\sqrt{8}$
(b) $5\sqrt{12}$
(c) $2\sqrt{24}$
(d) $4\sqrt{44}$
(e) $3\sqrt{45}$
(f) $6\sqrt{52}$
(g) $2\sqrt{300}$
(h) $2\sqrt{96}$

5. Write each expression as a single square root. (For example, $3\sqrt{2}=\sqrt{9}\times\sqrt{2}=\sqrt{18}$.)

(a) $2\sqrt{5}$
(b) $5\sqrt{2}$
(c) $8\sqrt{2}$
(d) $6\sqrt{3}$
(e) $5\sqrt{5}$
(f) $4\sqrt{7}$
(g) $2\sqrt{17}$
(h) $7\sqrt{10}$

6. Simplify fully:

(a) $\sqrt{8}+\sqrt{2}$
(b) $\sqrt{12}-\sqrt{3}$
(c) $\sqrt{50}-\sqrt{18}$
(d) $\sqrt{54}+\sqrt{24}$
(e) $\sqrt{45}-\sqrt{20}$
(f) $\sqrt{90}-\sqrt{40}+\sqrt{10}$
(g) $\sqrt{27}+\sqrt{75}-\sqrt{48}$
(h) $\sqrt{45}+\sqrt{80}-\sqrt{125}$
(i) $\sqrt{2}+\sqrt{32}+\sqrt{72}$

CHALLENGE

7. Simplify fully:

(a) $\sqrt{600}+\sqrt{300}-\sqrt{216}$
(b) $4\sqrt{18}+3\sqrt{12}-2\sqrt{50}$
(c) $2\sqrt{175}-5\sqrt{140}-3\sqrt{28}$

8. Find the value of x if:

(a) $\sqrt{63}-\sqrt{28}=\sqrt{x}$
(b) $\sqrt{80}-\sqrt{20}=\sqrt{x}$
(c) $2\sqrt{150}-3\sqrt{24}=\sqrt{x}$

2 E Further Simplification of Surds

This section deals with the simplification of more complicated surdic expressions. The usual rules of algebra, together with the methods of simplifying surds given in the last section, are all that is needed.

Simplifying Products of Surds: The product of two surds is found using the identity

$$\sqrt{a}\times\sqrt{b}=\sqrt{ab}\,.$$

It is important to check whether the answer needs further simplification.

WORKED EXERCISE: Simplify each product.

(a) $\sqrt{15}\times\sqrt{5}$
(b) $5\sqrt{6}\times 7\sqrt{10}$

SOLUTION:

(a)
$$\begin{aligned}\sqrt{15}\times\sqrt{5}&=\sqrt{75}\\&=\sqrt{25\times 3}\\&=5\sqrt{3}\end{aligned}$$

(b)
$$\begin{aligned}5\sqrt{6}\times 7\sqrt{10}&=35\sqrt{60}\\&=35\sqrt{4\times 15}\\&=35\times 2\sqrt{15}\\&=70\sqrt{15}\end{aligned}$$

Using Binomial Expansions: All the usual algebraic methods of expanding binomial products can be applied to surdic expressions.

WORKED EXERCISE:

Expand these products and then simplify them.

(a) $\left(\sqrt{15}+2\right)\left(\sqrt{3}-3\right)$ (b) $\left(\sqrt{15}-\sqrt{6}\right)^2$

SOLUTION:

(a) $\left(\sqrt{15}+2\right)\left(\sqrt{3}-3\right) = \sqrt{15}\left(\sqrt{3}-3\right)+2\left(\sqrt{3}-3\right)$

$= \sqrt{45}-3\sqrt{15}+2\sqrt{3}-6$

$= 3\sqrt{5}-3\sqrt{15}+2\sqrt{3}-6$

(b) $\left(\sqrt{15}-\sqrt{6}\right)^2 = 15-2\sqrt{90}+6$ (Use the identity $(A-B)^2 = A^2-2AB+B^2$.)

$= 21-2\times 3\sqrt{10}$

$= 21-6\sqrt{10}$

Exercise 2E

1. Simplify:

(a) $(\sqrt{3})^2$ (b) $\sqrt{2}\times\sqrt{3}$ (c) $\sqrt{7}\times\sqrt{7}$ (d) $\sqrt{6}\times\sqrt{5}$ (e) $2\times 3\sqrt{2}$ (f) $2\sqrt{5}\times 5$ (g) $2\sqrt{3}\times 3\sqrt{5}$ (h) $6\sqrt{2}\times 5\sqrt{7}$ (i) $(2\sqrt{3})^2$ (j) $(3\sqrt{7})^2$ (k) $5\sqrt{2}\times 3\sqrt{2}$ (l) $6\sqrt{10}\times 4\sqrt{10}$

2. Simplify:

(a) $\sqrt{15}\div\sqrt{3}$ (b) $\sqrt{42}\div\sqrt{6}$ (c) $3\sqrt{5}\div 3$ (d) $2\sqrt{7}\div\sqrt{7}$ (e) $3\sqrt{10}\div\sqrt{5}$ (f) $6\sqrt{33}\div 6\sqrt{11}$ (g) $10\sqrt{14}\div 5\sqrt{2}$ (h) $15\sqrt{35}\div 3\sqrt{7}$

3. Expand:

(a) $\sqrt{5}(\sqrt{5}+1)$ (b) $\sqrt{2}(\sqrt{3}-1)$ (c) $\sqrt{3}(2-\sqrt{3})$ (d) $2\sqrt{2}(\sqrt{5}-\sqrt{2})$ (e) $\sqrt{7}(7-2\sqrt{7})$ (f) $\sqrt{6}(3\sqrt{6}-2\sqrt{5})$

DEVELOPMENT

4. Simplify fully:

(a) $\sqrt{6}\times\sqrt{2}$ (b) $\sqrt{5}\times\sqrt{10}$ (c) $\sqrt{3}\times\sqrt{15}$ (d) $\sqrt{2}\times 2\sqrt{22}$ (e) $4\sqrt{12}\times\sqrt{3}$ (f) $3\sqrt{8}\times 2\sqrt{5}$

5. Expand and simplify:

(a) $\sqrt{2}(\sqrt{10}-\sqrt{2})$ (b) $\sqrt{6}(3+\sqrt{3})$ (c) $\sqrt{5}(\sqrt{15}+4)$ (d) $\sqrt{6}(\sqrt{8}-2)$ (e) $3\sqrt{3}(9-\sqrt{21})$ (f) $3\sqrt{7}(\sqrt{14}-2\sqrt{7})$

6. Expand and simplify:

(a) $(\sqrt{3}+1)(\sqrt{2}-1)$ (b) $(\sqrt{5}-2)(\sqrt{7}+3)$ (c) $(\sqrt{5}+\sqrt{2})(\sqrt{3}+\sqrt{2})$ (d) $(\sqrt{6}-1)(\sqrt{6}-2)$ (e) $(\sqrt{7}-2)(2\sqrt{7}+5)$ (f) $(3\sqrt{2}-1)(\sqrt{6}-\sqrt{3})$

7. Use the special expansion $(a+b)(a-b)=a^2-b^2$ to expand and simplify:

(a) $(\sqrt{5}+1)(\sqrt{5}-1)$ (b) $(3-\sqrt{7})(3+\sqrt{7})$ (c) $(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})$ (d) $(3\sqrt{2}-\sqrt{11})(3\sqrt{2}+\sqrt{11})$ (e) $(2\sqrt{6}+3)(2\sqrt{6}-3)$ (f) $(7-2\sqrt{5})(7+2\sqrt{5})$

8. Expand and simplify the following, using the special expansions

$$(a+b)^2 = a^2 + 2ab + b^2 \quad \text{and} \quad (a-b)^2 = a^2 - 2ab + b^2.$$

(a) $(\sqrt{3}+1)^2$ (d) $(\sqrt{7}-\sqrt{5}\,)^2$ (g) $(2\sqrt{7}+\sqrt{5}\,)^2$
(b) $(\sqrt{5}-1)^2$ (e) $(2\sqrt{3}-1)^2$ (h) $(3\sqrt{2}-2\sqrt{3}\,)^2$
(c) $(\sqrt{3}+\sqrt{2}\,)^2$ (f) $(2\sqrt{5}+3)^2$ (i) $(3\sqrt{5}+\sqrt{10}\,)^2$

CHALLENGE

9. Simplify fully:

(a) $\dfrac{\sqrt{40}}{\sqrt{10}}$ (c) $\dfrac{2\sqrt{6}\times\sqrt{5}}{\sqrt{10}}$ (e) $\dfrac{\sqrt{15}\times\sqrt{20}}{\sqrt{12}}$

(b) $\dfrac{\sqrt{18}}{\sqrt{50}}$ (d) $\dfrac{5\sqrt{7}\times\sqrt{3}}{\sqrt{28}}$ (f) $\dfrac{6\sqrt{3}\times 8\sqrt{2}}{\sqrt{32}\times\sqrt{27}}$

10. Use Pythagoras' theorem to find the hypotenuse of the right-angled triangle in which the lengths of the other two sides are:

(a) $\sqrt{2}$ and $\sqrt{7}$ (c) $\sqrt{7}+1$ and $\sqrt{7}-1$
(b) $\sqrt{5}$ and $2\sqrt{5}$ (d) $2\sqrt{3}-\sqrt{6}$ and $2\sqrt{3}+\sqrt{6}$

11. Simplify by forming the lowest common denominator:

(a) $\dfrac{1}{\sqrt{3}+1}+\dfrac{1}{\sqrt{3}-1}$ (b) $\dfrac{3}{2\sqrt{5}-\sqrt{7}}-\dfrac{3}{2\sqrt{5}+\sqrt{7}}$

2 F Rationalising the Denominator

When dealing with surdic expressions, it is usual to remove any surds from the denominator, a process called *rationalising the denominator*. There are two cases.

The Denominator has a Single Term: In the first case, the denominator is a surd or a multiple of a surd.

13 RATIONALISING A SINGLE-TERM DENOMINATOR:

In an expression like $\dfrac{\sqrt{7}}{2\sqrt{3}}$, multiply top and bottom by $\sqrt{3}$.

WORKED EXERCISE: Simplify each expression by rationalising the denominator.

(a) $\dfrac{\sqrt{7}}{2\sqrt{3}}$ (b) $\dfrac{55}{\sqrt{11}}$

SOLUTION:

(a) $$\frac{\sqrt{7}}{2\sqrt{3}} = \frac{\sqrt{7}}{2\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{21}}{2\times 3} = \frac{\sqrt{21}}{6}$$

(b) $$\frac{55}{\sqrt{11}} = \frac{55}{\sqrt{11}}\times\frac{\sqrt{11}}{\sqrt{11}} = \frac{55\sqrt{11}}{11} = 5\sqrt{11}$$

The Denominator has Two Terms: The second case involves a denominator with two terms, one or both of which contain a surd. The method uses the difference of squares identity

$$(A+B)(A-B) = A^2 - B^2$$

to square the unwanted surds and convert them to integers.

14 RATIONALISING A BINOMIAL DENOMINATOR:

In an expression like $\dfrac{3}{5+\sqrt{3}}$, multiply top and bottom by $5-\sqrt{3}$.

WORKED EXERCISE:

Rationalise the denominator in each expression.

(a) $\dfrac{3}{5+\sqrt{3}}$ (b) $\dfrac{1}{2\sqrt{3}-3\sqrt{2}}$

SOLUTION:

(a) $$\frac{3}{5+\sqrt{3}} = \frac{3}{5+\sqrt{3}} \times \frac{5-\sqrt{3}}{5-\sqrt{3}} = \frac{15-3\sqrt{3}}{25-3} = \frac{15-3\sqrt{3}}{22}$$

Using the difference of squares,
$$\left(5+\sqrt{3}\right)\left(5-\sqrt{3}\right) = 5^2 - \left(\sqrt{3}\right)^2 = 25-3.$$

(b) $$\frac{1}{2\sqrt{3}-3\sqrt{2}} = \frac{1}{2\sqrt{3}-3\sqrt{2}} \times \frac{2\sqrt{3}+3\sqrt{2}}{2\sqrt{3}+3\sqrt{2}} = \frac{2\sqrt{3}+3\sqrt{2}}{4\times 3 - 9\times 2} = -\frac{2\sqrt{3}+3\sqrt{2}}{6}$$

Using the difference of squares,
$$\left(2\sqrt{3}-3\sqrt{2}\right)\left(2\sqrt{3}+3\sqrt{2}\right) = \left(2\sqrt{3}\right)^2 - \left(3\sqrt{2}\right)^2 = 4\times 3 - 9\times 2.$$

Exercise 2F

1. Rewrite each fraction with a rational denominator.

(a) $\dfrac{1}{\sqrt{3}}$ (b) $\dfrac{1}{\sqrt{7}}$ (c) $\dfrac{3}{\sqrt{5}}$ (d) $\dfrac{5}{\sqrt{2}}$ (e) $\dfrac{\sqrt{2}}{\sqrt{3}}$ (f) $\dfrac{\sqrt{5}}{\sqrt{7}}$ (g) $\dfrac{2\sqrt{11}}{\sqrt{5}}$ (h) $\dfrac{3\sqrt{7}}{\sqrt{2}}$

2. Rewrite each fraction with a rational denominator.

(a) $\dfrac{1}{\sqrt{3}-1}$ (b) $\dfrac{1}{\sqrt{7}+2}$ (c) $\dfrac{1}{3+\sqrt{5}}$ (d) $\dfrac{1}{4-\sqrt{7}}$ (e) $\dfrac{1}{\sqrt{5}-\sqrt{2}}$ (f) $\dfrac{1}{\sqrt{10}+\sqrt{6}}$ (g) $\dfrac{1}{2\sqrt{3}+1}$ (h) $\dfrac{1}{5-3\sqrt{2}}$

DEVELOPMENT

3. Simplify each expression by rationalising the denominator.

(a) $\dfrac{2}{\sqrt{2}}$ (b) $\dfrac{5}{\sqrt{5}}$ (c) $\dfrac{6}{\sqrt{3}}$ (d) $\dfrac{21}{\sqrt{7}}$ (e) $\dfrac{3}{\sqrt{6}}$ (f) $\dfrac{5}{\sqrt{15}}$ (g) $\dfrac{8}{\sqrt{6}}$ (h) $\dfrac{14}{\sqrt{10}}$

4. Rewrite each fraction with a rational denominator.

(a) $\dfrac{1}{2\sqrt{5}}$ (c) $\dfrac{3}{5\sqrt{2}}$ (e) $\dfrac{10}{3\sqrt{2}}$ (g) $\dfrac{\sqrt{3}}{2\sqrt{10}}$

(b) $\dfrac{1}{3\sqrt{7}}$ (d) $\dfrac{2}{7\sqrt{3}}$ (f) $\dfrac{9}{4\sqrt{3}}$ (h) $\dfrac{2\sqrt{11}}{5\sqrt{7}}$

5. Rewrite each fraction with a rational denominator.

(a) $\dfrac{3}{\sqrt{5}+1}$ (d) $\dfrac{3\sqrt{3}}{\sqrt{5}+\sqrt{3}}$ (g) $\dfrac{\sqrt{3}-1}{\sqrt{3}+1}$ (j) $\dfrac{3\sqrt{2}+\sqrt{5}}{3\sqrt{2}-\sqrt{5}}$

(b) $\dfrac{4}{2\sqrt{2}-\sqrt{3}}$ (e) $\dfrac{2\sqrt{7}}{2\sqrt{7}-5}$ (h) $\dfrac{\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{2}}$ (k) $\dfrac{\sqrt{10}-\sqrt{6}}{\sqrt{10}+\sqrt{6}}$

(c) $\dfrac{\sqrt{7}}{5-\sqrt{7}}$ (f) $\dfrac{\sqrt{5}}{\sqrt{10}-\sqrt{5}}$ (i) $\dfrac{3-\sqrt{7}}{3+\sqrt{7}}$ (l) $\dfrac{7+2\sqrt{11}}{7-2\sqrt{11}}$

CHALLENGE

6. Simplify each expression by rationalising the denominator.

(a) $\dfrac{\sqrt{3}-1}{2-\sqrt{3}}$ (b) $\dfrac{2\sqrt{5}-\sqrt{2}}{\sqrt{5}+\sqrt{2}}$

7. Show that each expression is rational by first rationalising the denominators.

(a) $\dfrac{3}{\sqrt{2}}+\dfrac{3}{2+\sqrt{2}}$ (c) $\dfrac{4}{2+\sqrt{2}}+\dfrac{1}{3-2\sqrt{2}}$

(b) $\dfrac{1}{3+\sqrt{6}}+\dfrac{2}{\sqrt{6}}$ (d) $\dfrac{8}{3-\sqrt{7}}-\dfrac{6}{2\sqrt{7}-5}$

8. If $x=\dfrac{\sqrt{5}+1}{2}$, show that $1+\dfrac{1}{x}=x$.

9. The expression $\dfrac{\sqrt{6}+1}{\sqrt{3}+\sqrt{2}}$ can be written in the form $a\sqrt{3}+b\sqrt{2}$. Find a and b.

2G Chapter Review Exercise

1. Classify each positive integer as prime or composite.

(a) 21 (c) 37 (e) 51

(b) 23 (d) 39 (f) 61

2. Write each number as a product of its prime factors.

(a) 15 (c) 34 (e) 75

(b) 8 (d) 42 (f) 90

3. Find the highest common factor of the numerator and denominator, then use it to cancel each fraction down to lowest terms.

(a) $\dfrac{8}{18}$ (c) $\dfrac{30}{45}$ (e) $\dfrac{32}{60}$

(b) $\dfrac{9}{12}$ (d) $\dfrac{18}{42}$ (f) $\dfrac{48}{80}$

4. Find the lowest common denominator, then simplify:

(a) $\dfrac{1}{3}+\dfrac{1}{5}$ (c) $\dfrac{1}{6}-\dfrac{1}{9}$ (e) $\dfrac{3}{8}+\dfrac{1}{12}$

(b) $\dfrac{3}{4}-\dfrac{1}{8}$ (d) $\dfrac{1}{6}+\dfrac{1}{10}$ (f) $\dfrac{5}{12}-\dfrac{2}{9}$

5. Express each fraction as a decimal by rewriting it with denominator 10, 100 or 1000.

(a) $\dfrac{2}{5}$ (c) $\dfrac{3}{25}$ (e) $\dfrac{3}{8}$

(b) $\dfrac{1}{4}$ (d) $\dfrac{13}{20}$ (f) $\dfrac{7}{40}$

6. Express each fraction as a recurring decimal by dividing the numerator by the denominator.

(a) $\dfrac{2}{9}$ (c) $\dfrac{2}{11}$ (e) $\dfrac{1}{12}$

(b) $\dfrac{7}{9}$ (d) $\dfrac{7}{11}$ (f) $\dfrac{5}{12}$

7. Write each terminating decimal as a fraction in lowest terms.

(a) $0{\cdot}6$ (c) $0{\cdot}08$ (e) $0{\cdot}012$

(b) $0{\cdot}05$ (d) $0{\cdot}38$ (f) $0{\cdot}675$

8. Write each recurring decimal as a fraction in lowest terms.

(a) $0{\cdot}\dot{3}$ (d) $0{\cdot}\dot{4}\dot{5}$ (g) $0{\cdot}3\dot{6}$

(b) $0{\cdot}\dot{8}$ (e) $0{\cdot}\dot{2}7\dot{9}$ (h) $0{\cdot}5\dot{7}$

(c) $0{\cdot}\dot{2}\dot{5}$ (f) $0{\cdot}\dot{2}9\dot{7}$ (i) $0{\cdot}1\dot{6}\dot{3}$

9. Classify each of these real numbers as rational or irrational. Express those that are rational in the form $\dfrac{a}{b}$, where a and b are integers.

(a) 7 (c) $\sqrt{9}$ (e) $\sqrt[3]{15}$ (g) $-0{\cdot}16$

(b) $-2\frac{1}{4}$ (d) $\sqrt{10}$ (f) $\sqrt[4]{16}$ (h) π

10. Use a calculator to write each number correct to:
(i) two decimal places, (ii) two significant figures.

(a) $\sqrt{17}$ (c) $1{\cdot}16^{7}$ (e) $7{\cdot}3^{-2}$

(b) $\sqrt[3]{102}$ (d) $\dfrac{49}{64}$ (f) $\pi^{5\cdot5}$

11. Evaluate, correct to three significant figures:

(a) $\dfrac{7{\cdot}93}{8{\cdot}22-3{\cdot}48}$

(b) $-4{\cdot}9\times(-5{\cdot}8-8{\cdot}5)$

(c) $\sqrt[4]{1{\cdot}6\times2{\cdot}6}$

(d) $\dfrac{13^{5}}{11^{6}+17^{4}}$

(e) $\dfrac{\frac{4}{9}-\frac{2}{7}}{\frac{5}{8}-\frac{3}{10}}$

(f) $\sqrt{2{\cdot}4^{-1\cdot6}}$

(g) $\sqrt{\dfrac{1{\cdot}347}{2{\cdot}518-1{\cdot}679}}$

(h) $\dfrac{2{\cdot}7\times10^{-2}}{1{\cdot}7\times10^{-5}}$

(i) $\dfrac{\sqrt{\frac{1}{2}}+\sqrt[3]{\frac{1}{3}}}{\sqrt[4]{\frac{1}{4}}+\sqrt[5]{\frac{1}{5}}}$

12. Simplify:

(a) $\sqrt{24}$ (c) $\sqrt{50}$ (e) $3\sqrt{18}$

(b) $\sqrt{45}$ (d) $\sqrt{500}$ (f) $2\sqrt{40}$

13. Simplify:
(a) $\sqrt{5}+\sqrt{5}$
(b) $\sqrt{5}\times\sqrt{5}$
(c) $(2\sqrt{7})^2$
(d) $2\sqrt{5}+\sqrt{7}-3\sqrt{5}$
(e) $\sqrt{35}\div\sqrt{5}$
(f) $6\sqrt{55}\div 2\sqrt{11}$
(g) $\sqrt{8}\times\sqrt{2}$
(h) $\sqrt{10}\times\sqrt{2}$
(i) $2\sqrt{6}\times 4\sqrt{15}$

14. Simplify:
(a) $\sqrt{27}-\sqrt{12}$
(b) $\sqrt{18}+\sqrt{32}$
(c) $3\sqrt{2}+3\sqrt{8}-\sqrt{50}$
(d) $\sqrt{54}-\sqrt{20}+\sqrt{150}-\sqrt{80}$

15. Expand:
(a) $\sqrt{7}(3-\sqrt{7})$
(b) $\sqrt{5}(2\sqrt{6}+3\sqrt{2})$
(c) $\sqrt{15}(\sqrt{3}-5)$
(d) $\sqrt{3}(\sqrt{6}+2\sqrt{3})$

16. Expand and simplify:
(a) $(\sqrt{5}+2)(3-\sqrt{5})$
(b) $(2\sqrt{3}-1)(3\sqrt{3}+5)$
(c) $(\sqrt{7}-3)(2\sqrt{5}+4)$
(d) $(\sqrt{10}-3)(\sqrt{10}+3)$
(e) $(2\sqrt{6}+\sqrt{11})(2\sqrt{6}-\sqrt{11})$
(f) $(\sqrt{7}-2)^2$
(g) $(\sqrt{5}+\sqrt{2})^2$
(h) $(4-3\sqrt{2})^2$

17. Write with a rational denominator:
(a) $\dfrac{1}{\sqrt{5}}$
(b) $\dfrac{3}{\sqrt{2}}$
(c) $\dfrac{\sqrt{3}}{\sqrt{11}}$
(d) $\dfrac{1}{5\sqrt{3}}$
(e) $\dfrac{5}{2\sqrt{7}}$
(f) $\dfrac{\sqrt{2}}{3\sqrt{10}}$

18. Write with a rational denominator:
(a) $\dfrac{1}{\sqrt{5}+\sqrt{2}}$
(b) $\dfrac{1}{3-\sqrt{7}}$
(c) $\dfrac{1}{2\sqrt{6}-\sqrt{3}}$
(d) $\dfrac{\sqrt{3}}{\sqrt{3}+1}$
(e) $\dfrac{3}{\sqrt{11}+\sqrt{5}}$
(f) $\dfrac{3\sqrt{7}}{2\sqrt{5}-\sqrt{7}}$

19. Rationalise the denominator of each fraction.
(a) $\dfrac{\sqrt{7}-\sqrt{2}}{\sqrt{7}+\sqrt{2}}$
(b) $\dfrac{3\sqrt{3}+5}{3\sqrt{3}-5}$

20. Find the value of x if $\sqrt{18}+\sqrt{8}=\sqrt{x}$.

21. Simplify $\dfrac{3}{\sqrt{5}-2}+\dfrac{2}{\sqrt{5}+2}$ by forming the lowest common denominator.

22. Find the values of p and q such that $\dfrac{\sqrt{5}}{\sqrt{5}-2}=p+q\sqrt{5}$.

23. Show that $\dfrac{2}{6-3\sqrt{3}}-\dfrac{1}{2\sqrt{3}+3}$ is rational by first rationalising each denominator.

Functions and their Graphs

The principal purpose of this course is the study of functions. This chapter develops the idea of functions and relations and their graphs, with a review of known graphs. The final section explains how the graph of a known function or relation can be shifted or reflected, allowing a wide variety of new graphs to be obtained.

Curve-sketching software is useful in emphasising the basic idea that a function has a graph. It is also particularly helpful in understanding how a graph can be shifted or reflected, because a large number of similar graphs can be examined in a short time. Nevertheless, readers must eventually be able to construct a graph from its equation on their own.

3 A Functions and Relations

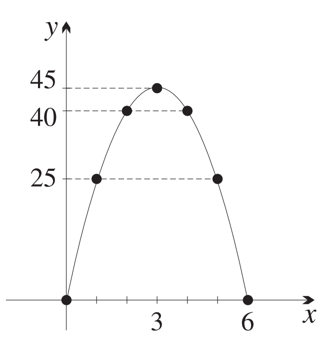
Functions and their graphs have been studied at least since early secondary school. These notes are intended to make the idea and the notation a little more precise.

A Function and its Graph: When a quantity y is completely determined by some other quantity x as a result of any *rule* whatsoever, we say that y *is a function of* x. For example, the height y of a ball thrown vertically upwards will be a function of the time x after the ball is thrown. In units of metres and seconds, a possible rule is

$$y = 5x(6 - x), \quad \text{where } 0 \le x \le 6.$$

We can construct a *table of values* of this function by choosing just a few values of the time x and calculating the corresponding heights y:

x	0	1	2	3	4	5	6
y	0	25	40	45	40	25	0



Each x-value and its corresponding y-value can then be put into an ordered pair ready to plot on a graph of the function. The seven ordered pairs calculated here are:

$$(0, 0),\ (1, 25),\ (2, 40),\ (3, 45),\ (4, 40),\ (5, 25),\ (6, 0),$$

and the graph is sketched opposite. The seven representative points have been plotted, but there are infinitely many such ordered pairs, and they join up to make the nice smooth curve shown to the right.

1 INFORMALLY, A FUNCTION IS A RULE:
- When a quantity y is completely determined by some other quantity x as a result of any rule whatsoever, we say that *y is a function of x*.
- The function rule is almost always an *equation*, possibly with a restriction. We shall identify the function and the equation and write, for example:

  'The function $y = 5x(6 - x)$, where $0 \le x \le 6$.'

A Function as a Set of Ordered Pairs: Alternatively, and more formally, one can identify a function completely with the ordered pairs generated by its function rule.

2 MORE FORMALLY, A FUNCTION IS A SET OF ORDERED PAIRS:
- A *function* is a set of ordered pairs in which no two ordered pairs have the same first or x-coordinate.
- That is, a function is identical to the ordered pairs of its graph.

The condition that no two ordered pairs have the same x-coordinate is crucial. In the case of the ball, no x-value can occur twice because at any one time the ball can only be in one position. A particular y-value, however, may occur many times.

Domain and Range: The time variable x in our example cannot be negative, because the ball had not been thrown then, and cannot be greater than 6, because the ball hits the ground again after 6 seconds. The *domain* is the set of possible x-values, so the domain is the closed interval $0 \le x \le 6$.

The height of the ball never exceeds 45 metres and is never negative. The *range* is the set of possible y-values, so the range is the closed interval $0 \le y \le 45$.

3 THE DOMAIN AND RANGE OF A FUNCTION:
- The *domain* of a function is the set of all possible x-coordinates.
- The *range* of a function is the set of all possible y-coordinates.

The Natural Domain: When the equation of a function is given with no restriction, we assume *as a convention* that the domain is the *natural domain*, consisting of all x-values that can validly be substituted into the equation.

For example, 'the function $y = \sqrt{x-2}$' means

'the function $y = \sqrt{x-2}$, where $x \ge 2$',

because one cannot take square roots of negative numbers.

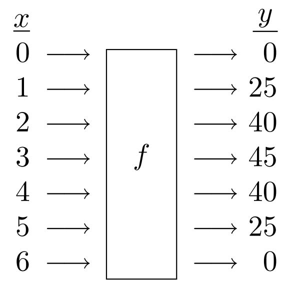
Again, 'the function $y = \dfrac{1}{x-2}$' means

'the function $y = \dfrac{1}{x-2}$, where $x \ne 2$',

because division by 0 is impossible.

4 THE NATURAL DOMAIN:
If no restriction is given, the domain is assumed to be the *natural domain*, which consists of all x-values that can validly be substituted into the equation.

The Function Machine and the Function Rule: A function can be regarded as a 'machine' with inputs and outputs. For example, the numbers in the right-hand column opposite are the outputs from the function $y = 5x(6-x)$ when the numbers 0, 1, 2, 3, 4, 5 and 6 are the inputs.



This model of a function has become far more intuitive in the last few decades because computers and calculators routinely produce output from a given input.

If the name f is given to our function, we can write the results of the input/output routines as follows:

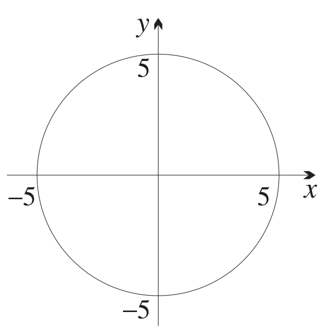
$$f(0) = 0, \quad f(1) = 25, \quad f(2) = 40, \quad f(3) = 45, \quad f(4) = 40, \quad \ldots$$

When x is the input, the output is $5x(6-x)$. Thus, using the well-known notation introduced by Euler in 1735, we can write the *function rule* as

$$f(x) = 5x(6-x), \text{ where } 0 \le x \le 6.$$

Relations: We shall often be dealing with graphs such as the following. These objects are sets of ordered pairs, but they are not functions. The more general word 'relation' is used for any curve or region in the plane, whether a function or not.

5 RELATIONS:
- Any set of ordered pairs is called a *relation*.
- A relation that is not a function will have two or more outputs for some inputs.
- Relations, like functions, have a *domain* and *range*.

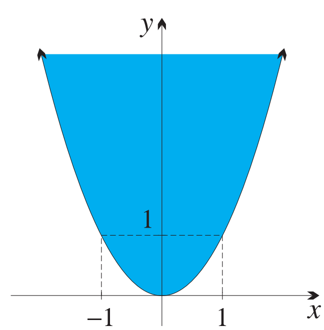


$x^2 + y^2 = 25$

Here the input $x = 0$ gives two outputs, $y = 5$ and $y = -5$, because the vertical line $x = 0$ meets the graph at $(0, 5)$ and at $(0, -5)$.

Domain: $-5 \le x \le 5$

Range: $-5 \le y \le 5$



$y \ge x^2$

Here the input $x = 0$ gives as output all numbers $y \ge 0$, because the vertical line $x = 0$ meets the graphed region at every point from the origin upwards.

Domain: all real numbers

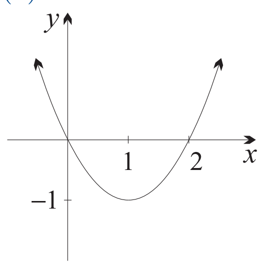
Range: $y \ge 0$

The Vertical Line Test: Once a relation is graphed, it is then quite straightforward to decide whether or not it is a function.

6 VERTICAL LINE TEST: A relation is a function if and only if no vertical line crosses the graph more than once.

Exercise 3A

1. Use the vertical line test to find which of the following graphs represent functions.

(a)


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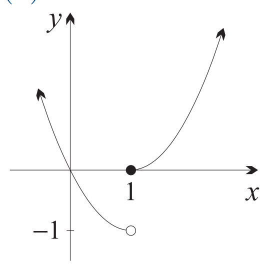

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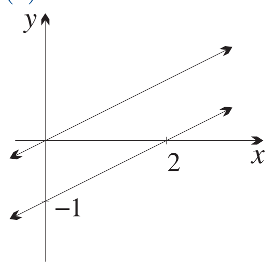
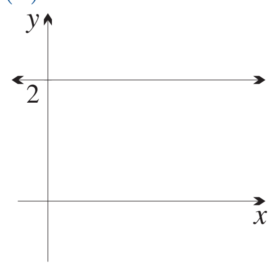
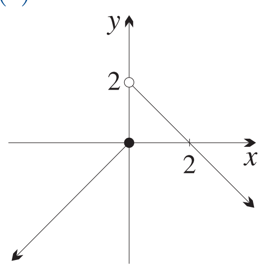


(d)


(e)


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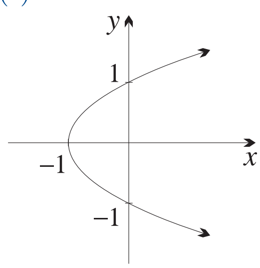
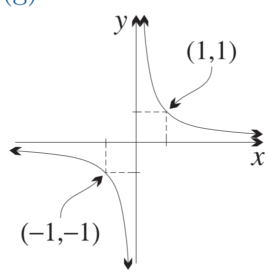
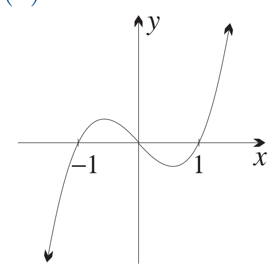
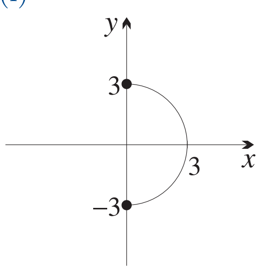
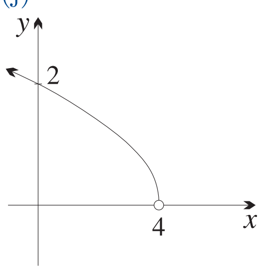


(g)


(h)


(i)


(j)


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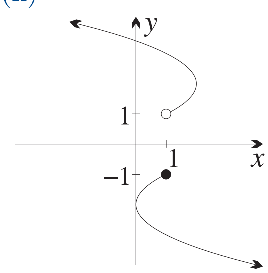

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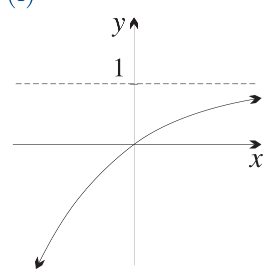


2. What are the domain and range of each relation in question 1?

3. Find $f(2)$, $f(0)$ and $f(-2)$ for each function.
(a) $f(x) = 3x - 1$ (b) $f(x) = 4 - x^2$ (c) $f(x) = x^3 + 8$ (d) $f(x) = 2^x$

4. Find $h(-3)$, $h(\frac{1}{2})$ and $h(5)$ for each function.
(a) $h(x) = 2x + 2$ (b) $h(x) = \frac{1}{x}$ (c) $h(x) = 3x - x^2$ (d) $h(x) = \sqrt{x + 4}$

5. (a) Copy and complete the table of values for each function.

(i) $f(x) = 2x + 1$

x	-1	0	1
$f(x)$			

(iii) $f(x) = 1 - x^2$

x	-2	-1	0	1	2
$f(x)$					

(ii) $f(x) = x^2 - 2x$

x	-1	0	1	2	3
$f(x)$					

(iv) $f(x) = x^3 - 4x$

x	-3	-2	-1	0	1	2	3
$f(x)$							

(b) Plot the points in each table in part (a) and hence sketch each function.

(c) Write down the domain and the range of each function in part (a).

6. Use the fact that $\frac{1}{0}$ is undefined to find the natural domain of each function.

(a) $f(x) = \dfrac{1}{x}$ (b) $f(x) = \dfrac{1}{x-3}$ (c) $f(x) = \dfrac{1}{x+1}$ (d) $f(x) = \dfrac{1}{2+x}$

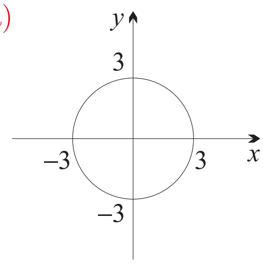
7. Use the fact that a negative number does not have a square root to find the natural domain of each function.

(a) $f(x) = \sqrt{x}$ (b) $f(x) = \sqrt{x-2}$ (c) $f(x) = \sqrt{x+3}$ (d) $f(x) = \sqrt{5+x}$

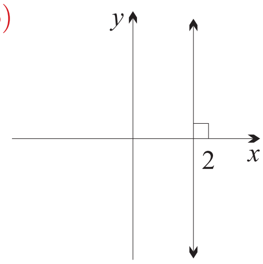
DEVELOPMENT

8. The following graphs show relations that are NOT functions. Write down the coordinates of two points on each graph that have the same x-coordinate.

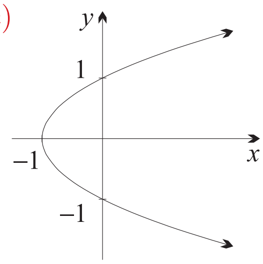
(a)



(b)



(c)



9. Find $g(a)$, $g(-a)$ and $g(a+1)$ for each function.

(a) $g(x) = 2x - 4$ (b) $g(x) = 2 - x$ (c) $g(x) = x^2$ (d) $g(x) = \dfrac{1}{x-1}$

10. Find $F(t^2)$, $F(t) - 2$ and $F(t-2)$ for each function.

(a) $F(x) = 5x + 2$ (b) $F(x) = \sqrt{x}$ (c) $F(x) = x^2 + 2x$ (d) $F(x) = 2 - x^2$

11. Find the natural domain of each function.

(a) $f(x) = 4x$ (d) $f(x) = \dfrac{3}{2x-1}$ (g) $f(x) = \dfrac{1}{\sqrt{x}}$ (j) $f(x) = \dfrac{1}{\sqrt{2x-3}}$

(b) $f(x) = 7 - 3x$ (e) $f(x) = \sqrt{2x+1}$ (h) $f(x) = \dfrac{1}{\sqrt{x+1}}$ (k) $f(x) = \dfrac{3}{\sqrt{3-4x}}$

(c) $f(x) = \dfrac{1}{4-x}$ (f) $f(x) = \sqrt{5-x}$ (i) $f(x) = \dfrac{2}{\sqrt{1-x}}$ (l) $f(x) = \dfrac{2}{\sqrt{2+3x}}$

12. Given that $P(x) = x^2 - 2x - 4$, find the value of:

(a) $P(\sqrt{2}\,)$ (b) $P(\sqrt{7}\,)$ (c) $P(\sqrt{3} - 1)$ (d) $P(1 + \sqrt{5}\,)$

13. Given that $f(x) = x^2 - 3x + 5$, find the value of:

(a) $\frac{1}{4}\left(f(-1) + 2f(0) + f(1)\right)$ (b) $\frac{1}{6}\left(f(0) + 4f(\tfrac{1}{2}) + f(1)\right)$

CHALLENGE

14. (a) Let $f(x) = \begin{cases} x, & \text{for } x \le 0, \\ 2 - x, & \text{for } x > 0. \end{cases}$ Create a table of values for $-3 \le x \le 3$, and confirm that the graph is that shown in question 1(e) above.

(b) Let $f(x) = \begin{cases} (x-1)^2 - 1, & \text{for } x < 1, \\ (x-1)^2, & \text{for } x \ge 1. \end{cases}$ Create a table of values for $-1 \le x \le 3$, and confirm that the graph is that shown in question 1(b) above.

15. Suppose that $f(x) = \begin{cases} -3x, & \text{for } x < 0, \\ 2x + 1, & \text{for } x \ge 0. \end{cases}$ Find the value of:

(a) $f(-5)$ (b) $f(0)$ (c) $f(13)$ (d) $f(a^2)$

16. If $F(x) = x^2 + 5x$, find in simplest form:
(a) $\dfrac{F(h) - F(-h)}{2h}$ (b) $\dfrac{F(p) - F(q)}{p - q}$ (c) $\dfrac{F(a+b) - F(a-b)}{2b}$

17. State the natural domain of each function.
(a) $f(x) = \dfrac{x}{\sqrt{x+2}}$ (c) $f(x) = \dfrac{1}{x^2 + x}$ (e) $f(x) = \sqrt{x^2 - 4}$
(b) $f(x) = \dfrac{2}{x^2 - 4}$ (d) $f(x) = \dfrac{1}{x^2 - 5x + 6}$ (f) $f(x) = \dfrac{1}{\sqrt{1 - x^2}}$

3 B Review of Linear Graphs

Sections 3B–3E will briefly review functions and relations that have been introduced in previous years, and the sketching of their graphs. Linear graphs are the subject of this section.

Linear Functions and Relations: A relation is called *linear* if its graph is a straight line. Its equation then is something like

$$3x + 4y - 12 = 0,$$

which consists entirely of a term in x, a term in y and a constant. This particular equation is a *function*, because it can be solved for y:

$$\boxed{+12 - 3x} \quad 4y = -3x + 12$$
$$\boxed{\div 4} \quad y = -\tfrac{3}{4}x + 3$$

If the coefficient of y is zero, however, as in the relation

$$x + 3 = 0$$

then the equation cannot be solved for y and the relation is not a function.

LINEAR RELATIONS AND LINEAR FUNCTIONS:

- A *linear relation* has a graph that is a straight line.
- A relation is *linear* if and only if its equation can be written as
$$ax + by + c = 0,$$
7 where the coefficients a and b are not both zero.
- The relation is a *linear function* if and only if $b \neq 0$, because then it can be solved for y and written as
$$y = -\frac{ax}{b} - \frac{c}{b}.$$

Sketching Linear Relations: When the coefficients a, b and c are all non-zero, the easiest way to sketch a linear function is to find the intercepts with the axes.

SKETCHING $ax + by + c = 0$ WHEN a AND b AND c ARE ALL NON-ZERO:

8
- Find the x-intercept by putting $y = 0$.
- Find the y-intercept by putting $x = 0$.

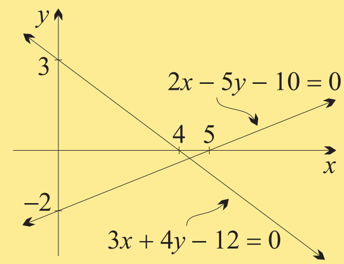
WORKED EXERCISE:

Sketch on one set of axes: (a) $3x + 4y - 12 = 0$ (b) $2x - 5y - 10 = 0$

SOLUTION:

(a) Consider $3x + 4y - 12 = 0$.
When $y = 0$, $3x - 12 = 0$
$x = 4$.
When $x = 0$, $4y - 12 = 0$
$y = 3$.

(b) Consider $2x - 5y - 10 = 0$.
When $y = 0$, $2x - 10 = 0$
$x = 5$.
When $x = 0$, $-5y - 10 = 0$
$y = -2$.



Three Special Cases: This method won't work when any of a, b and c is zero.

SKETCHING SPECIAL CASES OF LINEAR GRAPHS $ax + by + c = 0$:

(a) If $a = 0$, then the equation has the form $y = k$.
Its graph is a horizontal line with y-intercept k.

9 (b) If $b = 0$, then the equation has the form $x = \ell$.
Its graph is a vertical line with x-intercept ℓ.

(c) If $c = 0$, both intercepts are zero and the graph passes through the origin.
Find one more point on it, usually by putting $x = 1$.

WORKED EXERCISE:

Sketch the following three lines on one set of axes.

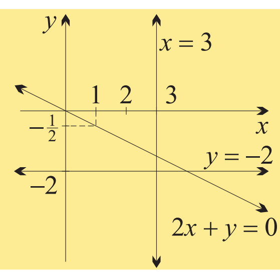
(a) $y = -2$ (b) $x = 3$ (c) $x + 2y = 0$

SOLUTION:

(a) The line $y = -2$ is horizontal with y-intercept -2.

(b) The line $x = 3$ is vertical with x-intercept 3.

(c) The line $x + 2y = 0$ passes through the origin,
and when $x = 1$, $y = -\frac{1}{2}$.



Exercise 3B

1. Consider the line $y = 2x - 2$.
(a) Find the x-intercept by putting $y = 0$.
(b) Find the y-intercept by putting $x = 0$.
(c) Sketch the line, showing both intercepts.

2. Consider the line $x + 2y - 4 = 0$.
(a) Find the x-intercept by putting $y = 0$.
(b) Find the y-intercept by putting $x = 0$.
(c) Sketch the line, showing both intercepts.

3. Repeat the steps in the previous two questions for each of these lines.
(a) $y = x + 1$
(b) $y = 4 - 2x$
(c) $y = \frac{1}{2}x - 3$
(d) $y = -3x - 6$
(e) $x + y - 1 = 0$
(f) $2x - y + 2 = 0$
(g) $x - 3y - 3 = 0$
(h) $x - 2y - 4 = 0$
(i) $2x - 3y - 12 = 0$
(j) $x + 4y + 6 = 0$
(k) $5x + 2y - 10 = 0$
(l) $-5x + 2y + 15 = 0$

4. Consider the line $y = -2x$.
(a) Show that the x-intercept and the y-intercept are both zero.
(b) Put $x = 1$ in order to find a second point on the line, then sketch the line.

5. Repeat the steps in the previous question for each of these lines.
(a) $y = x$
(b) $y = 3x$
(c) $y = -4x$
(d) $x + y = 0$
(e) $x - 2y = 0$
(f) $3x + 2y = 0$

6. Sketch the following vertical and horizontal lines.
(a) $x = 1$
(b) $y = 2$
(c) $x = -2$
(d) $y = 0$
(e) $2y = -3$
(f) $3x = 5$

DEVELOPMENT

7. Determine, by substitution, whether or not the given point lies on the given line.
(a) $(3, 1)$ $y = x - 2$
(b) $(7, 4)$ $y = 20 - 2x$
(c) $(1, -2)$ $y = -3x + 1$
(d) $(-5, 3)$ $2x + 3y + 1 = 0$
(e) $(-1, -4)$ $3x - 2y - 5 = 0$
(f) $(-6, -4)$ $4x - 5y - 4 = 0$

8. Consider the lines $x + y = 5$ and $x - y = 1$.
(a) Graph the lines on a number plane, using a scale of 1 cm to 1 unit on each axis.
(b) Read off the point of intersection of the two lines.
(c) Confirm your answer to part (b) by solving the two equations simultaneously.

9. Repeat the previous question for the following pairs of lines.
(a) $x + y = 2$, $x - y = -4$
(b) $x - y = 3$, $2x + y = 0$
(c) $x + 2y = -4$, $2x - y = -3$

CHALLENGE

10. (a) Show that the line $y - 2 = m(x - 1)$ passes through the point $(1, 2)$.
(b) Sketch on one number plane the four lines corresponding to the following values of m:
(i) $m = 1$ (ii) $m = 2$ (iii) $m = -\frac{1}{2}$ (iv) $m = 0$

3 C Review of Quadratic Graphs

This section covers only quadratics that can be factored — a more complete account will be given in Chapter 9. A *quadratic* is a function of the form

$$f(x) = ax^2 + bx + c, \quad \text{where } a, b \text{ and } c \text{ are constants, and } a \neq 0.$$

Its graph is a *parabola* with axis of symmetry parallel to the y-axis. Any sketch of the parabola should normally show four points:
- the y-intercept,
- the two x-intercepts (which may coincide or may not exist), and
- the vertex.

There are four steps in the sketching process.

THE FOUR STEPS IN SKETCHING A QUADRATIC $y = ax^2 + bx + c$ THAT CAN BE FACTORED:

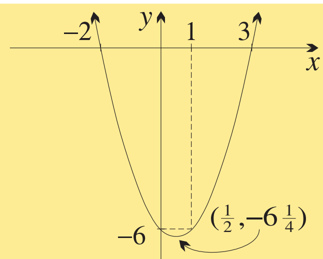
10

1. If a is positive, the parabola is concave up.
   If a is negative, the parabola is concave down.
2. To find the y-intercept, put $x = 0$. (The y-intercept is thus the constant c.)
3. To find the x-intercepts, factor the quadratic and put $y = 0$.
4. Find the axis of symmetry by finding the average of the x-intercepts.
   Then find the y-coordinate of the vertex by substituting back into y.

WORKED EXERCISE:

Sketch the graph of $y = x^2 - x - 6$.

SOLUTION:

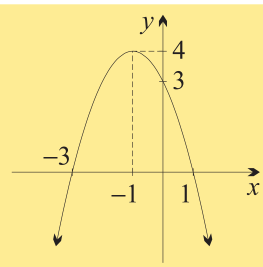


1. Since $a = 1$, the parabola is concave up.
2. The y-intercept is -6.
3. Factoring, $y = (x - 3)(x + 2)$,
   so the x-intercepts are $x = 3$ and $x = -2$.
4. The axis of symmetry is $x = \dfrac{3 - 2}{2}$ (Take the average of the x-intercepts.)
   $x = \frac{1}{2}$.
   When $x = \frac{1}{2}$, $y = -6\frac{1}{4}$, so the vertex is $(\frac{1}{2}, -6\frac{1}{4})$.

WORKED EXERCISE:

Sketch the graph of $y = -x^2 - 2x + 3$.

SOLUTION:



1. Since $a < 1$, the curve is concave down.
2. The y-intercept is $y = 3$.
3. Factoring, $y = -(x^2 + 2x - 3)$
   $= -(x + 3)(x - 1)$,
   so the x-intercepts are $x = -3$ and $x = 1$.
4. The axis of symmetry is $x = \dfrac{-3 + 1}{2}$ (Take the average of the x-intercepts.)
   $x = -1$.
   When $x = -1$, $y = 4$, so the vertex is $(-1, 4)$.

Exercise 3C

1. Consider the parabola $y = x^2 - 4$.
- (a) Find the y-intercept by putting $x = 0$.
- (b) Find the x-intercepts by putting $y = 0$.
- (c) Sketch the parabola, showing all intercepts.
- (d) State the domain and range.

2. Repeat the steps in the previous question for each of these parabolas.
(a) $y = x^2 - 9$ (b) $y = 1 - x^2$ (c) $y = x^2 - 25$ (d) $y = 36 - x^2$

3. Sketch the parabola $y = (x-1)(x-3)$ after carrying out these steps.
(a) Write down the two x-intercepts.
(b) Find the y-intercept.
(c) Take the average of the x-intercepts to find the axis of symmetry.
(d) Find the coordinates of the vertex.

4. Repeat the steps in the previous question for each of these parabolas.
(a) $y = (x-1)(x-5)$
(b) $y = (x+1)(x-3)$
(c) $y = x(x-4)$
(d) $y = (x+2)(x-6)$

5. Sketch the parabola $y = -(x+2)(x-4)$ after carrying out these steps.
(a) Write down the two x-intercepts.
(b) Find the y-intercept.
(c) Take the average of the x-intercepts to find the axis of symmetry.
(d) Find the coordinates of the vertex.

6. Repeat the steps in the previous question for each of these parabolas.
(a) $y = -(x+5)(x-3)$
(b) $y = -x(x-6)$

DEVELOPMENT

7. Sketch the parabola $y = x^2 + 2x - 8$ after carrying out these steps.
(a) Find the y-intercept.
(b) Factor the quadratic expression, and hence write down the x-intercepts.
(c) Find the coordinates of the vertex.

8. Repeat the steps in the previous question for each of these parabolas.
(a) $y = x^2 - 8x + 7$
(b) $y = x^2 - 2x - 15$
(c) $y = x^2 - x - 6$
(d) $y = 2x^2 + 5x$
(e) $y = 2x^2 + 3x - 2$
(f) $y = 4x^2 - 8x - 5$

9. Sketch the parabola $y = -x^2 - 2x + 3$ after carrying out these steps.
(a) Find the y-intercept.
(b) Factor the quadratic expression (take out a factor of -1 first), and hence write down the x-intercepts.
(c) Find the coordinates of the vertex.

10. Repeat the steps in the previous question for each of these parabolas.
(a) $y = -x^2 + 12x - 20$
(b) $y = -x^2 - 4x + 32$
(c) $y = -2x^2 + 3x$
(d) $y = -4x^2 - 8x + 21$

CHALLENGE

11. Sketch each parabola, showing the x-intercepts and the vertex.
(a) $y = x^2 - 3$
(b) $y = 5 - x^2$

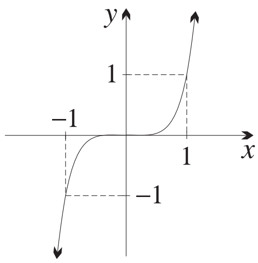
12. Sketch each parabola, showing the x and y intercepts and the vertex. (You will need to use the quadratic formula to find the x-intercepts.)
(a) $y = x^2 - 4x + 2$
(b) $y = x^2 + 2x - 5$

13. (a) Find b and c if the quadratic $y = x^2 + bx + c$ has zeroes $x = 4$ and $x = 7$.
(b) Find α if a quadratic with zeroes $x = 6$ and $x = \alpha$ has axis of symmetry $x = 1$.
(c) Find b and c if $x = 3$ is a zero of the quadratic $y = x^2 + bx + c$, and the y-intercept of the quadratic is 27.

3 D Higher Powers of x and Circles

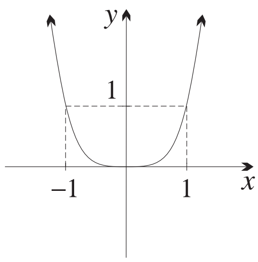
This section covers cubes and fourth powers of x and square roots of x, and also reviews circles and semicircles.

The Cube of x: Graphed to the right is the cubic function $y = x^3$. All odd powers look similar, becoming flatter near the origin as the index increases, and steeper further away.



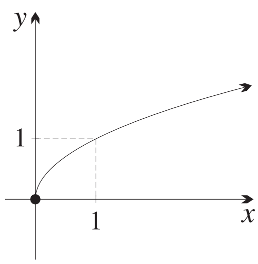
x	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
y	-8	-1	$-\frac{1}{8}$	0	$\frac{1}{8}$	1	8

The Fourth Power of x: The graph to the right shows $y = x^4$. All even powers look similar — they are always positive, and become flatter near the origin as the index increases, and steeper further away.



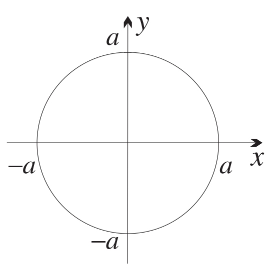
x	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
y	16	1	$\frac{1}{16}$	0	$\frac{1}{16}$	1	16

The Function $y = \sqrt{x}$: The graph of $y = \sqrt{x}$ is the upper half of a parabola on its side, as can be seen by squaring both sides to give $y^2 = x$. Remember that the symbol $\sqrt{x}$ means the *positive* square root of x, so the lower half is excluded:



x	0	$\frac{1}{4}$	1	2	4
y	0	$\frac{1}{2}$	1	$\sqrt{2}$	2

Circles and Semicircles: The graph of $x^2 + y^2 = a^2$ is a *circle* with radius $a > 0$ and centre the origin, as sketched to the right. This graph fails the vertical line test and so is not a function. This can also be seen algebraically — solving the equation for y yields

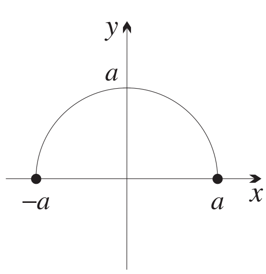


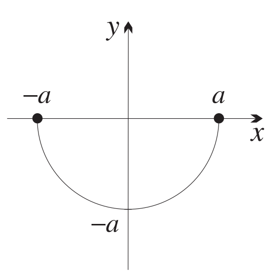
$$y = \sqrt{a^2 - x^2} \quad \text{or} \quad y = -\sqrt{a^2 - x^2}\,,$$

giving two values of y for some values of x.

The *positive square root* $y = \sqrt{a^2 - x^2}$, however, is a function, whose graph is the *upper semicircle* on the left below.

Similarly, the *negative square root* $y = -\sqrt{a^2 - x^2}$ is also a function, whose graph is the *lower semicircle* on the right below.





Exercise 3D

1. Write down the coordinates of the centre and the radius of each circle.
(a) $x^2 + y^2 = 16$ (b) $x^2 + y^2 = 49$ (c) $x^2 + y^2 = \frac{1}{9}$ (d) $x^2 + y^2 = 1{\cdot}44$

2. Sketch graphs of these circles, marking all intercepts with the axes, then write down the domain and range of each.
(a) $x^2 + y^2 = 1$ (b) $x^2 + y^2 = 9$ (c) $x^2 + y^2 = \frac{1}{4}$ (d) $x^2 + y^2 = \frac{9}{4}$

3. Consider the curve $y = x^3$.
(a) Copy and complete the following table of values:

x	$-1{\cdot}5$	-1	$-0{\cdot}5$	0	$0{\cdot}5$	1	$1{\cdot}5$
y							

(b) Plot the points in the table, using a scale of 2 cm to 1 unit on each axis, and then sketch the curve.

4. Repeat the previous question for the curve $y = x^4$.

5. Consider the curve $y = \sqrt{x}$.
(a) Copy and complete the following table of values:

x	0	$0{\cdot}25$	1	$2{\cdot}25$	4
y					

(b) Plot the points in the table, using a scale of 2 cm to 1 unit on each axis, and then sketch the curve.

DEVELOPMENT

6. Consider the circle $x^2 + y^2 = 25$.
(a) Copy and complete the following table of values, correct to one decimal place where necessary. (Remember that a positive number has TWO square roots.)

x	0	1	2	3	4	5
y						

(b) Plot the points in the table, using a scale of 1 cm to 1 unit on each axis.
(c) Reflect the points plotted in part (b) in the y-axis, and then sketch the circle.

7. Sketch each semicircle, and state the domain and range:
(a) $y = \sqrt{4 - x^2}$ (c) $y = -\sqrt{1 - x^2}$ (e) $y = -\sqrt{\frac{9}{4} - x^2}$
(b) $y = -\sqrt{4 - x^2}$ (d) $y = \sqrt{\frac{25}{4} - x^2}$ (f) $y = \sqrt{0{\cdot}64 - x^2}$

CHALLENGE

8. Carefully identify the radius of each circle or semicircle, then sketch its graph. Indicate on each sketch the coordinates of any points that have integer (that is, whole number) coordinates.
(a) $x^2 + y^2 = 5$ (b) $y = \sqrt{2 - x^2}$ (c) $y = -\sqrt{10 - x^2}$ (d) $x^2 + y^2 = 17$

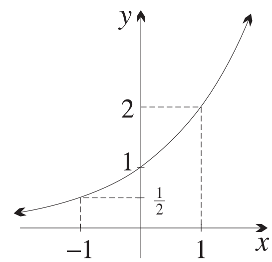
3 E Two Graphs that have Asymptotes

This section reviews exponential graphs and rectangular hyperbolas. They are grouped together because both these types of graphs have asymptotes.

Exponential Functions: Functions of the form $y = a^x$, where the base a is positive, are called *exponential functions*, because the variable x is in the *exponent* or *index*.

Here is a sketch of the function $y = 2^x$.

x	-4	-3	-2	-1	0	1	2	3	4
y	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16



Limits and Asymptotes of Exponential Functions: On the far left, as x becomes a very large negative number, $y = 2^x$ becomes very small. Indeed, we can make y 'as small as we like' by choosing sufficiently large negative values of x. We say that 'y approaches the limit zero as x approaches negative infinity', which we write as

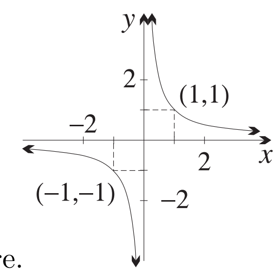
$$y \to 0 \text{ as } x \to -\infty.$$

The x-axis is called an *asymptote* of the curve (from the Greek word *asymptotos*, meaning 'apt to fall together'), because the curve gets 'as close as we like' to the x-axis for sufficiently large negative values of x.

Rectangular Hyperbolas: The *reciprocal function* $y = \dfrac{1}{x}$ has a graph that is called a *rectangular hyperbola*:

x	0	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{2}$	1	2	5	10
y	$*$	10	5	2	1	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{10}$

x	-10	-5	-2	-1	$-\frac{1}{2}$	$-\frac{1}{5}$	$-\frac{1}{10}$
y	$-\frac{1}{10}$	$-\frac{1}{5}$	$-\frac{1}{2}$	-1	-2	-5	-10



The star ($*$) at $x = 0$ indicates that the function is not defined there.

Limits and Asymptotes of Rectangular Hyperbolas: The x-axis is an asymptote to this curve on both sides of the graph. We can make y 'as small we like' by choosing sufficiently large positive or negative values of x. We say that 'y approaches the limit zero as x approaches ∞ and as x approaches $-\infty$', which we write as

$$y \to 0 \text{ as } x \to \infty \qquad \text{and} \qquad y \to 0 \text{ as } x \to -\infty.$$

The y-axis is a second asymptote to the graph. On the right-hand side of the origin, when x is a very small positive number, y becomes very large. We can make y 'as large as we like' by taking sufficiently small but still positive values of x. We say that 'y approaches ∞ as x approaches zero from above', which we write as

$$y \to \infty \text{ as } x \to 0^+.$$

On the left-hand side of the origin, y is negative and can be made 'as large negative as we like' by taking sufficiently small negative values of x. We say that 'y approaches $-\infty$ as x approaches zero from below', which we write as

$$y \to -\infty \text{ as } x \to 0^-.$$

Exercise 3E

1. (a) Copy and complete the following table of values of the hyperbola $y = \dfrac{2}{x}$.

x	-4	-2	-1	$-\frac{1}{2}$	$\frac{1}{2}$	1	2	4
y								

(b) Plot the points, using a scale of 1 cm to 1 unit on each axis, then sketch the hyperbola.
(c) Write down the equations of the two asymptotes of the hyperbola.
(d) Write down the domain and range of the function.

2. Construct a table of values for each hyperbola, then sketch it. Also, write down the domain and range of each hyperbola and the equations of the two asymptotes.
(a) $y = -\dfrac{1}{x}$ (b) $y = \dfrac{4}{x}$ (c) $y = \dfrac{3}{x}$ (d) $y = -\dfrac{2}{x}$

3. Consider the exponential curve $y = 3^x$.
(a) Copy and complete the following table of values of the exponential function $y = 3^x$. Give answers correct to one decimal place where necessary.

x	-2	$-1{\cdot}5$	-1	$-0{\cdot}5$	0	$0{\cdot}5$	1	$1{\cdot}5$	2
y									

(b) Plot these points, using scales of 2 cm to 1 unit on both axes, then sketch the curve.
(c) Write down the equation of the asymptote.
(d) Write down the domain and range of the function.

4. Construct a table of values for each exponential function, then sketch its graph. Also, write down the domain and range in each part and the equation of the asymptote.
(a) $y = 4^x$ (b) $y = 2^{-x}$ (c) $y = 1{\cdot}5^x$ (d) $y = 2{\cdot}5^{-x}$

DEVELOPMENT

5. Make y the subject of each equation, then sketch its graph.
(a) $xy = 6$ (b) $xy = -10$

6. Sketch the graph of each function.
(a) $y = 5^x$ (b) $y = 3^{-x}$

CHALLENGE

7. (a) In question 3 above, the line $y = 0$ is an asymptote to $y = 3^x$. Write a statement using limits to justify this. [HINT: See the theory on the previous page.]
(b) In question 4(b), the line $y = 0$ is an asymptote to the exponential curve $y = 2^{-x}$. Write a statement using limits to justify this. [HINT: See the theory on the previous page.]
(c) In question 1 above, the lines $y = 0$ and $x = 0$ are asymptotes to the hyperbola $y = \dfrac{2}{x}$. Write four statements using limits to justify this. [HINT: See the theory on the previous page.]

8. Does the equation $xy = 0$ represent a hyperbola? Explain your answer.

9. (a) Construct a table of values, and hence sketch the graph of the function $y = (\frac{1}{2})^x$.
(b) Explain why $y = (\frac{1}{2})^x$ is not a rising exponential curve by showing that $(\frac{1}{2})^x$ is the same as 2^{-x}.

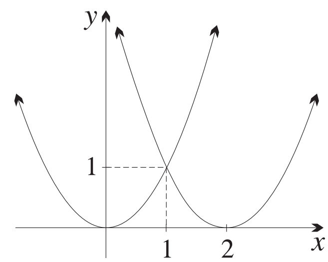
3 F Transformations of Known Graphs

Once a graph is drawn, there are several simple ways to manipulate the function and the graph to produce further graphs.

- The graph can be *shifted* (or *translated*) vertically or horizontally.
- The graph can be *reflected* in the x-axis or the y-axis.

Using these processes on known graphs considerably extends the range of functions and relations whose graphs can be quickly recognised and drawn.

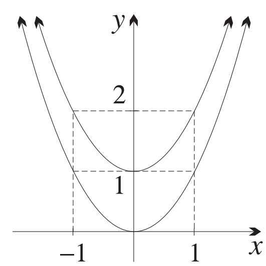
Shifting Left and Right: The graphs of $y = x^2$ and $y = (x-2)^2$ are sketched opposite from their tables of values. They make it clear that the graph of $y = (x-2)^2$ is obtained by shifting the graph of $y = x^2$ to the right by 2 units.



x	-2	-1	0	1	2	3	4
x^2	4	1	0	1	4	9	16
$(x-2)^2$	16	9	4	1	0	1	4

11 SHIFTING (OR TRANSLATING) LEFT AND RIGHT:
- To shift the graph k units to the *right*, replace x by $x - k$.
- Alternatively, if the graph is a function, the new function rule is $y = f(x-k)$.

Shifting Up and Down: The graph of $y = x^2 + 1$ is produced by shifting the graph of $y = x^2$ upwards 1 unit, because the values in the table for $y = x^2 + 1$ are all 1 more than the values in the table for $y = x^2$:



x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
x^2+1	10	5	2	1	2	5	10

Writing the transformed graph as $y - 1 = x^2$ makes it clear that the shifting has been obtained by replacing y by $y - 1$, giving a rule that is completely analogous to that for horizontal shifting.

12 SHIFTING (OR TRANSLATING) UP AND DOWN:
- To shift the graph ℓ units *upwards*, replace y by $y - \ell$.
- Alternatively, if the graph is a function, the new function rule is $y = f(x) + \ell$.

WORKED EXERCISE:

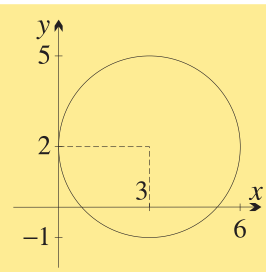
Find the centre and radius of the circle $x^2 + y^2 - 6x - 4y + 4 = 0$, then sketch it.

SOLUTION:

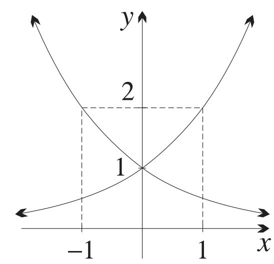
Completing the square in both x and y,

$$(x^2 - 6x + 9) + (y^2 - 4y + 4) + 4 = 9 + 4$$
$$(x-3)^2 + (y-2)^2 = 9.$$

This is just $x^2 + y^2 = 3^2$ shifted right 3 and up 2, so the centre is $(3, 2)$ and the radius is 3.



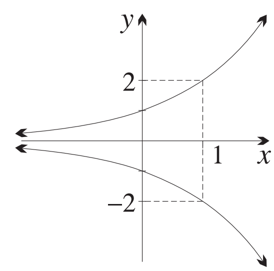
Reflection in the y-axis: When the tables of values for $y = 2^x$ and $y = 2^{-x}$ are both written down, it is clear that the graphs of $y = 2^x$ and $y = 2^{-x}$ must be reflections of each other in the y-axis.



x	-3	-2	-1	0	1	2	3
2^x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
2^{-x}	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

13 REFLECTION IN THE y-AXIS:
- To reflect the graph in the y-axis, replace x by $-x$.
- Alternatively, if the graph is a function, the new function rule is $y = f(-x)$.

Reflection in the x-axis: All the values in the table below for the function $y = -2^x$ are the opposites of the values in the table for $y = 2^x$. This means that the graphs of $y = -2^x$ and $y = 2^x$ are reflections of each other in the x-axis.



x	-3	-2	-1	0	1	2	3
2^x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
-2^x	$-\frac{1}{8}$	$-\frac{1}{4}$	$-\frac{1}{2}$	-1	-2	-4	-8

Writing the transformed graph as $-y = 2^x$ makes it clear that the reflection has been obtained by replacing y by $-y$, giving a rule that is completely analogous to that for reflection in the y-axis.

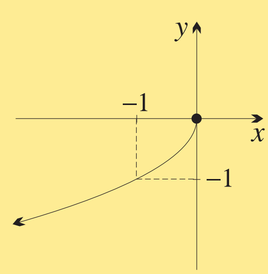
14 REFLECTION IN THE x-AXIS:
- To reflect the graph in the x-axis, replace y by $-y$.
- Alternatively, if the graph is a function, the new function rule is $y = -f(x)$.

WORKED EXERCISE: From the graph of $y = \sqrt{x}$, deduce the graph of $y = -\sqrt{-x}$.

SOLUTION: The equation can be rewritten as

$$-y = \sqrt{-x}$$

so the graph is reflected successively in both axes.



Rotation of 180° About the Origin: The previous worked exercise shows that reflection in both the x-axis and the y-axis is the same as a rotation of 180° about the origin, and that the order in which the reflections are done does not matter.

15 ROTATION OF 180° ABOUT THE ORIGIN:
- Successive reflections in the x-axis and the y-axis are the same as a rotation of 180° about the origin.
- The order in which these successive reflections are done does not matter.
- The rotation of 180° about the origin is also called the *reflection in the origin*, because every point is moved through the origin to a point the same distance from it on the opposite side.

Exercise 3F

1. How far and in what direction has the parabola $y = x^2$ been shifted to produce each of these parabolas?
(a) $y = x^2 + 2$
(b) $y = (x - 3)^2$
(c) $y = x^2 - 5$
(d) $y = (x + 4)^2$

2. How far and in what direction has the hyperbola $y = \frac{1}{x}$ been shifted to produce each of these hyperbolas?
(a) $y = \frac{1}{x - 2}$
(b) $y = \frac{1}{x} - 4$
(c) $y = \frac{1}{x + 3}$
(d) $y = \frac{1}{x} + 5$

3. In which of the coordinate axes has the exponential curve $y = 2^x$ been reflected to produce each of these exponential curves?
(a) $y = 2^{-x}$
(b) $y = -2^x$

4. Sketch each of these parabolas by shifting $y = x^2$ either horizontally or vertically. Mark all intercepts with the axes.
(a) $y = x^2 + 1$
(b) $y = (x - 1)^2$
(c) $y = x^2 - 1$
(d) $y = (x + 1)^2$

5. Sketch each of these hyperbolas by shifting $y = \frac{1}{x}$ either horizontally or vertically. Mark any intercepts with the axes.
(a) $y = \frac{1}{x + 1}$
(b) $y = \frac{1}{x} + 1$
(c) $y = \frac{1}{x - 1}$
(d) $y = \frac{1}{x} - 1$

6. Sketch each of these circles by shifting $x^2 + y^2 = 1$ either horizontally or vertically. Mark all intercepts with the axes.
(a) $(x - 1)^2 + y^2 = 1$
(b) $x^2 + (y + 1)^2 = 1$
(c) $(x + 1)^2 + y^2 = 1$
(d) $x^2 + (y - 1)^2 = 1$

7. Sketch each of these exponential curves by reflecting $y = 3^x$ in either the x-axis or the y-axis. Mark the y-intercept in each case.
(a) $y = -3^x$
(b) $y = 3^{-x}$

DEVELOPMENT

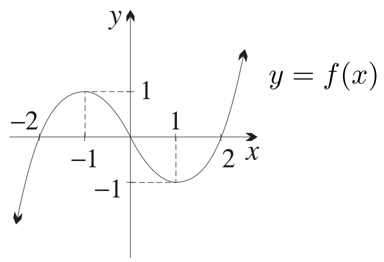
8. Write down the new equation for each function or relation after the given translation has been applied. Sketch the graph of the new function or relation.
(a) $y = x^2$: right 1 unit
(b) $y = 2^x$: down 3 units
(c) $y = x^3$: left 1 unit
(d) $y = \frac{1}{x}$: right 3 units
(e) $x^2 + y^2 = 9$: up 1 unit
(f) $y = x^2 - 4$: left 1 unit
(g) $xy = 1$: down 1 unit
(h) $y = \sqrt{x}$: up 2 units

9. Write down the new equation for each function or relation after reflection in the given line. Sketch the graph of the new function or relation.

(a) $y = x^2$: x-axis
(b) $y = 2^x$: y-axis
(c) $y = x^3$: x-axis
(d) $y = \frac{1}{x}$: x-axis
(e) $x^2 + y^2 = 9$: y-axis
(f) $y = x^2 - 4$: y-axis
(g) $xy = 1$: x-axis
(h) $y = \sqrt{x}$: y-axis

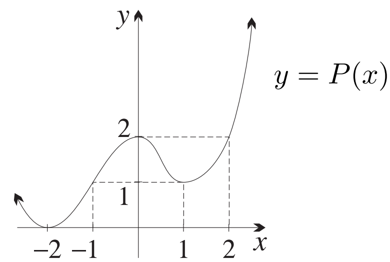
10. In each case an unknown function has been drawn. Draw the functions specified below.

(a)



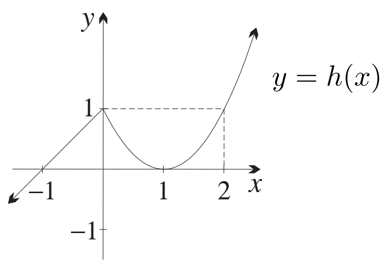
(i) $y = f(x - 2)$
(ii) $y = f(x + 1)$

(b)



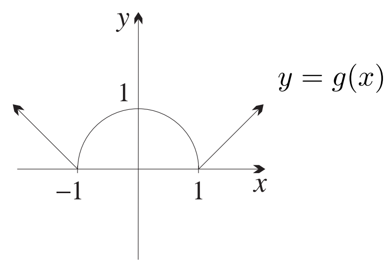
(i) $y = P(x + 2)$
(ii) $y = P(x + 1)$

(c)



(i) $y - 1 = h(x)$
(ii) $y = h(x) - 1$

(d)



(i) $y - 1 = g(x)$
(ii) $y = g(x - 1)$

CHALLENGE

11. Consider the straight line equation $x + 2y - 4 = 0$.
(a) The line is translated 2 units left. Find the equation of the new line.
(b) The original line is translated 1 unit down. Find the equation of this third line.
(c) Comment on your answers, and draw the lines on the same number plane.

12. Sketch $y = \dfrac{1}{x}$, then use shifting to sketch the following graphs. Find any x-intercepts and y-intercepts, and mark them on your graphs.

(a) $y = \dfrac{1}{x - 2}$
(b) $y = 1 + \dfrac{1}{x - 2}$
(c) $y = \dfrac{1}{x - 2} - 2$
(d) $y = \dfrac{1}{x + 1} - 1$
(e) $y = 3 + \dfrac{1}{x + 2}$
(f) $y = \dfrac{1}{x - 3} + 4$

13. Use shifting, and completion of the square where necessary, to determine the centre and radius of each circle.

(a) $(x+1)^2 + y^2 = 4$
(b) $(x-1)^2 + (y-2)^2 = 1$
(c) $x^2 - 2x + y^2 - 4y - 4 = 0$
(d) $x^2 + 6x + y^2 - 8y = 0$
(e) $x^2 - 10x + y^2 + 8y + 32 = 0$
(f) $x^2 + 14x + 14 + y^2 - 2y = 0$

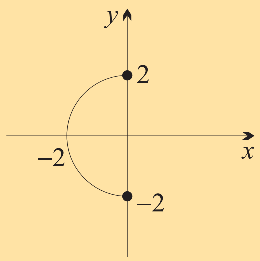
14. In each part, explain how the graph of each subsequent equation is a transformation of the first graph (there may be more than one answer), then sketch each curve.

(a) From $y = 2x$: (i) $y = 2x + 4$ (ii) $y = 2x - 4$ (iii) $y = -2x + 4$
(b) From $y = x^2$: (i) $y = x^2 + 9$ (ii) $y = x^2 - 9$ (iii) $y = (x-3)^2$
(c) From $y = x^2$: (i) $y = (x+1)^2$ (ii) $y = -(x+1)^2$ (iii) $y = -(x+1)^2 + 2$
(d) From $y = \sqrt{x}$: (i) $y = \sqrt{x+4}$ (ii) $y = -\sqrt{x+4}$ (iii) $y = -\sqrt{x}$
(e) From $y = \dfrac{1}{x}$: (i) $y = \dfrac{1}{x} + 1$ (ii) $y = \dfrac{1}{x+2} + 1$ (iii) $y = -\dfrac{1}{x}$

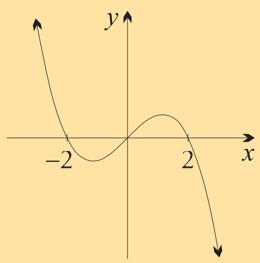
3G Chapter Review Exercise

1. Determine which of the following relations are functions.

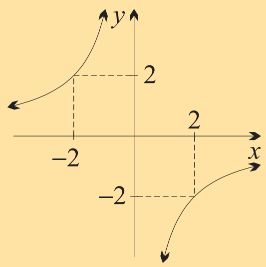
(a)



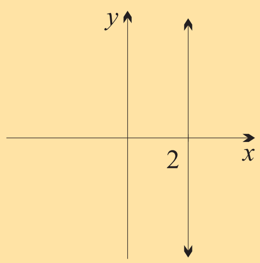
(b)



(c)



(d)



2. Write down the domain and range of each relation in the previous question.

3. Find $f(3)$ and $f(-2)$ for each function.

(a) $f(x) = x^2 + 4x$
(b) $f(x) = x^3 - 3x^2 + 5$

4. Find the natural domain of each function.

(a) $f(x) = \dfrac{1}{x-2}$
(b) $f(x) = \sqrt{x-1}$
(c) $f(x) = \sqrt{3x+2}$
(d) $f(x) = \dfrac{1}{\sqrt{2-x}}$

5. Find $F(a) - 1$ and $F(a-1)$ for each function.

(a) $F(x) = 2x + 3$
(b) $F(x) = x^2 - 3x - 7$

6. Suppose that $f(x) = \begin{cases} x+1, & \text{for } x < 0, \\ 1-x, & \text{for } x \geq 0. \end{cases}$
Complete the table of values for $f(x)$, then sketch its graph.

x	-3	-2	-1	0	1	2	3
y							

7. Find the x-intercept and the y-intercept of each line, then sketch it.
(a) $y = 2x + 2$ (b) $x - 3y + 6 = 0$

8. Sketch each of these lines through the origin.
(a) $y = 2x$ (b) $x + 2y = 0$

9. Sketch each vertical or horizontal line.
(a) $y = -1$ (b) $x - 3 = 0$

10. Sketch each parabola, showing the x-intercepts and the y-intercept and the coordinates of the vertex. Also state the domain and range.
(a) $y = 16 - x^2$ (c) $y = (x-2)(x-6)$ (e) $y = x^2 + 2x - 24$
(b) $y = x(x+2)$ (d) $y = -(x+5)(x-1)$ (f) $y = -x^2 + 6x + 40$

11. Sketch each circle.
(a) $x^2 + y^2 = 9$ (b) $x^2 + y^2 = 100$

12. Sketch each semicircle, and write down the domain and range in each case.
(a) $y = \sqrt{16 - x^2}$ (b) $y = -\sqrt{25 - x^2}$

13. Construct a table of values for each hyperbola, then sketch it. Write down the domain and range in each case.
(a) $y = \dfrac{8}{x}$ (b) $y = -\dfrac{4}{x}$

14. Construct a table of values for each exponential function, then sketch it. Write down the domain and range in each case.
(a) $y = 2^x$ (b) $y = 3^{-x}$

15. Construct a table of values for each function, then sketch it.
(a) $y = x^3 - 1$ (b) $y = x^4 - 1$ (c) $y = \sqrt{x} + 1$

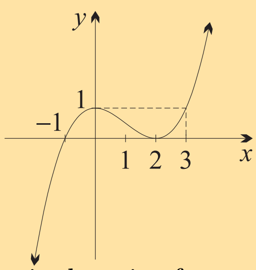
16. Sketch each parabola by shifting $y = x^2$. Mark the intercepts with the axes.
(a) $y = x^2 - 4$ (b) $y = (x-2)^2$ (c) $y = x^2 + 4$ (d) $y = (x+2)^2$

17. Sketch each hyperbola by shifting $y = \dfrac{1}{x}$. Mark any intercepts with the axes.
(a) $y = \dfrac{1}{x} - 2$ (b) $y = \dfrac{1}{x-2}$

18. Sketch each circle by shifting $x^2 + y^2 = 4$. Mark the intercepts with the axes.
(a) $(x+2)^2 + y^2 = 4$ (b) $(x-2)^2 + (y+2)^2 = 4$

19. Sketch each exponential function by reflecting $y = 2^x$ in either the x-axis or the y-axis. Mark the y-intercept.
(a) $y = 2^{-x}$ (b) $y = -2^x$

20. The graph drawn to the right shows the curve $y = f(x)$. Use this graph to sketch the graph of:



(a) $y = f(x+1)$
(b) $y = f(x) + 1$
(c) $y = f(x-1)$
(d) $y = f(x) - 1$

21. [A revision medley of curve sketches] Sketch each set of graphs on a single pair of axes, showing all significant points. Use transformations, tables of values, or any other convenient method.

(a) $y = 2x$, $y = 2x + 3$, $y = 2x - 1$
(b) $y = -\frac{1}{2}x$, $y = -\frac{1}{2}x + 1$, $y = -\frac{1}{2}x - 2$
(c) $y = x^2$, $y = (x+2)^2$, $y = (x-1)^2$
(d) $x + y = 0$, $x + y = 2$, $x + y = -3$
(e) $y = x^2$, $y = 2x^2$, $y = \frac{1}{2}x^2$
(f) $x - y = 0$, $x - y = 1$, $x - y = -2$
(g) $x^2 + y^2 = 4$, $x^2 = 1 - y^2$, $y^2 = 25 - x^2$
(h) $y = 3x$, $x = 3y$, $y = 3x + 1$, $x = 3y + 1$
(i) $y = 2^x$, $y = 3^x$, $y = 4^x$
(j) $y = -x$, $y = 4 - x$, $y = x - 4$, $x = -4 - y$
(k) $y = x^2 - x$, $y = x^2 - 4x$, $y = x^2 + 3x$
(l) $(x-1)^2 + y^2 = 1$, $(x+1)^2 + y^2 = 1$, $x^2 + (y-1)^2 = 1$
(m) $y = x^2 - 1$, $y = 1 - x^2$, $y = 4 - x^2$, $y = -1 - x^2$
(n) $y = (x+2)^2$, $y = (x+2)^2 - 4$, $y = (x+2)^2 + 1$
(o) $y = x^2 - 1$, $y = x^2 - 4x + 3$, $y = x^2 - 8x + 15$
(p) $y = \sqrt{9 - x^2}$, $x = -\sqrt{4 - y^2}$, $y = -\sqrt{1 - x^2}$
(q) $y = \dfrac{1}{x}$, $y = 1 + \dfrac{1}{x}$, $y = -\dfrac{1}{x}$
(r) $y = \sqrt{x}$, $y = \sqrt{x} + 1$, $y = \sqrt{x + 1}$
(s) $y = 2^x$, $y = 2^x - 1$, $y = 2^{x-1}$
(t) $y = \dfrac{1}{x}$, $y = \dfrac{1}{x - 2}$, $y = \dfrac{1}{x + 1}$
(u) $y = x^3$, $y = x^3 + 1$, $y = (x+1)^3$
(v) $y = x^4$, $y = (x-1)^4$, $y = x^4 - 1$
(w) $y = \sqrt{x}$, $y = -\sqrt{x}$, $y = 4 - \sqrt{x}$
(x) $y = 2^{-x}$, $y = 2^{-x} - 2$, $y = 3 + 2^{-x}$
(y) $y = x^2$, $x = y^2$, $x = y^2 - 1$

CHAPTER **FOUR**

Graphs and Inequations

The interrelationship between algebra and graphs is the theme of this chapter. Graphs are used to solve various equations and inequations, including those involving absolute value. Conversely, algebraic techniques are used to graph unfamiliar curves and regions in the coordinate plane.

Computer sketching of curves may be useful in demonstrating how the features of a graph are related to the algebraic properties of its equation, and to gain some familiarity with the variety of graphs that appear in this chapter.

4 A Inequations and Inequalities

Statements involving the four symbols $<$ and $\le$ and $>$ and $\ge$ are called either *inequalities* or *inequations*.

An *inequality* should be true for all values of the variables. For example:

- $x - 5 < x$ is true for all values of x.
- $x^2 + y^2 \ge 0$ holds for all values of x and y, because squares can't be negative.

An *inequation*, on the other hand, requires a solution, which may be expressed algebraically or as a graph. For example:

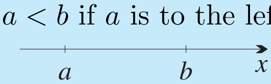
- $x - 5 > 0$ is an inequation whose solution is $x > 5$, meaning that it is true for all numbers greater than 5, but not for any other numbers.
- $x^2 + y^2 < 25$ is an inequation whose solution is the inside of a circle with centre the origin and radius 5.

The Meaning of 'less than': Suppose that a and b are real numbers.

1

THE GEOMETRIC MEANING OF $a < b$:

We say that $a < b$ if a is to the left of b on the number line:



THE ARITHMETIC MEANING OF $a < b$:

We say that $a < b$ if $b - a$ is positive.

For example, $-5 < 1$ because $1 - (-5) = 6$, which is positive.

The first interpretation requires geometry, because it relies on the idea of a 'line' and of one point being 'on the left-hand side of' another.

The second interpretation requires only arithmetic. It assumes that the term 'positive number' has already been understood.

Solving Linear Inequations: The rules for adding to and subtracting from both sides of an inequation, and for multiplying or dividing both sides, are exactly the same as for equations, with one qualification — the inequality symbol reverses when multiplying or dividing by a negative number.

2 REVERSING THE INEQUALITY SYMBOL: When multiplying or dividing both sides of an inequation by a *negative* number, the inequality symbol is reversed.
For example, we know that $5 < 8$.
But when both sides are multiplied by -2, $-10 > -16$.

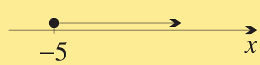
An inequation may relate three expressions. For example, the statement

'$2 < x < 5$' (Read this as '2 is less than x, and x is less than 5.')

means that x could be any number between 2 and 5.

WORKED EXERCISE:
Solve each inequation and graph its solution on a number line.
(a) $3x \le 8x + 25$ (b) $20 > 2 - 3x \ge 8$

SOLUTION:
(a) $3x \le 8x + 25$
$\boxed{-\,8x}$ $-5x \le 25$
$\boxed{\div(-5)}$ $x \ge -5$ (Reverse the inequality because of the division by -5.)



(The *closed circle* at -5 indicates that -5 is *included*.)

(b) $20 > 2 - 3x \ge 8$
$\boxed{-\,2}$ $18 > -3x \ge 6$ (Subtract 2 from all three terms.)
$\boxed{\div(-3)}$ $-6 < x \le -2$ (Reverse both inequalities because of the division by -3.)

(The *open circle* at -6 indicates that -6 is *excluded*.)

Using a Counter-example to Prove that an Inequality is False: An inequality has to be true for all values of the variables. Thus only one counter-example is needed to prove that an inequality is false.

WORKED EXERCISE:
Use a counter-example to prove that each statement is false.
(a) '$5x \ge x$, for all values of x.' (b) 'If $a < b$, then $a^2 < b^2$.'

SOLUTION:
(a) Take as a counter-example $x = -3$.
Then LHS $= -15$ and RHS $= -3$.
But $-15 < -3$, contradicting the statement.

(b) Take as a counter-example $a = -3$ and $b = -1$, and notice that here $a < b$.
Then LHS $= 9$ and RHS $= 1$.
But $9 > 3$, contradicting the statement.

Exercise 4A

1. Graph each set of real numbers on the number line.

(a) $x > 1$ (b) $x \le 2$ (c) $x \ge -3$ (d) $0 < x < 3$ (e) $-2 < x \le 1$ (f) $-5 \le x < 10$

2. Solve each inequation, and graph your solution on the number line.

(a) $x - 2 < 3$ (b) $3x \ge -6$ (c) $5x + 5 > 15$ (d) $4x - 3 \le -7$ (e) $2x + 3 \ge x + 7$ (f) $6x - 5 < 3x - 17$

3. Solve each inequation.

(a) $-2x < 6$ (b) $-5x \ge -50$ (c) $-x \le 1$ (d) $3 - 2x > 7$ (e) $11 - 3x < 2$ (f) $-4 - x \ge 1$ (g) $2 - x > 2x - 4$ (h) $3 - 3x \le 19 + x$ (i) $12 - 7x > -2x - 18$

DEVELOPMENT

4. Solve each double inequation, and graph your solution on the number line.

(a) $3 < x + 2 < 6$ (b) $-5 < x - 3 \le 4$ (c) $-8 \le 4x < 12$ (d) $-1 \le 2x \le 3$ (e) $-10 < 3x - 1 < 2$ (f) $-7 < 5x + 3 \le 3$

5. Solve each double inequation.

(a) $-4 < -2x < 8$ (b) $-2 \le -x \le 1$ (c) $-7 \le 5 - 3x < 4$ (d) $-4 < 1 - \frac{1}{3}x \le 3$

6. Solve each inequation.

(a) $\dfrac{x}{5} - \dfrac{x}{2} < 3$ (b) $\frac{1}{4}x + 1 \ge \frac{1}{2}x$ (c) $\dfrac{x+1}{4} - \dfrac{2x-1}{3} \le 1$ (d) $\frac{1}{6}(2 - x) - \frac{1}{3}(2 + x) > 2$

7. Write down and solve a suitable inequation to find the values of x for which the line $y = 5x - 4$ is below the line $y = 7 - \frac{1}{2}x$.

8. (a) Sketch the lines $y = 1 - x$, $y = 2$ and $y = -1$ on a number plane and find the points of intersection.

(b) Solve the inequation $-1 < 1 - x \le 2$ and relate the answer to the graph.

CHALLENGE

9. If $-1 \le t < 3$, what is the range of values of:

(a) $4t$ (b) $-t$ (c) $t + 7$ (d) $2t - 1$ (e) $\frac{1}{2}(t + 1)$ (f) $\frac{1}{2}(3t - 1)$

10. Determine whether or not each statement is true for all values of x. If it is not true, give a counter-example.

(a) $x^2 > 0$ (b) $x^2 \ge x$ (c) $2^x > 0$ (d) $x \ge \dfrac{1}{x}$ (e) $2x \ge x$ (f) $x + 2 > x$ (g) $x \ge -x$ (h) $2x - 3 > 2x - 7$

11. Determine whether or not each statement is true for all vaues of a and b. If it is not true, give a counter-example.

(a) If $0 < a < b$, then $\dfrac{1}{a} > \dfrac{1}{b}$.

(b) If $a < b$, then $a^2 < b^2$.

(c) If $a^2 + b^2 = 0$, then $a = b = 0$.

(d) If $a < b$ and $a, b \ne 0$, then $\dfrac{1}{a} > \dfrac{1}{b}$.

(e) If $a < b$, then $-a > -b$.

(f) If $0 < a < b$, then $\sqrt{a^2 + b^2} = a + b$.

4 B Solving Quadratic Inequations

A quadratic inequation should be solved by sketching the graph of the associated parabola and examining where the graph is above or below the x-axis.

3 SOLVING A QUADRATIC INEQUATION:
- Move everything to the LHS.
- Sketch the graph of the LHS, showing the x-intercepts.
- Read the solution off the graph by asking the question 'Where is the graph above or below the x-axis?'

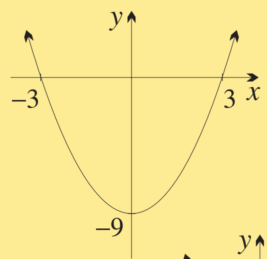
WORKED EXERCISE:

Solve each quadratic inequation by sketching a suitable graph and then reading the solution off the graph.

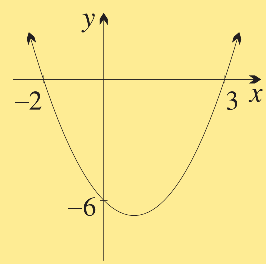
(a) $x^2 > 9$ (b) $x^2 \le x + 6$

SOLUTION:

(a) Moving everything onto the left, $x^2 - 9 > 0$.
Factoring the LHS, $(x - 3)(x + 3) > 0$.
The graph of $y = \text{LHS}$ is sketched opposite.
Where is the graph above the x-axis?
From the graph opposite, $x > 3$ or $x < -3$.

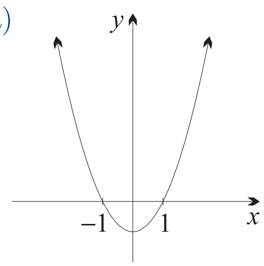


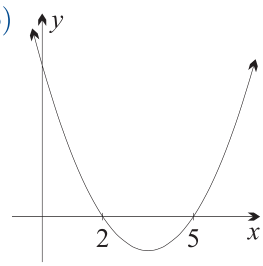
(b) Moving everything onto the left, $x^2 - x - 6 \le 0$.
Factoring the LHS, $(x - 3)(x + 2) \le 0$.
The graph of $y = \text{LHS}$ is sketched opposite.
Where is the graph below the x-axis?
From the graph opposite, $-2 \le x \le 3$.

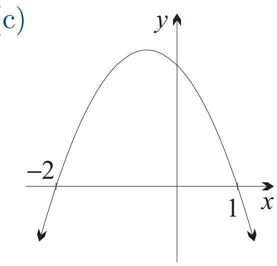


Exercise 4B

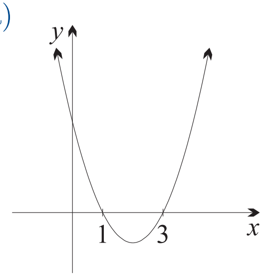
1. Write down the values of x for which each parabola is:
(i) below the x-axis, (ii) above the x-axis.

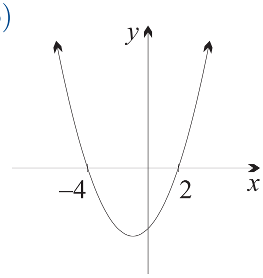
(a)


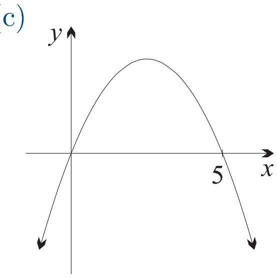
(b)


(c)


2. For each parabola, state the values of x for which: (i) $y = 0$, (ii) $y > 0$, (iii) $y < 0$.

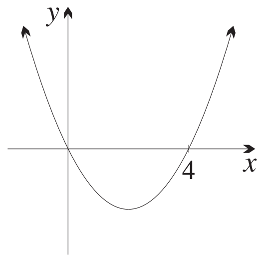
(a)


(b)


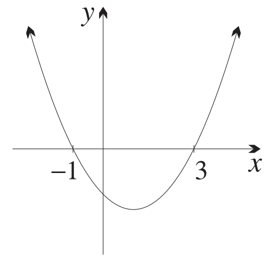
(c)


3. Use the given graph of the LHS to help solve each inequation.

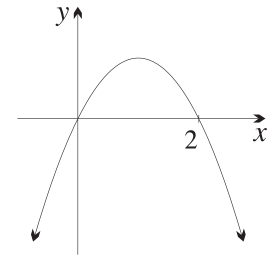
(a) $x(x-4) < 0$



(b) $(x-3)(x+1) \geq 0$



(c) $x(2-x) \leq 0$



4. Sketch the associated parabola and hence solve:

(a) $(x+2)(x-4) < 0$
(b) $(x-3)(x+1) > 0$
(c) $(2-x)(x-5) \geq 0$
(d) $(x+1)(x+3) \geq 0$
(e) $(2x-1)(x-5) > 0$
(f) $(3x+5)(x+4) \leq 0$

DEVELOPMENT

5. Factor the LHS, then sketch an appropriate parabola in order to solve:

(a) $x^2 + 2x - 3 < 0$
(b) $x^2 - 5x + 4 \geq 0$
(c) $x^2 + 6x + 8 > 0$
(d) $x^2 - x - 6 \leq 0$
(e) $2x^2 - x - 3 \leq 0$
(f) $4 + 3x - x^2 > 0$

6. Solve:

(a) $x^2 \leq 1$
(b) $x^2 > 3x$
(c) $x^2 \geq 144$
(d) $x^2 > 0$
(e) $x^2 + 9 \leq 6x$
(f) $4x - 3 \geq x^2$

CHALLENGE

7. What range of values may $x^2 + 3$ take if: (a) $2 < x < 4$ (b) $-1 < x \leq 3$

8. (a) Draw a sketch of the curve $y = x^2 - 4$ and hence solve the inequation $x^2 - 4 \geq 0$. Hence write down the natural domain of $\sqrt{x^2 - 4}$.

(b) Solve the inequation $x^2 - 4 > 0$ and hence write down the natural domain of $\dfrac{5}{\sqrt{x^2 - 4}}$.

9. Use the methods of the previous question to write down the domain of:

(a) $\sqrt{4 - x^2}$
(b) $\dfrac{1}{\sqrt{4 - x^2}}$
(c) $\sqrt{25 - x^2}$
(d) $\dfrac{1}{\sqrt{25 - x^2}}$
(e) $3\sqrt{x^2 - 4}$
(f) $\dfrac{5}{\sqrt{x^2 - 4}}$

4 C Intercepts and Sign

The graph of a function lies above the x-axis when its values are positive. The graph lies below the x-axis when its values are negative. This principle is used in the present section to solve inequations and to sketch functions.

Most functions in this section are *polynomials*, meaning that they can each be written as a sum of multiples of powers of x. For example,

$$y = 5x^3 - 2x^2 + 7x + 1$$

is a *cubic polynomial* because the term with the highest index is $5x^3$. The word 'polynomial' means 'many terms'.

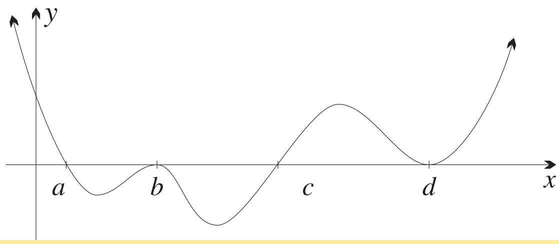
Solving an Inequation from a Graph: Once the graph of a function has been drawn, it is easy to read off the graph where the function is positive and where it is negative.

4 **READING THE SOLUTION TO AN INEQUATION OFF A GRAPH:**
- A function is positive when its graph is above the x-axis.
- A function is negative when its graph is below the x-axis.

WORKED EXERCISE:

A function $y = f(x)$ is graphed below. From the graph, solve:

(a) $y < 0$ (b) $y \ge 0$



SOLUTION:

(a) To solve $y < 0$, ask where the graph is below the x-axis.
The solution is $a < x < b$ or $b < x < c$.

(b) To solve $y \ge 0$, ask where the graph is above or on the x-axis.
The solution is $x \le a$ or $x = b$ or $x \ge c$.

Finding the x-intercepts and the y-intercept: Now consider the reverse problem of sketching the graph of an unknown function. The points where the graph meets the x-axis and the y-axis are found by putting the other variable equal to zero.

5 **FINDING THE x-INTERCEPTS AND THE y-INTERCEPT:**
- To find the y-intercept, substitute $x = 0$.
- To find the x-intercepts, substitute $y = 0$.

The x-intercepts are also called the *zeroes* of the function. Finding them will usually involve factoring the function.

WORKED EXERCISE:

(a) Find the y-intercept of the function $y = x^3 - x^2 - x + 1$.

(b) Factor $y = x^3 - x^2 - x + 1$ and hence find the x-intercepts of its graph.

SOLUTION:

(a) When $x = 0$, $y = 1$. (This is the y-intercept.)

(b) Factoring by grouping,
$$\begin{aligned} y &= x^2(x-1) - (x-1) \\ &= (x^2-1)(x-1) \\ &= (x-1)(x+1)(x-1) \\ &= (x+1)(x-1)^2. \end{aligned}$$
Hence $y = 0$ when $x = -1$ or 1. (These are the x-intercepts.)
The complete graph is sketched in the next worked exercise.

The Sign of the Function: Once the zeroes have been found, the sign of the function as x varies can be found by using a set of test values. This method requires an important theorem called the *intermediate value theorem*.

6 CONTINUOUS FUNCTIONS AND THE INTERMEDIATE VALUE THEOREM:
- A function is called *continuous* at a point if its graph can be drawn through the point without lifting the pen off the paper.
- A continuous function can only change sign at an x-intercept (or zero).

For example, the function sketched on the previous page is continuous, because it can be drawn without taking the pen off the paper.

- It changes sign at the zeroes $x = a$ and $x = c$.
- It does not change sign, however, at the zeroes $x = b$ and $x = d$.

7 EXAMINING THE SIGN OF A FUNCTION:
Draw up a table of test values around any zeroes.

NOTE: If a function is not continuous for some value of x (called a *discontinuity*), then test values will also need to be taken on both sides of this value.

WORKED EXERCISE:

(a) State the zeroes of the function $y = (x + 1)(x - 1)^2$.
(b) Take test points around the zeroes to examine the sign of the function.
(c) Hence sketch the graph of the function.

SOLUTION:

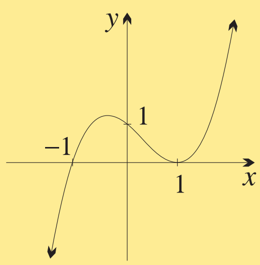
(a) The function has zeroes at $x = 1$ and $x = -1$, and there are no discontinuities.

(b)

x	-2	-1	0	1	2
y	-9	0	1	0	3
sign	$-$	0	$+$	0	$+$

Hence y is positive for $-1 < x < 1$ or $x > 1$, and y is negative for $x < -1$.

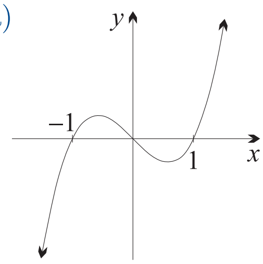
(c) The graph is sketched opposite.



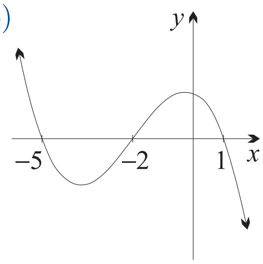
Exercise 4C

1. Write down the values of x for which each curve is:
(i) below the x-axis, (ii) above the x-axis.

(a)

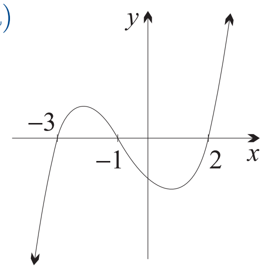


(b)

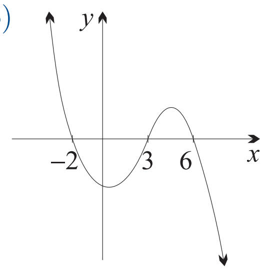


2. For each curve, write down the values of x for which: (i) $y = 0$, (ii) $y > 0$, (iii) $y < 0$.

(a)

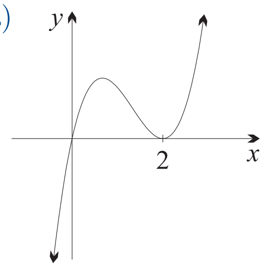


(b)

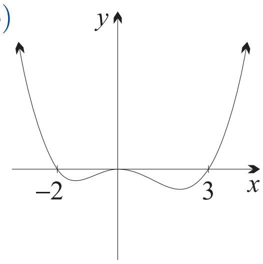


3. For each curve, state the values of x for which: (i) $f(x) = 0$, (ii) $f(x) > 0$, (iii) $f(x) < 0$.

(a)

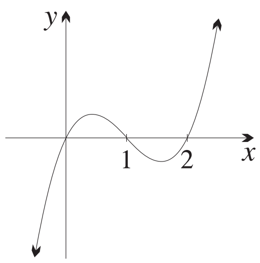


(b)

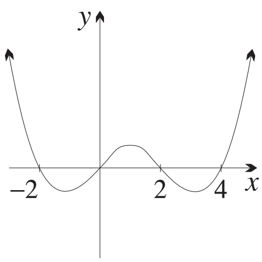


4. Use the given graph of the LHS to help solve each inequation.

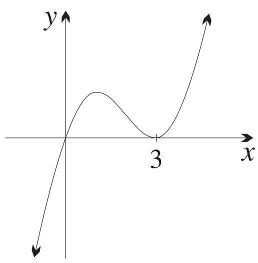
(a) $x(x-1)(x-2) \le 0$



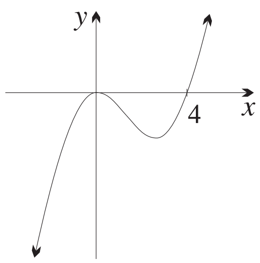
(b) $x(x+2)(x-2)(x-4) < 0$



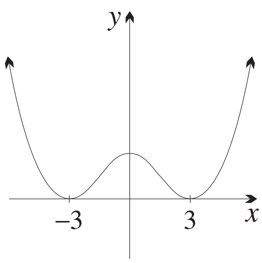
(c) $x(x-3)^2 > 0$



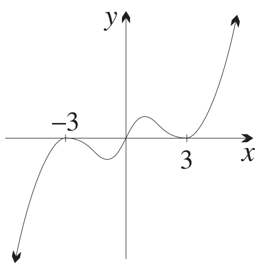
(d) $x^2(x-4) \ge 0$



(e) $(x-3)^2(x+3)^2 \le 0$



(f) $x(x-3)^2(x+3)^2 \ge 0$



DEVELOPMENT

5. Explain why the zeroes (that is, the x-intercepts) of $y = (x+1)^2(x-1)$ are $x = 1$ and $x = -1$. Then copy and complete the table of values and sketch the graph.

x	-2	-1	0	1	2
y					
sign					

6. Apply the method used in the previous question to sketch the following quadratics, cubics and quartics. Mark all x-intercepts and y-intercepts.

(a) $y = (x+1)(x+3)$
(b) $y = x(x-2)(x-4)$
(c) $y = (x-1)(x+2)^2$
(d) $y = x(x-2)(x+2)$
(e) $y = (x-2)x(x+2)(x+4)$
(f) $y = (x-1)^2(x-3)^2$

7. First factor each polynomial completely, then use the method of the previous two questions to sketch its graph. (Take out any common factors first.)

(a) $f(x) = x^3 - 4x$
(b) $f(x) = x^3 - 5x^2$
(c) $f(x) = x^3 - 4x^2 + 4x$

8. From the graphs in the previous question, or from the tables of values used to construct them, solve the following inequations. Begin by getting all terms onto the LHS.
(a) $x^3 > 4x$ (b) $x^3 < 5x^2$ (c) $x^3 + 4x \le 4x^2$

CHALLENGE

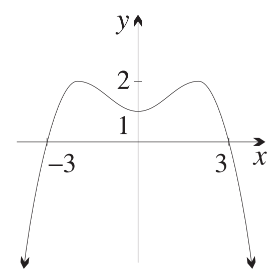
9. Collect all terms of each inequation on the LHS if necessary and then factor the LHS. Then solve the inequation by sketching the graph of the LHS.
(a) $(x-1)(x-3)(x-5) < 0$
(b) $(x+3)(x-1)(x-4) \ge 0$
(c) $(x-1)^2(x-3)^2 > 0$
(d) $(x+2)x(x-2)(x-4) > 0$
(e) $x^3 > 9x$
(f) $x^3 \ge 5x^2$

4 D Odd and Even Symmetry

It has been said that all mathematics is the study of symmetry. Two simple types of symmetry occur so often in the functions of this course that every function should be tested routinely for them.

Even Functions and Symmetry in the y-axis:

A graph with *line symmetry in the y-axis* is called *even*. This means that the graph is unchanged by reflection in the y-axis, like the graph to the right.

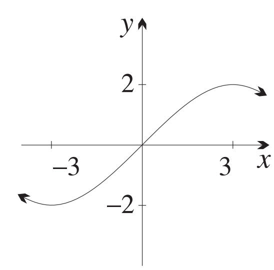


As explained in Section 3F, when the graph of $y = f(x)$ is reflected in the y-axis, the new curve has equation $y = f(-x)$. Hence for a function to be *even*, the graphs of $y = f(x)$ and $y = f(-x)$ must coincide.

8 EVEN FUNCTIONS:
- A function $f(x)$ is called *even* if $f(-x) = f(x)$, for all x in its domain.
- A function is even if and only if its graph has *line symmetry in the y-axis.*

Odd Functions and Symmetry in the Origin:

A graph with *point symmetry in the origin* is called *odd*. This means that the graph is unchanged by a rotation of 180° about the origin, or equivalently, by successive reflections in the x-axis and the y-axis.



When the graph of $y = f(x)$ is reflected in the x-axis and then in the y-axis, the new curve has equation $y = -f(-x)$. Hence for function to be *odd*, the graphs of $f(-x)$ and $-f(x)$ must coincide.

9 ODD FUNCTIONS:
- A function $f(x)$ is called *odd* if $f(-x) = -f(x)$, for all x in its domain.
- A function is odd if and only if its graph has *point symmetry in the origin*.

Testing for Evenness and Oddness: One test will pick up both types of symmetry.

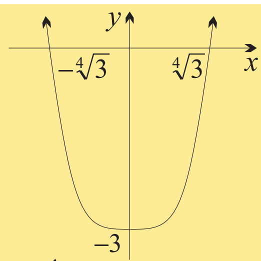
10 TESTING FOR EVENNESS AND ODDNESS (OR NEITHER):
- Simplify $f(-x)$ and note whether it is $f(x)$, $-f(x)$ or neither.

NOTE: Most functions are neither even nor odd.

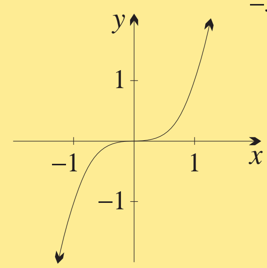
WORKED EXERCISE:
Test each function for evenness or oddness, then sketch it.
(a) $f(x) = x^4 - 3$ (b) $f(x) = x^3$ (c) $f(x) = x^2 - 2x$

SOLUTION:

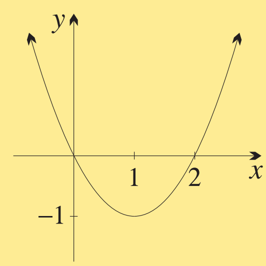
(a) Here $f(x) = x^4 - 3$.
Substituting $-x$ for x, $f(-x) = (-x)^4 - 3$
$= x^4 - 3$
$= f(x)$.
Hence $f(x)$ is an even function.



(b) Here $f(x) = x^3$.
Substituting $-x$ for x, $f(-x) = (-x)^3$
$= -x^3$
$= -f(x)$.
Hence $f(x)$ is an odd function.

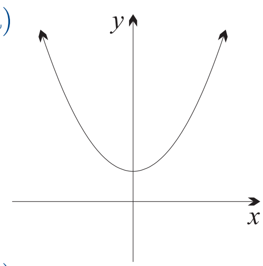


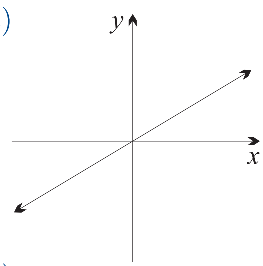
(c) Here $f(x) = x^2 - 2x$.
Substituting $-x$ for x, $f(-x) = (-x)^2 - 2(-x)$
$= x^2 + 2x$.
Since $f(-x)$ is equal neither to $f(x)$ nor to $-f(x)$, the function is neither even nor odd.
[The curve does, however, have line symmetry in $x = 1$.]

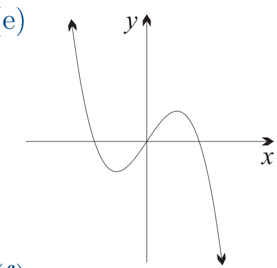


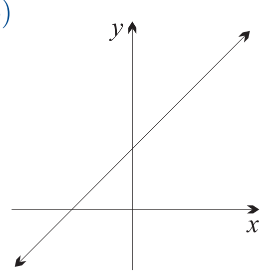
Exercise 4D

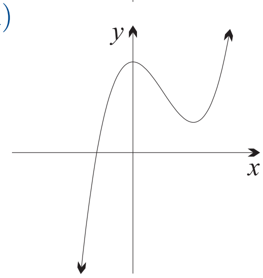
1. Classify each function $y = f(x)$ as even, odd or neither.

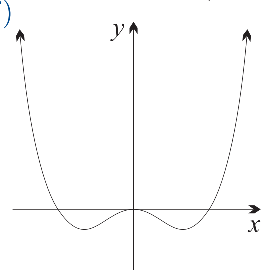
(a)


(c)


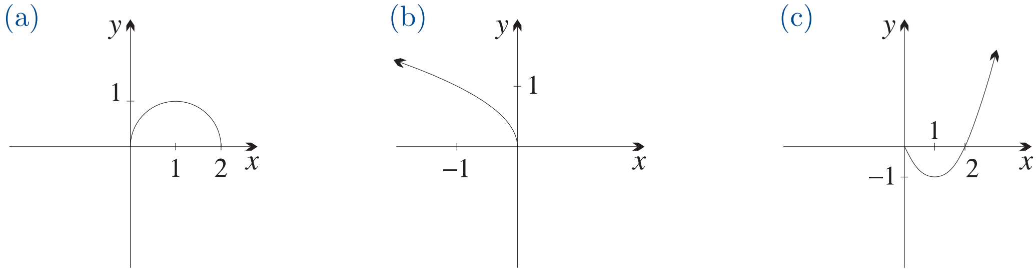
(e)


(b)


(d)


(f)


2. In each diagram below, complete the graph so that: (i) $f(x)$ is even, (ii) $f(x)$ is odd.



3. Simplify $f(-x)$ for each function, and hence determine whether it is even, odd or neither.

(a) $f(x) = x^2 - 9$
(b) $f(x) = x^2 - 6x + 5$
(c) $f(x) = x^3 - 25x$
(d) $f(x) = x^4 - 4x^2$
(e) $f(x) = x^3 + 5x^2$
(f) $f(x) = x^5 - 16x$
(g) $f(x) = x^5 - 8x^3 + 16x$
(h) $f(x) = x^4 + 3x^3 - 9x^2 - 27x$

4. On the basis of the previous question, copy and complete these sentences:

(a) 'A polynomial function is odd if . . . '.
(b) 'A polynomial function is even if . . . '.

DEVELOPMENT

5. Factor each polynomial in parts (a)–(f) of question 3 above and write down its zeroes (that is, its x-intercepts). Then use a table of test points to sketch its graph. Confirm that the graph exhibits the symmetry established above.

6. Determine whether each function is even, odd or neither.

(a) $f(x) = 3^x$
(b) $f(x) = 3^{-x}$
(c) $f(x) = \sqrt{3 - x^2}$
(d) $f(x) = \dfrac{1}{x^2 + 1}$
(e) $f(x) = \dfrac{4x}{x^2 + 4}$
(f) $f(x) = 2^x + 2^{-x}$
(g) $f(x) = 2^x - 2^{-x}$
(h) $f(x) = 2^x + x^2$

7. (a) Pick up a book, reflect it in its vertical axis and then in its horizontal axis, and observe that it is now rotated 180° from its original position about the centre of the book.

(b) Repeat this procedure, but this time reflect the book first in its horizontal axis and then in its vertical axis. Did the changed order change the result?

(c) Use parts (a) and (b) to explain why the graph of $y = -f(-x)$ is the graph of $y = f(x)$ rotated by 180°about the origin.

CHALLENGE

8. (a) Prove that if $f(x)$ is an odd function defined at $x = 0$, then $y = f(x)$ must pass through the origin. [HINT: Apply the condition for a function to be odd to $f(0)$.]

(b) If $f(x)$ is an even function defined at $x = 0$, does the graph of $y = f(x)$ have to pass through the origin? Either prove the statement or give a counter-example.

9. (a) Suppose that $h(x) = f(x) \times g(x)$.

(i) Show that if f and g are both even or both odd, then $h(x)$ is even.
(ii) Show that if one of $f(x)$ and $g(x)$ is even and the other odd, then $h(x)$ is odd.

(b) Suppose that $h(x) = f(x) + g(x)$.

(i) Show that if $f(x)$ and $g(x)$ are both even, then $h(x)$ is even.
(ii) Show that if $f(x)$ and $g(x)$ are both odd, then $h(x)$ is odd.

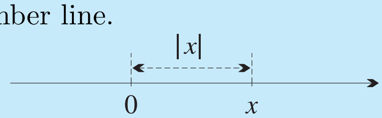
4 E The Absolute Value Function

Often it is the size or magnitude of a number that is significant, rather than whether it is positive or negative. *Absolute value* is the mathematical name for this concept.

Absolute Value as Distance: Distance is the clearest way to define absolute value.

11 **ABSOLUTE VALUE AS DISTANCE:**

- The absolute value $|x|$ of a number x is the distance from x to the origin on the number line.



For example, $|-5| = 5$ and $|0| = 0$ and $|5| = 5$.

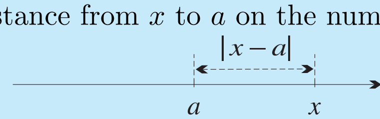
- Since distance is always positive or zero, so also is the absolute value of x.

Thus absolute value is a measure of the *size* or *magnitude* of a number. In the examples above, the numbers -5 and $+5$ both have the same magnitude, 5, and differ only in their signs.

Distance Between Numbers: Replacing x by $x - a$ in the previous definition gives a measure of the distance from x to a on the number line.

12 **DISTANCE BETWEEN NUMBERS:**

The distance from x to a on the number line is $|x - a|$.



For example, the distance between 5 and -2 is $|5 - (-2)| = 7$

Solving Equations and Inequations Using Distance: In this course, most equations and inequations involving absolute values are simple enough to be solved using distances on the number line.

13 **METHOD FOR SOLVING SIMPLE ABSOLUTE VALUE EQUATIONS AND INEQUATIONS:**

- Force the equation or inequation into one of the following forms:

$$|x - a| = b, \quad \text{or} \quad |x - a| < b, \quad \text{or} \quad |x - a| \geq b \quad \text{or} \quad \ldots$$

- Then find the solution using distance on a number line.

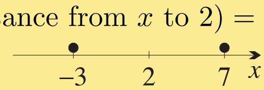
WORKED EXERCISE:

Solve each absolute value equation using distance on the number line.

(a) $|x - 2| = 5$ (b) $|x + 3| = 4$

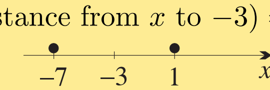
SOLUTION:

(a)
$$|x - 2| = 5$$
$$(\text{distance from } x \text{ to } 2) = 5$$



so $x = -3$ or $x = 7$.

(b)
$$|x + 3| = 4$$
$$|x - (-3)| = 4$$
$$(\text{distance from } x \text{ to } -3) = 4$$



so $x = -7$ or $x = 1$.

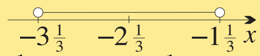
WORKED EXERCISE:
Solve each absolute value inequation using distance on the number line.
(a) $|3x + 7| < 3$ (b) $\left|7 - \frac{1}{4}x\right| \geq 3$

SOLUTION:

(a) $|3x + 7| < 3$

$\boxed{\div 3}$ $|x + 2\frac{1}{3}| < 1$ (The coefficient of x must be made 1 or -1.)

(distance from x to $-2\frac{1}{3}$) < 1 (Note that $x + 2\frac{1}{3} = x - (-2\frac{1}{3})$.)

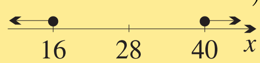


so $-3\frac{1}{3} < x < -1\frac{1}{3}$.

(b) $\left|7 - \frac{1}{4}x\right| \geq 3$

$\boxed{\times 4}$ $|28 - x| \geq 12$ (The coefficient of x must be made 1 or -1.)

(distance from x to 28) ≥ 12 (Note that $|28 - x| = |x - 28|$.)



so $x \leq 16$ or $x \geq 40$.

An Alternative Method: If $a \geq 0$ is a constant, then

$$|f(x)| = a \qquad \text{means} \qquad f(x) = a \text{ or } f(x) = -a.$$

This gives a useful alternative method of solving simple absolute value equations.

14 AN ALTERNATIVE METHOD OF SOLVING ABSOLUTE VALUE EQUATIONS AND INEQUATIONS:
Suppose that $f(x)$ is a function of x, and that $a \geq 0$ is a constant.
- To solve $|f(x)| = a$, write '$f(x) = a$ or $f(x) = -a$'.
- To solve $|f(x)| < a$, write '$-a < f(x) < a$'.
- To solve $|f(x)| > a$, write '$f(x) > a$ or $f(x) < -a$'.

WORKED EXERCISE:
Solve the equations of the previous two worked exercises the alternative way.

SOLUTION:

(a) $|x - 2| = 5$

$x - 2 = 5$ or $x - 2 = -5$

$x = -3$ or $x = 7$

(b) $|x + 3| = 4$

$x + 3 = -4$ or $x + 3 = 4$

$x = -7$ or $x = 1$

(c) $|3x + 7| < 3$

$-3 < 3x + 7 < 3$

$\boxed{-7}$ $-10 < 3x < -4$ (Subtract 7 from each of the three terms.)

$\boxed{\div 3}$ $-3\frac{1}{3} < x < -1\frac{1}{3}$ (Divide each term by 3.)

(d) $|7 - \frac{1}{4}x| \geq 3$

$7 - \frac{1}{4}x \geq 3$ or $7 - \frac{1}{4}x \leq -3$

$\boxed{-7}$ $-\frac{1}{4}x \geq -4$ or $-\frac{1}{4}x \leq -10$ (Subtract 7 from each of the three terms.)

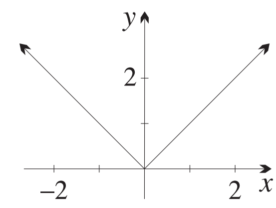
$\boxed{\times (-4)}$ $x \leq 16$ or $x \geq 40$ (The inequalities reverse because $-4 < 0$.)

An Expression for Absolute Value Involving Cases: If x is a negative number, then the absolute value of x is $-x$, the opposite of x. This gives an alternative definition:

15 ABSOLUTE VALUE INVOLVING CASES:

For any real number x, define $|x| = \begin{cases} x, & \text{for } x \geq 0, \\ -x, & \text{for } x < 0. \end{cases}$

The two cases lead directly to the graph of $y = |x|$.



Alternatively, a table of values makes clear the sharp point at the origin where the two branches meet at right angles:

x	-2	-1	0	1	2
$\lvert x \rvert$	2	1	0	1	2

- The domain is the set of all real numbers, and the range is $y \geq 0$.
- The function is even, since the graph has line symmetry in the y-axis.
- The function has a zero at $x = 0$, and is positive for all other values of x.

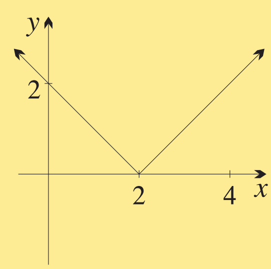
Graphing Functions with Absolute Value: Transformations can now be applied to the graph of $y = |x|$ to sketch many functions involving absolute value. More complicated functions, however, require the approach involving cases.

A short table of values is always an excellent safety check.

WORKED EXERCISE:
Sketch $y = |x - 2|$ using shifting, then give the equations of the two branches.

SOLUTION:
This is just $y = |x|$ shifted 2 units to the right.
From the expression using cases, the branches are:

$$y = \begin{cases} x - 2, & \text{for } x \geq 2, \\ -x + 2, & \text{for } x < 2. \end{cases}$$



Checking using a table of values gives:

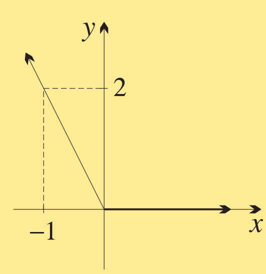
x	0	1	2	3	4
y	2	1	0	1	2

WORKED EXERCISE:
Use cases to sketch $y = |x| - x$. Check using a table of values.

SOLUTION:
Considering separately the cases $x \geq 0$ and $x < 0$,

$$y = \begin{cases} x - x, & \text{for } x \geq 0, \\ -x - x, & \text{for } x < 0, \end{cases}$$

that is, $y = \begin{cases} 0, & \text{for } x \geq 0, \\ -2x, & \text{for } x < 0. \end{cases}$



Checking using a table of values gives:

x	-2	-1	0	1	2
y	4	2	0	0	0

Absolute Value as the Square Root of the Square: Taking the absolute value of a number means stripping any negative sign from the number. An algebraic function capable of doing this job already exists.

16 ABSOLUTE VALUE AS THE POSITIVE SQUARE ROOT OF THE SQUARE:

- For all real numbers x, $|x|^2 = x^2$ and $|x| = \sqrt{x^2}$.

For example, $|-3|^2 = 9 = (-3)^2$ and $|-3| = \sqrt{9} = \sqrt{(-3)^2}$.

Identities Involving Absolute Value: Here are some standard identities.

17 IDENTITIES INVOLVING ABSOLUTE VALUE:

- $|-x| = |x|$, for all x.
- $|x - y| = |y - x|$, for all x and y.
- $|xy| = |x||y|$, for all x and y.
- $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}$, for all x, and for all $y \neq 0$.

Substitution of some positive and negative values for x and y should be sufficient to demonstrate these results.

Inequalities Involving Absolute Value: Here are two important inequalities.

18 INEQUALITIES INVOLVING ABSOLUTE VALUE:

- $|x| \geq 0$, for all x.
- $|x + y| \leq |x| + |y|$, for all x and y (called the *triangle inequality*).

Again, a variety of substitutions should be sufficient to demonstrate these.

Exercise 4E

1. Evaluate:

(a) $|5|$ (b) $|-3|$ (c) $|7 - 4|$ (d) $|4 - 7|$ (e) $|14 - 9 - 12|$ (f) $|-7 + 8|$ (g) $|3^2 - 5^2|$ (h) $|11 - 16| - 8$

2. Solve each absolute value equation or inequation, then graph the solution on a number line. In parts (e)–(h), you will need to divide through by the coefficient of x.

(a) $|x| = 1$ (b) $|x| = 3$ (c) $|4x| = 8$ (d) $|2x| = 10$ (e) $|x| < 2$ (f) $|x| \geq 2$ (g) $|2x| \leq 6$ (h) $|3x| > 12$

3. Solve each equation or inequation, and graph the solution on a number line.

(a) $|x - 4| = 1$ (b) $|x - 3| = 7$ (c) $|x - 6| = 11$ (d) $|x + 2| = 2$ (e) $|x + 1| = 6$ (f) $|x + 5| = 2$ (g) $|x - 2| < 3$ (h) $|x - 7| \geq 2$ (i) $|x - 2| \leq 5$ (j) $|x + 3| > 4$ (k) $|x + 10| \leq 6$ (l) $|x + 6| \geq 6$

4. (a) Copy and complete these tables of values of the functions $y = |x - 2|$ and $y = |x| - 2$.

x	-1	0	1	2	3
$\lvert x - 2\rvert$					

x	-1	0	1	2	3
$\lvert x\rvert - 2$					

(b) Draw the graphs of the two functions on the same number plane and observe the differences between them. How is each graph obtained by shifting $y = |x|$?

DEVELOPMENT

5. Show that each statement is true for $x = -3$.

(a) $|5x| = 5|x|$ (b) $|-x| = |x|$ (c) $|x|^2 = x^2$ (d) $|x - 7| = |7 - x|$ (e) $x \le |x|$ (f) $-|x| \le x$

6. Show that each statement is false for $x = -2$.

(a) $|x| = x$ (b) $|-x| = x$ (c) $|x + 2| = |x| + 2$ (d) $|x + 1| = x + 1$ (e) $|x - 1| < |x| - 1$ (f) $|x|^3 = x^3$

7. Solve for x:

(a) $|7x| = 35$ (b) $|2x - 1| = 11$ (c) $|7x - 3| = 11$ (d) $|3x + 2| = 8$ (e) $|5x + 2| = 9$ (f) $|3x - 5| \le 4$ (g) $|6x - 7| > 5$ (h) $|2x + 1| < 3$ (i) $|5x + 4| \ge 6$

8. Sketch each graph by transforming $y = |x|$ or by drawing up a table of values. Also write down the equations of the two branches in each case.

(a) $y = |2x|$ (b) $y = |\frac{1}{2}x|$ (c) $y = |x - 1|$ (d) $y = |x + 3|$ (e) $y = |x| - 1$ (f) $y = |x| + 3$ (g) $y = |2 - x|$ (h) $y = 2 - |x|$

CHALLENGE

9. (a) For what values of x is $y = \dfrac{|x|}{x}$ undefined?

(b) Use a table of values of x from -3 to 3 to sketch the graph.

(c) Hence write down the equations of the two branches of $y = \dfrac{|x|}{x}$.

10. Determine whether or not each statement is true for all values of x and y. If it is not true, give a counter-example.

(a) $|x + y| = |x| + |y|$ (b) $|x + y| \le |x| + |y|$ (c) $|x - y| \le |x| - |y|$ (d) $|x - y| \le |x| + |y|$ (e) $|x - y| \ge \big||x| - |y|\big|$ (f) $2^{|x|} = 2^x$

11. Draw up a table of values for $-3 \le x \le 3$ and sketch the graph. Also write down the equation of each branch of the function.

(a) $y = |x| + x$ (b) $y = |x| - x$ (c) $y = 2(x + 1) - |x + 1|$ (d) $y = x^2 - |2x|$

4 F Using Graphs to Solve Equations and Inequations

In this section, graphs are used to solve a variety of equations and inequations. Once the graphs are drawn, two questions can usually be answered from them:

- How many solutions are there — indeed are there any solutions at all?
- What are the approximate values of the solutions?

Exact solutions can often be calculated once the situation has been sorted out from the graph.

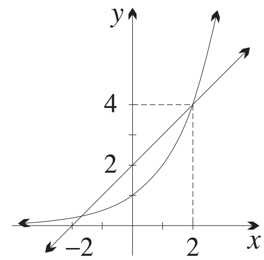
Constructing Two Functions from a Given Equation: Here is an equation that cannot be solved algebraically:

$$2^x = x + 2$$

One solution is $x = 2$, because $2^2 = 2 + 2$. But this is not the only solution.

The diagram to the right shows the graphs of $y = \text{LHS}$ and $y = \text{RHS}$, that is, of

$$y = 2^x \qquad \text{and} \qquad y = x + 2,$$



and clearly there is a second solution. The graph shows that the LHS and the RHS are equal:

- at $x = 2$ (where they are both 4), and
- at $x \doteqdot -1{\cdot}7$ (where they are both about $0{\cdot}3$).

These are the two solutions of the original equation.

19 GRAPHICAL SOLUTION OF AN EQUATION:
- Sketch the graphs of $y = \text{LHS}$ and $y = \text{RHS}$ on one pair of axes.
- Read off the x-coordinates of any points of intersection.

The original equation may need to be rearranged first.

WORKED EXERCISE:

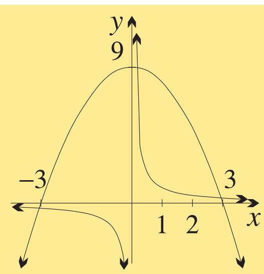
(a) Graph $y = \dfrac{1}{x}$ and $y = 9 - x^2$ on one set of axes.

(b) Find from your graph the number of solutions of the equation $x^2 + \dfrac{1}{x} = 9$.

(c) What are the approximate values of these solutions?

SOLUTION:

(a) The two graphs are sketched opposite.



(b) First rearrange the equation to $\dfrac{1}{x} = 9 - x^2$.

Its solutions are the x-coordinates of the intersections of

$$y = \frac{1}{x} \qquad \text{and} \qquad y = 9 - x^2.$$

From the graph there are three solutions.

(c) One solution lies between -4 and -3, one between 0 and 1, and one between 2 and 3.

Solving an Inequation using Graphs: Now consider the inequation

$$2^x < x + 2.$$

The curves $y = 2^x$ and $y = x + 2$ were sketched above.
The curve $y = 2^x$ is only below $y = x + 2$ between the two points of intersection, so the solution of the inequation is approximately $-1{\cdot}7 < x < 2$.

20 GRAPHICAL SOLUTION OF INEQUATIONS:
- Sketch the graphs of $y = \text{LHS}$ and $y = \text{RHS}$ on one pair of axes.
- Then determine which curve lies above the other at each value of x.

Absolute Value Equations — Graphical Solutions: If the graph can be used to sort out the situation, then the exact values can usually be found algebraically.

WORKED EXERCISE:

(a) Solve $|2x-5| = x+2$, using graphs of the LHS and RHS.

(b) Hence solve $|2x-5| \geq x+2$.

SOLUTION:

(a) The graphs of $y = |2x-5|$ and $y = x+2$ intersect at P and Q.

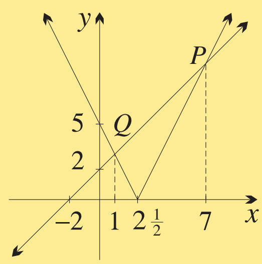
Here P is the intersection of $y = x+2$ with $y = 2x-5$:

$$x+2 = 2x-5$$
$$x = 7,$$

and Q is the intersection of $y = x+2$ with $y = -2x+5$:

$$x+2 = -2x+5$$
$$x = 1.$$

Hence the final solution is $x = 7$ or $x = 1$.



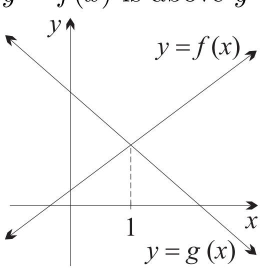
(b) Where is the graph of $y = |2x-5|$ above the graph of $y = x+2$?

This occurs to the right of P and to the left of Q, so the solution is $x \leq 1$ or $x \geq 7$.

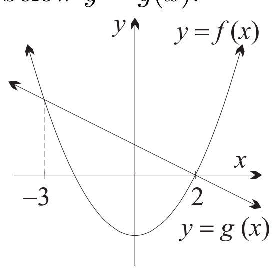
Exercise 4F

1. Write down the values of x for which:
(i) $y = f(x)$ is above $y = g(x)$, (ii) $y = f(x)$ is below $y = g(x)$.

(a)

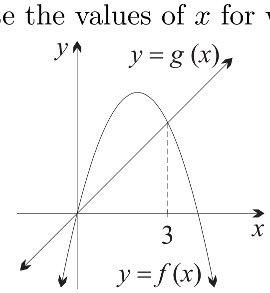


(b)

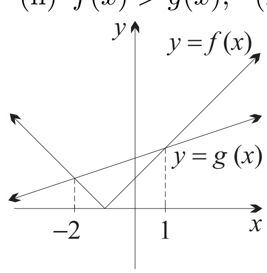


2. State the values of x for which: (i) $f(x) = g(x)$, (ii) $f(x) > g(x)$, (iii) $f(x) < g(x)$.

(a)

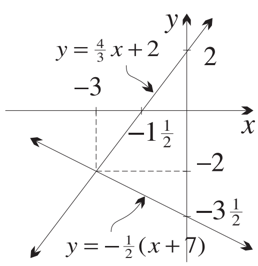


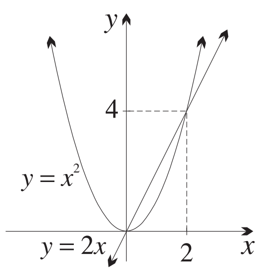
(b)

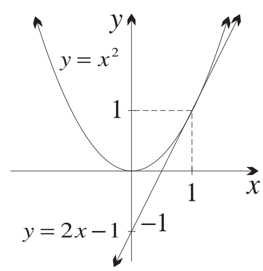


3. Use the given graphs to help solve these inequations.

(a) $\frac{4}{3}x + 2 \leq -\frac{1}{2}(x+7)$ (b) $x^2 \leq 2x$ (c) $x^2 \leq 2x - 1$

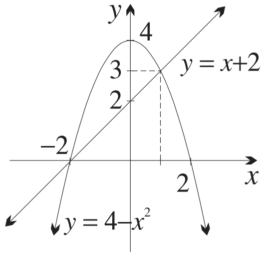


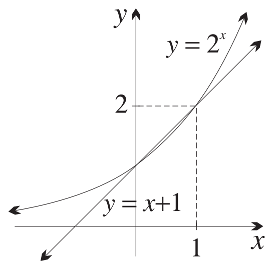


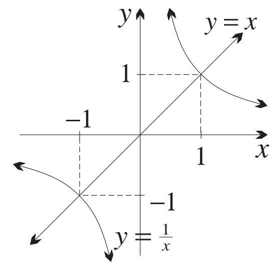


4. Solve these inequations using the graphs below.

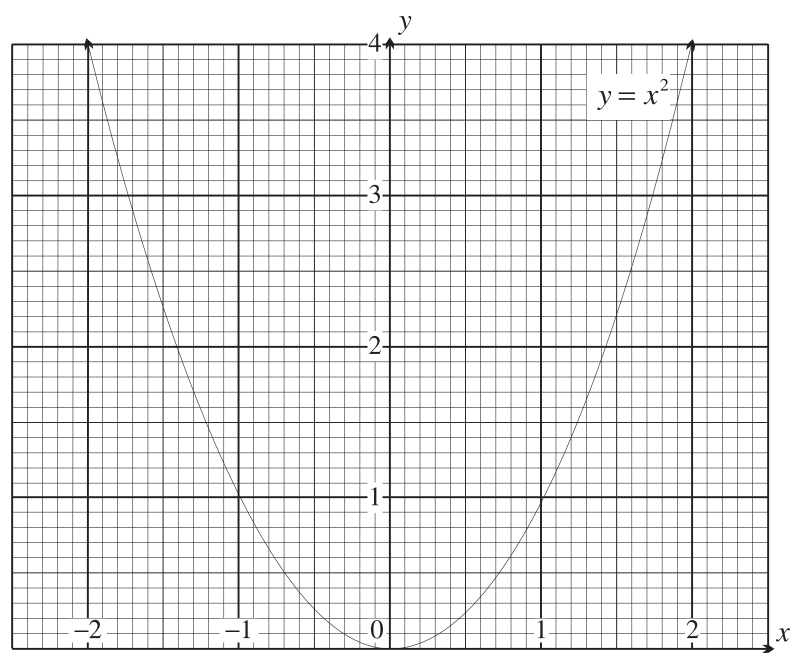
(a) $4 - x^2 < x + 2$ (b) $2^x \le x + 1$ (c) $\dfrac{1}{x} < x$







5.



In preparation for the following questions, photocopy the sketch above, which shows the graph of the function

$$y = x^2, \quad \text{for } -2 \le x \le 2.$$

(a) Read $\sqrt{2}$ and $\sqrt{3}$ off the graph, correct to one decimal place, by locating 2 and 3 on the y-axis and reading the answer off the x-axis. Check your approximations using a calculator.

(b) Draw on the graph the line $y = x + 2$, and hence read off the graph the solutions to $x^2 = x + 2$. Then check your solution by solving $x^2 = x + 2$ algebraically.

(c) From the graph, write down the solution of $x^2 > x + 2$.

(d) Draw a suitable line to solve $x^2 = 2 - x$ and $x^2 \le 2 - x$. Check your results by solving $x^2 = 2 - x$ algebraically.

(e) Draw $y = x + 1$, and hence solve $x^2 = x + 1$ approximately. Check your result algebraically.

(f) Find approximate solutions for these quadratic equations by rearranging each with x^2 as subject, and drawing a suitable line on the graph.

(i) $x^2 + x = 0$ (ii) $x^2 - x - \frac{1}{2} = 0$ (iii) $2x^2 - x - 1 = 0$

DEVELOPMENT

6. (i) Carefully sketch each pair of equations. (ii) Read off the points of intersection. (iii) Write down the equation satisfied by the x-coordinates of the points of intersection.

(a) $y = x - 2$ and $y = 3 - \frac{1}{4}x$

(b) $y = x$ and $y = 2x - x^2$

(c) $y = \dfrac{2}{x}$ and $y = x - 1$

(d) $y = x^3$ and $y = x$

7. Use your graphs from the previous question to solve the following inequations.

(a) $x - 2 \geq 3 - \frac{1}{4}x$

(b) $x < 2x - x^2$

(c) $\dfrac{2}{x} > x - 1$

(d) $x^3 > x$

8. (a) Sketch on the same number plane the functions $y = |x + 1|$ and $y = \frac{1}{2}x - 1$.

(b) Hence explain why all real numbers are solutions of the inequation $|x + 1| > \frac{1}{2}x - 1$.

9. Sketch each pair of equations and hence find the points of intersection.

(a) $y = |x + 1|$ and $y = 3$

(b) $y = |x - 2|$ and $y = x$

(c) $y = |2x|$ and $2x - 3y + 8 = 0$

(d) $y = |x| - 1$ and $y = 2x + 2$

10. Use your answers to the previous question to help solve:

(a) $|x + 1| \leq 3$

(b) $|x - 2| > x$

(c) $|2x| \geq \dfrac{2x + 8}{3}$

(d) $|x| > 2x + 3$

CHALLENGE

11. (a) Sketch $y = x^2 - 2$, $y = x$ and $y = -x$ on the same number plane and find all points of intersection of the three functions.

(b) Hence find the solutions of $x^2 - 2 = |x|$. (c) Hence solve $x^2 - 2 > |x|$.

12. (a) Sketch $y = x^2 - 6$ and $y = |x|$ on one set of axes.

(b) Find the x-coordinates of the points of intersection.

(c) Hence solve $x^2 - 6 \leq |x|$.

13. Sketch graphs of the LHS and RHS of each equation on the same number plane in order to find the number of solutions (tables of values may help). Do not attempt to solve them.

(a) $1 - \frac{1}{2}x = x^2 - 2x$

(b) $|2x| = 2^x$

(c) $x^3 - x = \frac{1}{2}(x + 1)$

(d) $4x - x^2 = \dfrac{1}{x}$

(e) $2^x = 2x - x^2$

(f) $2^{-x} - 1 = \dfrac{1}{x}$

14. Draw appropriate graphs, using a computer or graphics calculator, in order to find the solutions to these equations correct to one decimal place.

(a) $x^3 = 2(x - 2)^2$

(b) $x^3 = \sqrt{4 - x^2}$

(c) $2^x = -x(x + 2)$

(d) $2^{-x} = 2x - x^2$

4 G Regions in the Number Plane

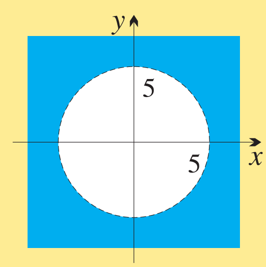
The circle $x^2 + y^2 = 25$ divides the plane into two *regions* — inside the circle and outside the circle. The graph of the inequation $x^2 + y^2 > 25$ will be one of these regions. It remains to work out which of these regions should be shaded.

Graphing Regions: To sketch the region of an inequation, use the following method.

GRAPHING THE REGION CORRESPONDING TO AN INEQUATION:

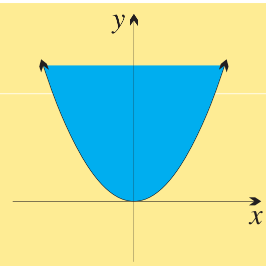
1. THE BOUNDARY:
   Replace the inequality symbol by an equals symbol and graph the curve.
   This will be the boundary of the region.
2. SHADING: Determine which parts of the plane are included and which are excluded, and shade the parts that are included. There are two methods:
   (a) [Always possible] Take one or more test points not on any boundary, and
21 substitute into the LHS and RHS of the original inequation.
   The origin is the easiest test point; the next easiest are points on the axes.
   (b) [Quicker, but not always possible] Solve the inequation for y if possible, and shade the region above or below the curve. Alternatively, solve for x, and sketch the region to the right or left of the curve.
3. CHECKING BOUNDARIES AND CORNERS:
   Boundaries are drawn unbroken if included, and broken if excluded.
   Corners have a closed circle if included, and an open circle if excluded.

WORKED EXERCISE:
Sketch the region $x^2 + y^2 > 25$.



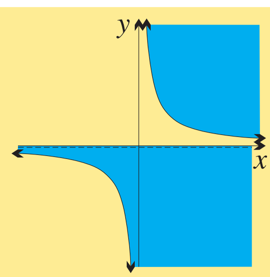
SOLUTION:
The boundary is $x^2 + y^2 = 25$, and is excluded.
Take a test point $(0, 0)$. Then RHS $= 25$,
LHS $= 0$.
So $(0, 0)$ does not lie in the region.

WORKED EXERCISE:
Sketch $y \geq x^2$.



SOLUTION:
The boundary is $y = x^2$, and is included.
Because the inequation is $y \geq x^2$,
the region involved is the region *above* the curve.

WORKED EXERCISE:
[A harder example] Sketch $x \geq \dfrac{1}{y}$.



SOLUTION:
The boundary is the hyperbola $x = \dfrac{1}{y}$,
and is included.
Because the inequation is $x \geq \dfrac{1}{y}$,
the region shaded is the region to the right of the curve.

Intersections of Regions: Some questions will ask explicitly for a sketch of the intersection of two given regions, as in the first worked exercise below.

In other questions the intersection will be implicit. For example, in the second worked exercise below,

$$|2x+3y|<6$$

means $-6<2x+3y<6$, which is the intersection of the regions

$$2x+3y>-6 \quad \text{and} \quad 2x+3y<6.$$

Any restriction on x or on y also requires taking an intersection of regions. For example, in the third worked exercise below,

$$x^2+y^2<25, \text{ where } x \le 3 \text{ and } y>-4,$$

means the intersection of the three different regions

$$x^2+y^2<25 \quad \text{and} \quad x \le 3 \quad \text{and} \quad y>-4.$$

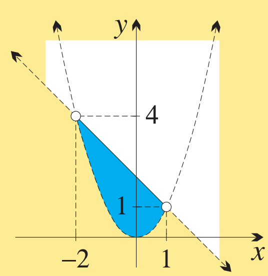
22 INTERSECTIONS OF REGIONS:
- Sketch each region using the methods described above.
- Then sketch their intersection, giving a clear code for the shading.
- Pay particular attention to whether corner points are included or excluded.

WORKED EXERCISE:
Graph the intersection of the regions

$$y>x^2 \quad \text{and} \quad x+y \le 2,$$

giving the coordinates of any corner points.

SOLUTION:
The boundary of the first region is $y=x^2$,
and the region lies above the curve (with boundary excluded).

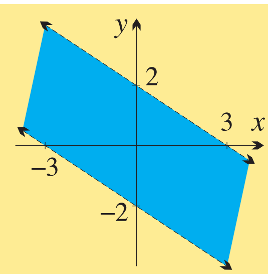


The boundary of the second region is $x+y=2$.
Solving for y gives $y \le 2-x$,
so the region lies below the curve (with boundary included).

By inspection, or by solving simultaneous equations,
we find that the parabola and the line meet at $(1,1)$ and $(-2,4)$.
These points are not in the intersection because they are not in the region $y>x^2$.

WORKED EXERCISE:
Graph the region $|2x+3y|<6$.

SOLUTION:
This is the region $-6<2x+3y<6$.



The boundaries are the parallel lines

$$2x+3y=6 \text{ and } 2x+3y=-6,$$

both of which are excluded.
The required region is the region between these two lines.

WORKED EXERCISE:

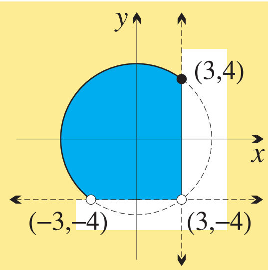
(a) Graph the region $x^2 + y^2 \le 25$, for $x \le 3$ and $y > -4$.

(b) Give the coordinates of each corner point.

SOLUTION:

(a) The boundaries are $x^2 + y^2 = 25$ (included), and the vertical and horizontal lines $x = 3$ (included) and $y = -4$ (excluded).

(b) The points of intersection are $(3, 4)$ (included), $(3, -4)$ (excluded) and $(-3, -4)$ (excluded).



Exercise 4G

1. For each inequation: (i) sketch the boundary, (ii) shade the region above or below the boundary, as required.

(a) $y < 1$
(b) $y \ge -3$
(c) $y > x - 1$
(d) $y \le 3 - x$
(e) $y \le 2x + 2$
(f) $y < \frac{1}{2}x - 1$

2. For each inequation: (i) sketch the boundary, (ii) shade the region to the left or right of the boundary, as required.

(a) $x < -2$
(b) $x > 1$
(c) $x \ge y + 2$
(d) $x < 2y - 1$
(e) $x > 3 - y$
(f) $x \le \frac{1}{2}y + 2$

3. For each inequation, sketch the boundary line, then use a suitable test point to decide which side of the line to shade.

(a) $2x + 3y - 6 > 0$
(b) $x - y + 4 \ge 0$
(c) $y - 2x + 3 < 0$

DEVELOPMENT

4. For each inequation, sketch the boundary circle, then use a suitable test point to decide which region to shade.

(a) $x^2 + y^2 < 4$
(b) $x^2 + y^2 \ge 1$
(c) $(x - 2)^2 + y^2 \le 4$
(d) $(x + 1)^2 + (y - 2)^2 > 9$

5. Sketch the following regions. (Factor the quadratics first.)

(a) $y \ge x^2 - 1$
(b) $y < x^2 - 2x - 3$
(c) $y \ge x^2 + 2x + 1$
(d) $y > 4 - x^2$
(e) $y \le x^2 + 3x$
(f) $y \le 2 + x - x^2$

6. Sketch the following regions of the number plane.

(a) $y > 2^x$
(b) $y \ge |x|$
(c) $y \le |x + 1|$
(d) $y > x^3$
(e) $y \le 2^{-x}$
(f) $y < |\frac{1}{2}x - 1|$

7. (a) Find the point of intersection of the lines $x = -1$ and $y = 2x - 1$.

(b) Hence sketch the intersection of the regions $x > -1$ and $y \le 2x - 1$, paying careful attention to the boundaries and their point of intersection.

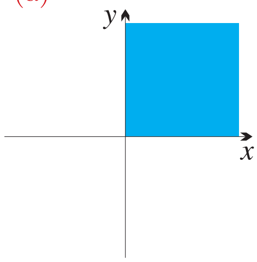
8. Sketch on separate number planes the regions $y < x$ and $y \ge -x$. Hence sketch the intersection of the two regions. Pay careful attention to the boundaries and their point of intersection.

9. Similarly, graph the intersection of:

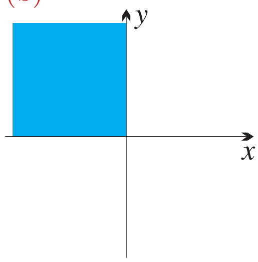
(a) $y > x$ and $y \le 2 - x$
(b) $y > \frac{1}{2}x + 1$ and $y \le -x - 2$

10. Identify the inequations that correspond to the following regions.

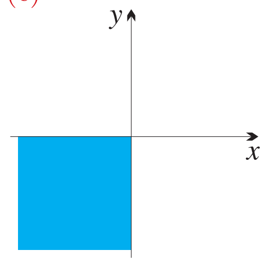
(a)



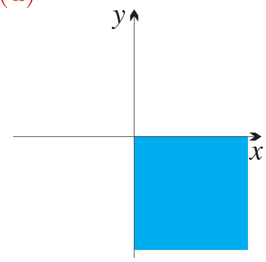
(b)



(c)



(d)



11. Sketch the intersection of $x^2 + y^2 > 1$ and $x^2 + y^2 \le 9$.

CHALLENGE

12. (a) Find the points at which the line $y = 4 - x$ intersects the circle $x^2 + y^2 = 16$.
(b) Hence sketch the intersection of the regions $y \ge 4 - x$ and $x^2 + y^2 < 16$.

13. (a) The inequation $|x| < 2$ implies the intersection of two regions in the number plane.
(i) Write down the inequations representing these two regions.
(ii) Hence sketch the region $|x| < 2$.
(b) The inequation $|x - y| \le 2$ implies the intersection of the two regions $x - y \le 2$ and $x - y \ge -2$. Sketch $|x - y| \le 2$.

14. Sketch the region $x^2 + y^2 \ge 5$ for the domain $x > -1$ and range $y < 2$, and give the coordinates of each corner.

15. Sketch $y = \dfrac{1}{x}$, and hence sketch the region $y > \dfrac{1}{x}$.

4H Chapter Review Exercise

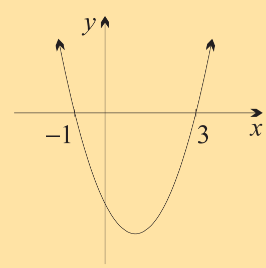
1. Solve each inequation, and graph your solution on the number line.

(a) $3x + 2 < 17$
(b) $\frac{1}{2}x - 6 \ge -4$
(c) $7x + 5 \le 3x + 1$
(d) $-4x > -8$
(e) $2 - 3x \le 5$
(f) $7 - 4x \ge 13 - x$

2. Solve each double inequation, and graph your solution on the number line.

(a) $-6 < 2x \le 4$
(b) $-3 \le x + 2 < 7$
(c) $-12 < -3x < 9$

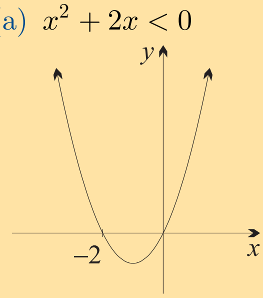
3. Examine the parabola sketched to the right. By considering where the curve is above the x-axis and where it is below the x-axis, write down the values of x for which:



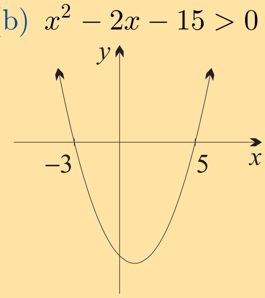
(a) $y = 0$
(b) $y > 0$
(c) $y < 0$

4. Solve each quadratic inequation, using the graph provided.

(a) $x^2 + 2x < 0$



(b) $x^2 - 2x - 15 > 0$



5. Solve each quadratic inequation.

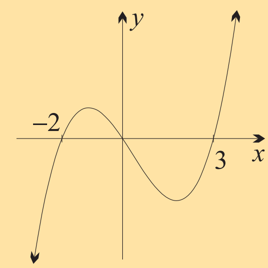
(a) $(x+4)(x-3)<0$ (b) $(x-1)(x+5)\ge 0$ (c) $x^2-4x>0$ (d) $x^2-9\le 0$ (e) $x^2-4x-12\ge 0$ (f) $2x^2+x-1<0$

6. Examine the given curve, then write down the values of x for which:

(a) $y=0$

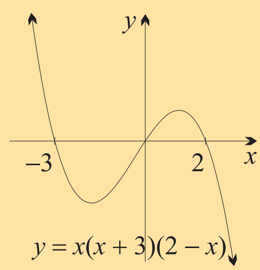
(b) $y>0$

(c) $y<0$

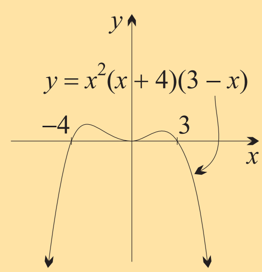


7. Solve each inequation, using the graph below it.

(a) $x(x+3)(2-x)>0$



(b) $x^2(x+4)(3-x)<0$



8. Draw up a table of test points on either side of the zeroes (that is, the x-intercepts) of each function, then sketch the graph of the function.

(a) $y=(x+2)(x-1)(x-4)$ (b) $y=(x+3)^2(x+1)(x-2)$

9. Use the graphs in the previous question to solve the following inequations.

(a) $(x+2)(x-1)(x-4)\le 0$ (b) $(x+3)^2(x+1)(x-2)\ge 0$

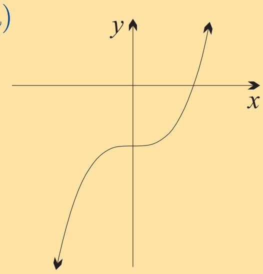
10. (a) Factor x^3-25x.

(b) Draw up a table of x-intercepts and test values, then sketch the curve $y=x^3-25x$.

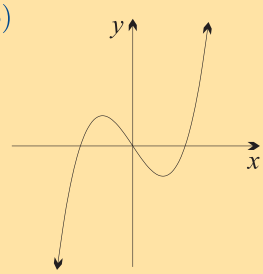
(c) Hence solve $x^3-25x>0$.

11. Classify each function $y=f(x)$ as even, odd or neither.

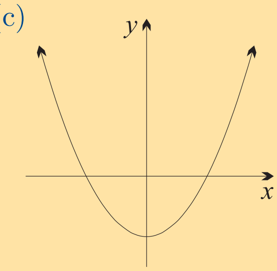
(a)



(b)

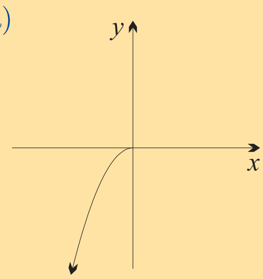


(c)

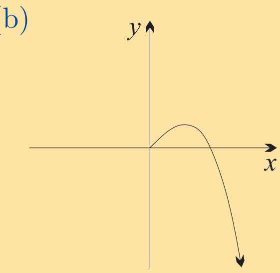


12. In each diagram below, complete the graph so that: (i) $f(x)$ is even, (ii) $f(x)$ is odd.

(a)



(b)



13. Find $f(-x)$ for each function, and then decide whether the function is even, odd or neither.

(a) $f(x)=x^3$ (b) $f(x)=x^4$ (c) $f(x)=x+3$ (d) $f(x)=2x^2-5$

14. Evaluate:
(a) $|-7|$ (c) $|3-8|$ (e) $|-2|-|-5|$
(b) $|4|$ (d) $|-2-(-5)|$ (f) $|13-9-16|$

15. Solve for x:
(a) $|x|=5$ (b) $|3x|=18$ (c) $|x-2|=4$ (d) $|x+3|=2$

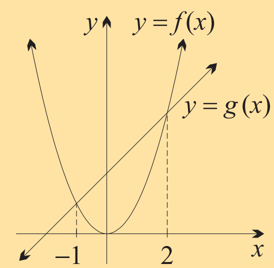
16. Solve each inequation, and graph the solution on the number line.
(a) $|x|\le 3$ (b) $|x|>3$ (c) $|x-5|<2$ (d) $|x+1|\ge 4$

17. Solve:
(a) $|2x-3|=5$ (b) $|2x+1|>7$ (c) $|3x-4|\le 7$

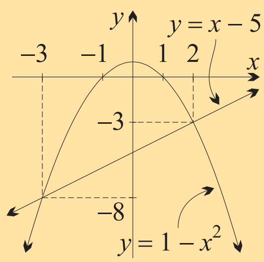
18. Sketch $y=|x|$ from a table of values.

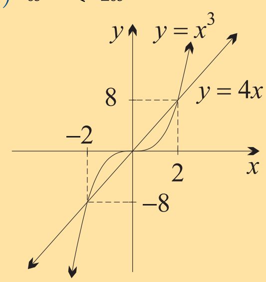
19. Sketch each graph by shifting $y=|x|$, or by using a table of values. Mark all x- and y-intercepts.
(a) $y=|x|-2$ (b) $y=|x-2|$ (c) $y=|x|+2$ (d) $y=|x+2|$

20. Examine the diagram to the right, where $y=f(x)$ and $y=g(x)$ are sketched on one set of axes. Then write down the values of x for which:
(a) $f(x)=g(x)$
(b) $f(x)>g(x)$
(c) $f(x)<g(x)$



21. Use the graph provided to solve each inequation.
(a) $1-x^2>x-5$ (b) $x^3<4x$





22. Carefully sketch each pair of equations and hence find their points of intersection.
(a) $y=x^2$ and $y=-x$ (b) $y=|x-3|$ and $y=\frac{1}{2}x$

23. Use the graphs in the previous question to solve the following inequations.
(a) $x^2>-x$ (b) $|x-3|<\frac{1}{2}x$

24. Sketch the following regions.
(a) $y>2$ (c) $y\ge 2x$ (e) $x>y-2$
(b) $y\le 2-x$ (d) $x\le 2$ (f) $x\le 2y+2$

25. Sketch each region.
(a) $y\ge 1-x^2$ (b) $x^2+y^2<1$ (c) $y\le 2^x$

26. Sketch the intersection of the regions $y<2x-2$ and $y\ge 1-x$.

CHAPTER **FIVE**

Trigonometry

Trigonometry is important because the graphs of the sine and cosine functions are waves. Waves appear everywhere in the natural world, for example as water waves, as sound waves, or as the electromagnetic waves that are responsible for radio, heat, light, ultraviolet radiation, X-rays and gamma rays. In quantum mechanics, a wave is associated with every particle.

Trigonometry began, however, in classical times as the study of the relationships between angles and lengths in geometrical figures. Its name, from the Greek words *trigos*, meaning 'land', and *metros*, meaning 'measurement', reminds us that trigonometry is fundamental to surveying and navigation. This introductory chapter establishes the geometric basis of the trigonometric functions and their graphs, developing them from the geometry of triangles and circles.

Some of this chapter will be quite new to most readers, in particular the extension of the trigonometric functions to angles of any magnitude, the graphs of these functions, and trigonometric identities and equations.

5 A Trigonometry with Right-Angled Triangles

This section and the next will review the definitions of the trigonometric functions for acute angles and apply them to problems involving right-angled triangles.

The Trigonometric Functions for Acute Angles: Let θ be any acute angle.

Construct a right-angled triangle with an acute angle θ and label the sides:

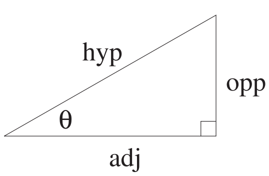
hyp — the *hypotenuse*, the side opposite the right angle,
opp — the side *opposite* the angle θ,
adj — the third side, *adjacent* to θ but not the hypotenuse.

THE TRIGONOMETRIC FUNCTIONS FOR AN ACUTE ANGLE θ:

1

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}}$$

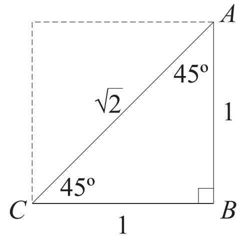
$$\operatorname{cosec}\theta = \frac{\text{hyp}}{\text{opp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$



Any two triangles with angles of 90° and θ are similar, by the AA similarity test. Hence the values of the six trigonometric functions are the same, whatever the size of the triangle. The full names of the six functions are:

sine, cosine, tangent, cosecant, secant, cotangent.

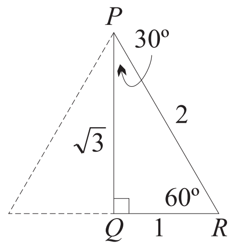
Special Angles: The values of the trigonometric functions for the three acute angles $30°$, $45°$ and $60°$ can be calculated exactly.



Take half a square with side length 1 .

The resulting right-angled triangle ABC has two angles of $45°$.

By Pythagoras' theorem, the hypotenuse AC has length $\sqrt{2}$.



Take half an equilateral triangle with side length 2 by dropping an altitude.

The resulting right-angled $\triangle PQR$ has angles of $60°$ and $30°$.

By Pythagoras' theorem, $PQ = \sqrt{3}$.

Applying the definitions on the previous page gives the values in the table below.

2 A TABLE OF EXACT VALUES θ:

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\operatorname{cosec}\theta$	$\sec\theta$	$\cot\theta$
$30°$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	2	$\frac{2}{\sqrt{3}}$	$\sqrt{3}$
$45°$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$60°$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}}$	2	$\frac{1}{\sqrt{3}}$

Trigonometric Functions of Other Angles: The calculator is usually used to approximate trigonometric functions of other angles. Make sure that you know:

- how to enter angles in degrees and minutes,
- how to change angles in decimals of a degree to angles in degrees and minutes.

$$\sin 53°47' \doteqdot 0{\cdot}8068 \qquad \text{and} \qquad \sin\theta = \tfrac{5}{8},\ \text{so } \theta \doteqdot 38°41'.$$

The Reciprocal Trigonometric Functions: The functions cosecant, secant and cotangent can mostly be avoided by using the sine, cosine and tangent functions.

3 AVOIDING THE RECIPROCAL TRIGONOMETRIC FUNCTIONS:

The three reciprocal trigonometric functions can often be avoided because

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}, \qquad \sec\theta = \frac{1}{\cos\theta}, \qquad \cot\theta = \frac{1}{\tan\theta}.$$

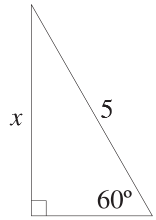
Finding an Unknown Side of a Triangle: The calculator only has the sine, cosine and tangent functions, so it is best to use only these three functions in problems.

4 TO FIND AN UNKNOWN SIDE OF A RIGHT-ANGLED TRIANGLE:

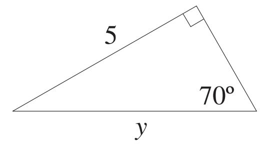
1. Start by writing $\dfrac{\text{unknown side}}{\text{known side}} = \ldots\ldots$ (Place the unknown at top left.)
2. Complete the RHS with sin, cos or tan, or the reciprocal of one of these.

WORKED EXERCISE:
Find the side marked with a pronumeral in each triangle. Give the answer in exact form if possible, or else correct to five significant figures.

(a)



(b)



SOLUTION:

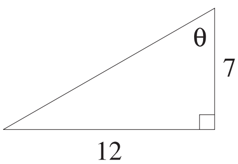
(a)
$$\frac{x}{5} = \sin 60^\circ \qquad \left(\frac{\text{opposite}}{\text{hypotenuse}}\right)$$
$$\boxed{\times 5} \quad x = 5\sin 60^\circ$$
$$= \frac{5\sqrt{3}}{2}$$

(b)
$$\frac{y}{5} = \frac{1}{\sin 70^\circ} \qquad \left(\frac{\text{hypotenuse}}{\text{opposite}}\right)$$
$$\boxed{\times 5} \quad y = \frac{5}{\sin 70^\circ}$$
$$\doteqdot 5{\cdot}3209$$

Finding an Unknown Angle: As before, use only sine, cosine and tangent.

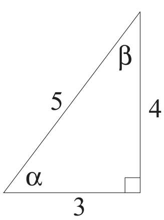
5 **FINDING AN UNKNOWN ANGLE, GIVEN TWO SIDES OF A RIGHT-ANGLED TRIANGLE:**
Work out from the known sides which one of $\cos\theta$, $\sin\theta$ or $\tan\theta$ to use.

WORKED EXERCISE:
Find θ in the triangle drawn to the right.



SOLUTION:
The given sides are the opposite and the adjacent sides, so $\tan\theta$ is known.
$$\tan\theta = \frac{12}{7} \qquad \left(\frac{\text{opposite}}{\text{adjacent}}\right)$$
$$\theta \doteqdot 59^\circ 45'.$$

Exercise 5A



1. From the diagram opposite, write down the values of:
(a) $\cos\alpha$ (c) $\sin\alpha$ (e) $\sin\beta$
(b) $\tan\beta$ (d) $\cos\beta$ (f) $\tan\alpha$

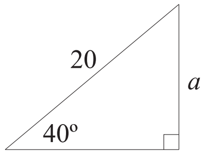
2. Use your calculator to find, correct to four decimal places:
(a) $\sin 24^\circ$ (c) $\tan 35^\circ$ (e) $\tan 2^\circ$ (g) $\sin 1^\circ$
(b) $\cos 61^\circ$ (d) $\sin 87^\circ$ (f) $\cos 33^\circ$ (h) $\cos 3^\circ$

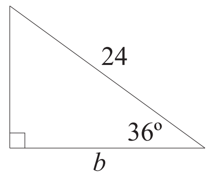
3. Use your calculator to find, correct to four decimal places:
(a) $\tan 57^\circ 30'$ (c) $\tan 78^\circ 40'$ (e) $\sin 43^\circ 6'$ (g) $\sin 8'$
(b) $\cos 32^\circ 24'$ (d) $\cos 16^\circ 51'$ (f) $\sin 5^\circ 50'$ (h) $\tan 57'$

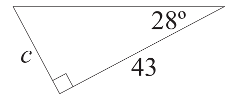
4. Use your calculator to find the acute angle θ, correct to the nearest degree, if:
(a) $\tan\theta = 4$ (c) $\sin\theta = \frac{1}{5}$ (e) $\cos\theta = \frac{7}{9}$
(b) $\cos\theta = 0{\cdot}7$ (d) $\sin\theta = 0{\cdot}456$ (f) $\tan\theta = 1\frac{3}{4}$

5. Use your calculator to find the acute angle α, correct to the nearest minute, if:
(a) $\cos\alpha = \frac{3}{4}$ (c) $\sin\alpha = 0{\cdot}1$ (e) $\sin\alpha = 0{\cdot}7251$
(b) $\tan\alpha = 2$ (d) $\tan\alpha = 0{\cdot}3$ (f) $\cos\alpha = \frac{7}{13}$

6. Find, correct to the nearest whole number, the value of each pronumeral.

(a)


(b)


(c)


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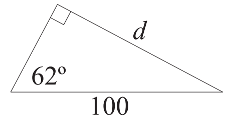
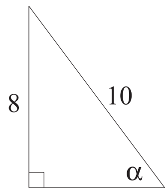


7. Find, correct to the nearest degree, the size of each angle marked with a pronumeral.

(a)


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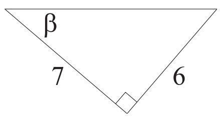

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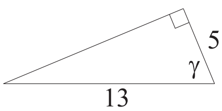

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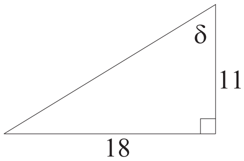
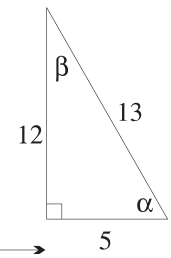
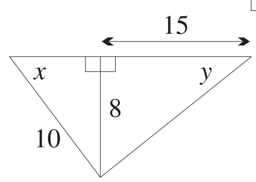


DEVELOPMENT



8. From the diagram opposite, write down the values of:

(a) $\sin\alpha$ (c) $\sec\beta$ (e) $\operatorname{cosec}\alpha$
(b) $\tan\beta$ (d) $\cot\alpha$ (f) $\sec\alpha$



9. (a) Use Pythagoras' theorem to find the third side in each of the three right triangles in the diagram opposite.

(b) Write down the values of:

(i) $\cos y$ (iii) $\cot x$ (v) $\sec x$
(ii) $\sin x$ (iv) $\operatorname{cosec} y$ (vi) $\cot y$

10. Draw the two special triangles containing the acute angles $30°$, $60°$ and $45°$. Hence write down the exact values of:

(a) $\sin 60°$ (b) $\tan 30°$ (c) $\cos 45°$ (d) $\sec 60°$ (e) $\operatorname{cosec} 45°$ (f) $\cot 30°$

11. Use your calculator to find, correct to four decimal places:

(a) $\operatorname{cosec} 20°$ (c) $\cot 56°$ (e) $\cot 28°30'$ (g) $\sec 67°43'$
(b) $\sec 48°$ (d) $\sec 5°$ (f) $\operatorname{cosec} 73°24'$ (h) $\operatorname{cosec} 81°13'$

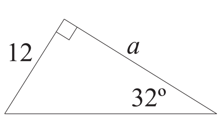
12. Use your calculator to find the acute angle θ, correct to the nearest degree, if:

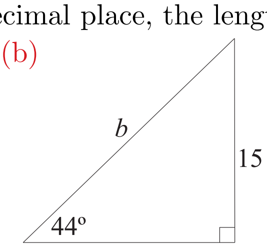
(a) $\sec\theta = 3$ (c) $\operatorname{cosec}\theta = 1{\cdot}75$ (e) $\operatorname{cosec}\theta = 5{\cdot}963$
(b) $\cot\theta = 2{\cdot}5$ (d) $\sec\theta = \frac{25}{4}$ (f) $\cot\theta = 2\frac{4}{7}$

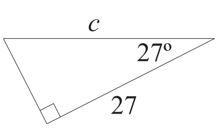
13. Use your calculator to find the acute angle α, correct to the nearest minute, if:

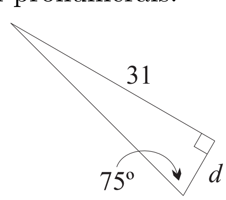
(a) $\cot\alpha = 0{\cdot}6$ (c) $\operatorname{cosec}\alpha = 7$ (e) $\operatorname{cosec}\alpha = \frac{20}{13}$
(b) $\sec\alpha = 1{\cdot}1$ (d) $\cot\alpha = 0{\cdot}23$ (f) $\sec\alpha = 3{\cdot}967$

14. Find, correct to one decimal place, the lengths of the sides marked with pronumerals.

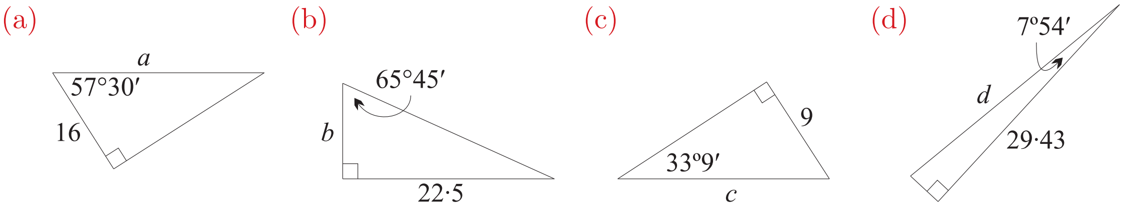
(a)


(b)


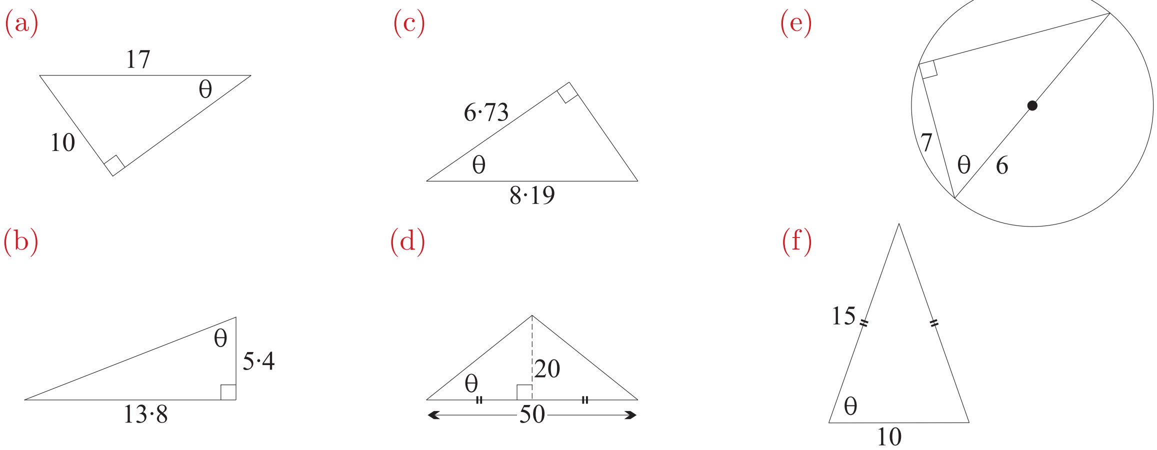
(c)


(d)


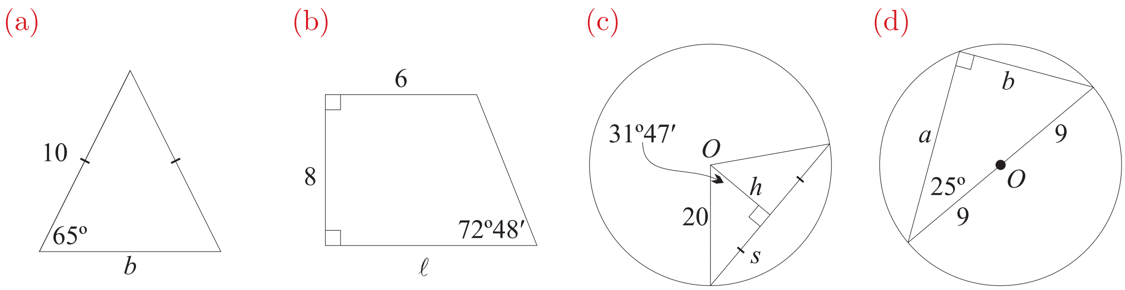
15. Find, correct to two decimal places, the lengths of the sides marked with pronumerals.



16. Find θ, correct to the nearest minute, in each diagram below.



17. Find the value of each pronumeral, correct to three decimal places.



CHALLENGE

18. It is given that α is an acute angle and that $\tan\alpha = \frac{\sqrt{5}}{2}$.
(a) Draw a right-angled triangle showing this information.
(b) Use Pythagoras' theorem to find the length of the unknown side.
(c) Hence write down the exact values of $\sin\alpha$ and $\cos\alpha$.
(d) Show that $\sin^2\alpha + \cos^2\alpha = 1$.

19. Suppose that β is an acute angle and $\sec\beta = \frac{\sqrt{11}}{3}$.
(a) Find the exact values of: (i) $\operatorname{cosec}\beta$ (ii) $\cot\beta$
(b) Show that $\operatorname{cosec}^2\beta - \cot^2\beta = 1$.

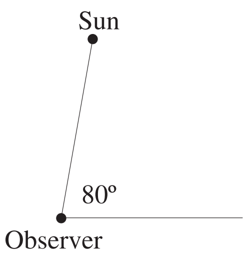
20. Find, without using a calculator, the value of:
(a) $\sin 45^\circ \cos 45^\circ + \sin 30^\circ$
(b) $\sin 60^\circ \cos 30^\circ - \cos 60^\circ \sin 30^\circ$
(c) $1 + \tan^2 60^\circ$
(d) $\operatorname{cosec}^2 30^\circ - \cot^2 30^\circ$

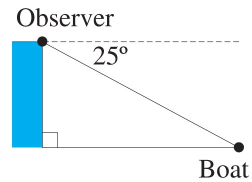
21. Without using a calculator, show that:
(a) $1 + \tan^2 45^\circ = \sec^2 45^\circ$
(b) $2\sin 30^\circ \cos 30^\circ = \sin 60^\circ$
(c) $\cos^2 60^\circ - \cos^2 30^\circ = -\frac{1}{2}$
(d) $\dfrac{2\tan 30^\circ}{1 - \tan^2 30^\circ} = \tan 60^\circ$

5 B Problems Involving Right-Angled Triangles

The trigonometry developed so far can be used to solve practical problems involving right-angled triangles. The two examples below are typical of problems involving compass bearings and angle of elevation or depression.

Angles of Elevation and Depression: Angles of elevation and depression are always measured from the horizontal, and are always acute angles.





The *angle of elevation* of the sun in the diagram above is 80°, because the angle at the observer between the sun and the horizontal is 80°.

For an observer on top of the cliff, the *angle of depression* of the boat is 25°, because the angle at the observer between boat and horizontal is 25°.

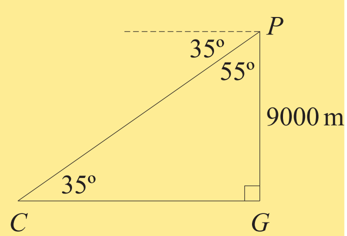
WORKED EXERCISE:

From a plane flying at 9000 metres above level ground, I can see a church at an angle of depression of 35° from the cabin of the plane. Find how far the church is from the plane, correct to the nearest 100 metres:

(a) measured along the ground, (b) measured along the line of sight.

SOLUTION:

The situation is illustrated in the diagram by $\triangle PGC$.



(a) $\dfrac{GC}{9000} = \tan 55°$

$GC = 9000 \tan 55°$

$\fallingdotseq 12\,900$ metres

(b) $\dfrac{PC}{9000} = \dfrac{1}{\cos 55°}$

$PC = \dfrac{9000}{\cos 55°}$

$\fallingdotseq 15\,700$ metres

WORKED EXERCISE:

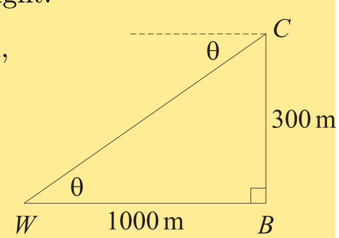
A walker on level ground is 1 kilometre from the base of a 300-metre vertical cliff.

(a) Find, correct to the nearest minute, the angle of elevation of the top.

(b) Find, correct to the nearest metre, the line-of-sight distance to the top.

SOLUTION:

The situation is illustrated by $\triangle CWB$ in the diagram to the right.



(a) $\tan\theta = \dfrac{CB}{WB}$

$= \dfrac{300}{1000}$

$= \frac{3}{10}$,

$\theta \fallingdotseq 16°42'$

(b) Using Pythagoras' theorem,

$CW^2 = CB^2 + WB^2$

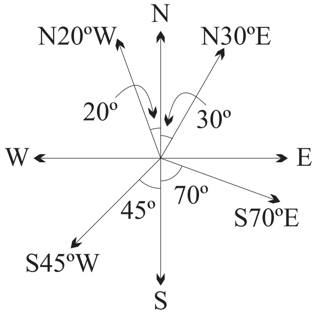
$= 1000^2 + 300^2$

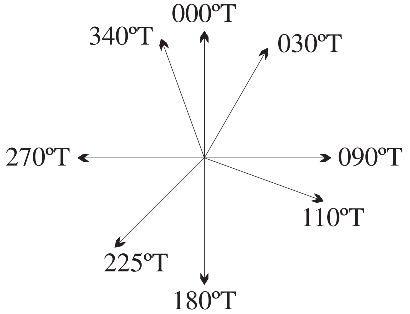
$= 1\,090\,000$

$CW \fallingdotseq 1044$ metres.

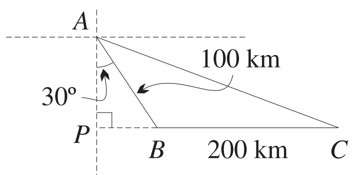
Compass Bearings and True Bearings: *Compass bearings* are based on north, south, east and west. Any other direction is specified by the deviation from north or south towards the east or west. The diagram to the left below gives four examples. Note that S45°W can also be written simply as SW (that is, south west).

True bearings are measured clockwise from north. The diagram to the right below gives the same four directions expressed as true bearings. Three digits are normally used, even for angles less than 100°.





WORKED EXERCISE: [Compass bearings and true bearings]
A plane flying at 400 km per hour flies from A to B in a direction S30°E for 15 minutes. The plane then turns sharply to fly due east for 30 minutes to C.



(a) Find how far south and east of A the point B is.
(b) Find the true bearing of C from A, correct to the nearest degree.

SOLUTION:

(a) The distances AB and BC are 100 km and 200 km respectively.
Working in $\triangle PAB$,

$$\frac{PB}{100} = \sin 30^\circ$$
$$PB = 100\sin 30^\circ$$
$$= 50\text{ km},$$
and $$AP = 100\cos 30^\circ$$
$$= 50\sqrt{3}\text{ km}.$$

(b) Using opposite over adjacent in $\triangle PAC$,

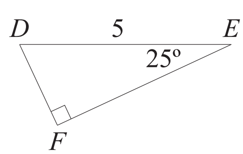
$$\tan \angle PAC = \frac{PC}{AP}$$
$$= \frac{50+200}{50\sqrt{3}}$$
$$= \frac{5}{\sqrt{3}}$$
$$\angle PAC \doteqdot 71^\circ.$$

Hence the bearing of C from A is about 109°T.

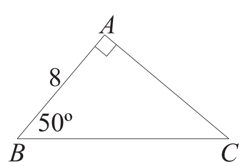
Exercise 5B

1. In each triangle below, find all unknown sides, correct to one decimal place, and all unknown angles, correct to the nearest degree where necessary.

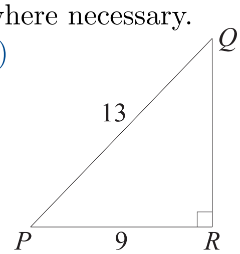
(a)



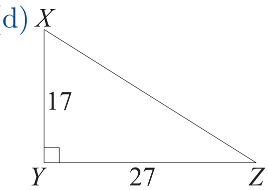
(b)



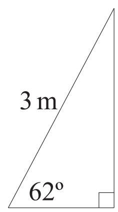
(c)



(d)

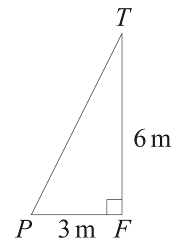


2.



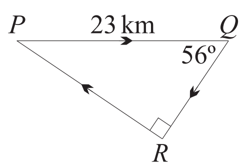
A ladder of length 3 metres is leaning against a wall and is inclined at 62° to the ground. How far does it reach up the wall? (Answer in metres correct to two decimal places.)

3.



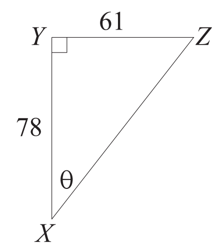
Determine, correct to the nearest degree, the angle of elevation of the top T of a 6 metre flagpole FT from a point P on level ground 3 metres from F.

4.



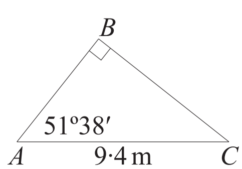
Ben cycles from P to Q to R and then back to P in a road-race. Find, correct to the nearest kilometre, the distance he has ridden.

5.



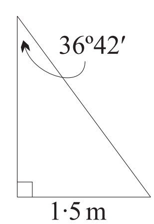
A ship sails 78 nautical miles due north from X to Y, then 61 nautical miles due east from Y to Z. Find θ, the bearing of Z from X, correct to the nearest degree.

6.



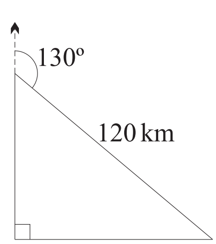
A tree snapped into two sections AB and BC in high winds and then fell. The section BA is inclined at 51°38′ to the horizontal and AC is 9·4 metres long. Find the height of the original tree, in metres correct to one decimal place.

7.



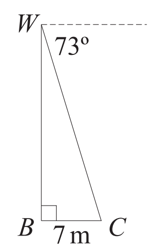
A ladder makes an angle of 36°42′ with a wall, and its foot is 1·5 metres out from the base of the wall. Find the length of the ladder, in metres correct to one decimal place.

8.



Eleni drives 120 km on a bearing of 130°T. She then drives due west until she is due south of her starting point. How far is she from her starting point, correct to the nearest kilometre?

9.

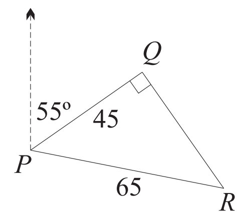


John is looking out a window W at a car C parked on the street below. If the angle of depression of C from W is 73° and the car is 7 metres from the base B of the building, find the height WB of the window, correct to the nearest metre.

10. Port Q is 45 nautical miles from port P on a bearing of 055°T. Port R is 65 nautical miles from port P, and $\angle PQR = 90°$.

(a) Find $\angle QPR$ to the nearest degree.

(b) Hence find the bearing of R from P, correct to the nearest degree.



DEVELOPMENT

11. A ladder of length 5 metres is placed on level ground against a vertical wall. If the foot of the ladder is 1·5 metres from the base of the wall, find, correct to the nearest degree, the angle at which the ladder is inclined to the ground.

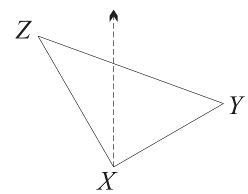
12. Find, correct to the nearest tenth of a metre, the height of a tower, if the angle of elevation of the top of the tower is $64°48'$ from a point on horizontal ground 10 metres from the base of the tower.

13. A boat is 200 metres out to sea from a vertical cliff of height 40 metres. Find, correct to the nearest degree, the angle of depression of the boat from the top of the cliff.

14. The bearings of towns Y and Z from town X are 060°T and 330°T respectively.

(a) Show that $\angle ZXY = 90°$.

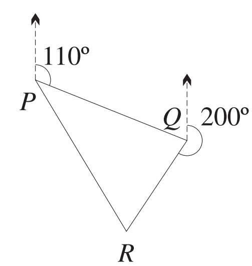
(b) Given that town Z is 80 km from town X and that $\angle XYZ = 50°$, find, correct to the nearest kilometre, how far town Y is from town X.



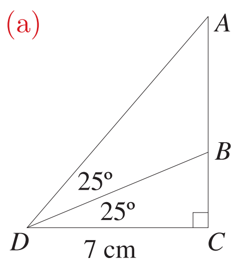
15. A ship leaves port P and travels 150 nautical miles to port Q on a bearing of 110°T. It then travels 120 nautical miles to port R on a bearing of 200°T.

(a) Explain why $\angle PQR = 90°$.

(b) Find, correct to 1°, the bearing of port R from port P.

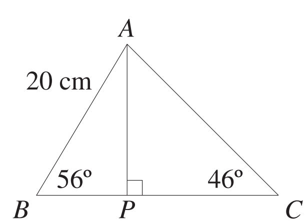


16. (a)



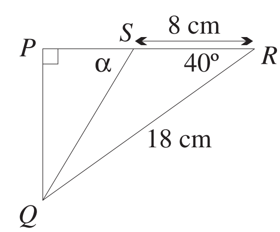
Show that $AC = 7\tan 50°$ and $BC = 7\tan 25°$, and hence find the length AB, correct to 1 mm.

(b)



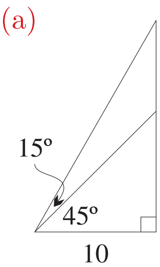
Show that $AP = 20\sin 56°$, and hence find the length of PC, giving your answer correct to 1 cm.

(c)



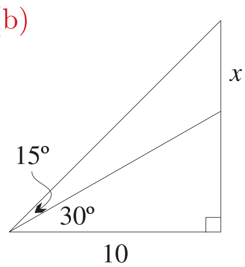
Show that $PR = 18\cos 40°$, find an expression for PQ, and hence find the angle α, correct to the nearest minute.

17. (a)



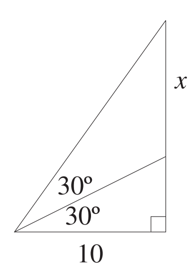
Show that $x = 10(\sqrt{3} - 1)$.

(b)



Show that $x = \frac{10}{3}(3 - \sqrt{3})$.

(c)



Show that $x = \frac{20}{3}\sqrt{3}$.

CHALLENGE

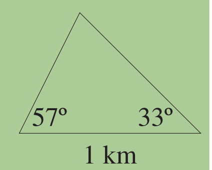
18. Answer correct to four significant figures, or correct to the nearest minute.

(a) A triangle has sides of 7 cm, 7 cm and 5 cm. What are the sizes of its angles?

(b) A rectangle has dimensions 7 cm × 12 cm. At what acute angle do the diagonals meet?

(c) The diagonals of a rhombus are 16 cm and 10 cm. Find the vertex angles.

19. From the ends of a straight horizontal road 1 km long, a balloon directly above the road is observed to have angles of elevation of 57° and 33° respectively. Find, correct to the nearest metre, the height of the balloon above the road.



20. From a ship sailing due north, a lighthouse is observed to be on a bearing of 042°T. Later, when the ship is 2 nautical miles from the lighthouse, the bearing of the lighthouse from the ship is 148°T. Find, correct to three significant figures, the distance of the lighthouse from the initial point of observation.

5 C Trigonometric Functions of a General Angle

The definitions of the trigonometric functions given in Section 5A only apply to acute angles, because the other two angles of a right-angled triangle have to be acute angles.

This section introduces more general definitions based on circles in the coordinate plane. The new definitions will apply to any angle, but will, of course, give the same values at acute angles as the previous definitions.

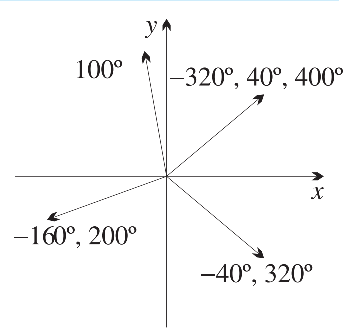
Putting a General Angle on the Cartesian Plane: Let θ be any angle — possibly negative, possibly obtuse or reflex, possibly greater than 360°. We shall associate with θ a ray with vertex at the origin.

6 THE RAY CORRESPONDING TO θ:

- The positive direction of the x-axis is the ray representing the angle 0°.
- For all other angles, rotate this ray anticlockwise through an angle θ.

If the angle is negative, the ray is rotated backwards, which means clockwise.

Here are some examples of angles and the corresponding rays. The angles have been written at the ends of the arrows representing the rays.



Notice that one ray can correspond to many angles. For example, all the following angles have the same ray as 40°:

$$\ldots,\ -680°, -320°,\ 400°,\ 760°,\ \ldots$$

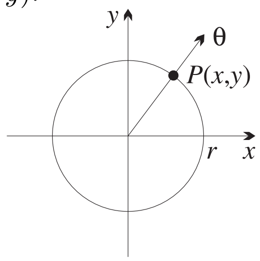
A given ray thus corresponds to infinitely many angles, all differing by multiples of 360°.

7 CORRESPONDING ANGLES AND RAYS:

- To each angle, there corresponds exactly one ray.
- To each ray, there correspond infinitely many angles, all differing from each other by multiples of 360°.

The Definitions of the Trigonometric Functions: Let θ be any angle.

Construct a circle with centre the origin and any positive radius r.
Let the ray corresponding to θ intersect the circle at the point $P(x, y)$.
The six trigonometric functions are now defined as follows:



8 DEFINITIONS OF THE SIX TRIGONOMETRIC FUNCTIONS:

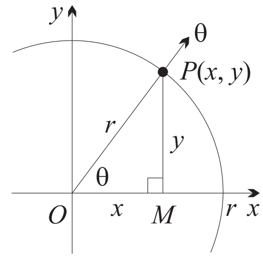
$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$

$$\operatorname{cosec}\theta = \frac{r}{y} \qquad \sec\theta = \frac{r}{x} \qquad \cot\theta = \frac{x}{y}$$

NOTE: We chose r to be 'any positive radius'. If a different radius were chosen, the two figures would be similar, so the lengths x, y and r would stay in the same ratio. Since the definitions depend only on the ratios of the lengths, the values of the trigonometric functions would not change.

Agreement with the Earlier Definition: Let θ be an acute angle.

Construct the ray corresponding to θ.
Let the perpendicular from P meet the x-axis at M.
Then $\theta = \angle POM$, so relating the sides to the angle θ,



$$\text{hyp} = OP = r, \qquad \text{opp} = PM = y, \qquad \text{adj} = OM = x.$$

Hence the old and the new definitions are in agreement.

NOTE: Most people find that the diagram above is the easiest way to learn the new definitions of the trigonometric functions. Take the old definitions in terms of hypotenuse, opposite and adjacent sides, and make the replacements

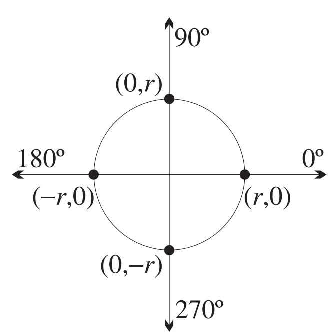
$$\text{hyp} \longleftrightarrow r, \qquad \text{opp} \longleftrightarrow y, \qquad \text{adj} \longleftrightarrow x.$$

Boundary Angles: Integer multiples of 90°, that is $\ldots, -90^\circ, 0^\circ, 90^\circ, 180^\circ, \ldots$, are called *boundary angles* because they lie on the boundaries between quadrants.

The values of the trigonometric functions at these boundary angles are not always defined, and are 0, 1 or -1 when they are defined. The diagram to the right below can be used to calculate them, and the results are shown in the table.

9 THE BOUNDARY ANGLES:

θ	0°	90°	180°	270°
x	r	0	$-r$	0
y	0	r	0	$-r$
r	r	r	r	r
$\sin\theta$	0	1	0	-1
$\cos\theta$	1	0	-1	0
$\tan\theta$	0	$*$	0	$*$
$\operatorname{cosec}\theta$	$*$	1	$*$	-1
$\sec\theta$	1	$*$	-1	$*$
$\cot\theta$	$*$	0	$*$	0



In practice, the answer to any question about the values of the trigonometric functions at these boundary angles should be read off the graphs of the functions. These graphs need to be known very well indeed.

The Domains of the Trigonometric Functions: The trigonometric functions are defined everywhere except where the denominator is zero.

10

DOMAINS OF THE TRIGONOMETRIC FUNCTIONS:

- $\sin\theta$ and $\cos\theta$ are defined for all angles θ.
- $\tan\theta$ and $\sec\theta$ are undefined when $x = 0$, that is, when
  $\theta = \dots,\ -90^\circ,\ 90^\circ,\ 270^\circ,\ 450^\circ,\ \dots .$
- $\cot\theta$ and $\operatorname{cosec}\theta$ are undefined when $y = 0$, that is, when
  $\theta = \dots,\ -180^\circ,\ 0^\circ,\ 180^\circ,\ 360^\circ,\ \dots .$

Exercise 5C

1. On a number plane, draw rays representing the following angles.

(a) 40° (b) 110° (c) 190° (d) 290° (e) 420° (f) 500°

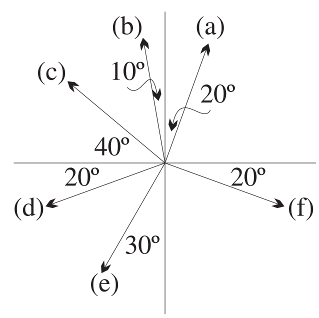
2. On another number plane, draw rays representing the following angles.

(a) -50° (b) -130° (c) -250° (d) -350° (e) -440° (f) -550°

3. For each of the angles in question 1, write down the size of the negative angle between -360° and 0° that is represented by the same ray.

4. For each of the angles in question 2, write down the size of the positive angle between 0° and 360° that is represented by the same ray.

5. Write down two positive angles between 0° and 720° and two negative angles between -720° and 0° that are represented by each of the rays in the diagram to the right.

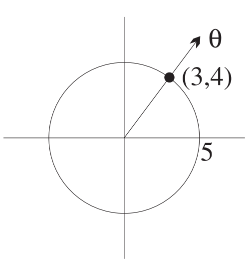


6. Use the definitions

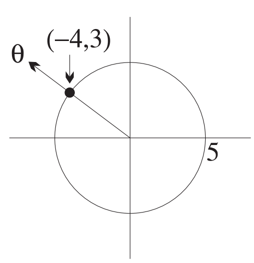
$$\sin\theta = \frac{y}{r} \qquad \text{and} \qquad \cos\theta = \frac{x}{r} \qquad \text{and} \qquad \tan\theta = \frac{y}{x}$$

to write down the values of the six trigonometric ratios of the angle θ in each diagram.

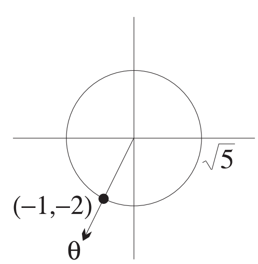
(a)



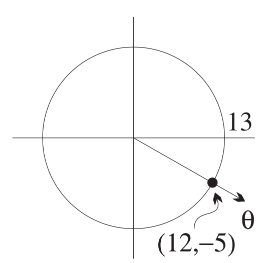
(b)



(c)

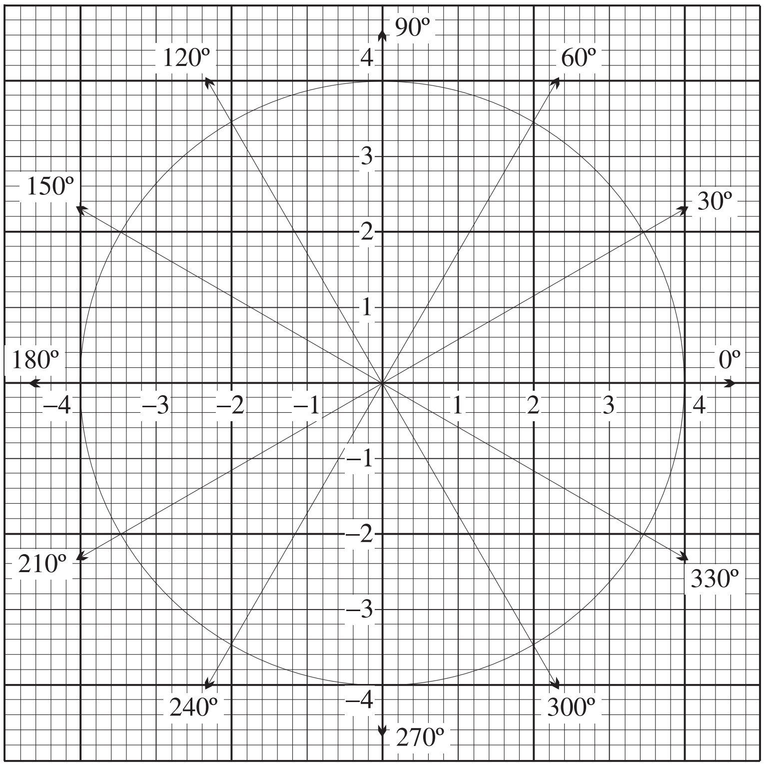


(d)



DEVELOPMENT

7. [The graphs of $\sin\theta$, $\cos\theta$ and $\tan\theta$] The diagram shows angles from 0° to 360° at 30° intervals. The circle has radius 4 units.



(a) Use the diagram and the definitions of the three trigonometric ratios to complete the following table. Measure the values of x and y correct to two decimal places, and use your calculator only to perform the necessary divisions.

θ	$-30°$	$0°$	$30°$	$60°$	$90°$	$120°$	$150°$	$180°$	$210°$	$240°$	$270°$	$300°$	$330°$	$360°$	$390°$
x															
y															
r															
$\sin\theta$															
$\cos\theta$															
$\tan\theta$															

(b) Use your calculator to check the accuracy of the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$ that you obtained in part (a).

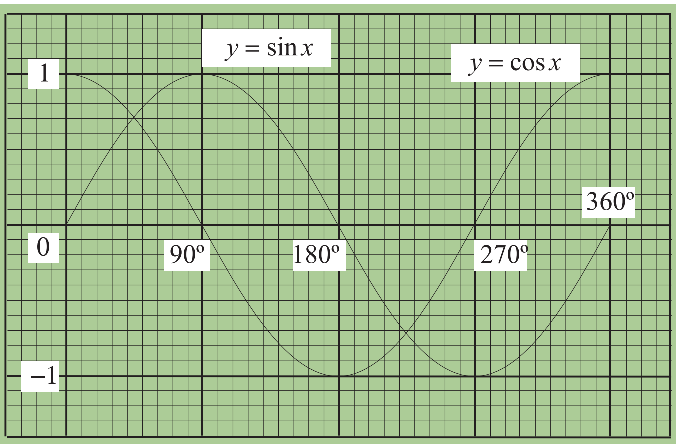
(c) Using the table of values in part (a), graph the curves $y = \sin\theta$, $y = \cos\theta$ and $y = \tan\theta$ as accurately as possible on graph paper. Use the following scales:

On the horizontal axis: let 2 mm represent 10°.

On the vertical axis: let 2 cm represents 1 unit.

CHALLENGE

8.



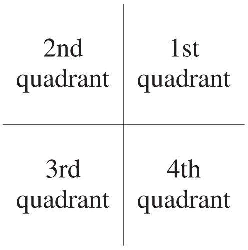
(a) Read off the diagram, correct to two decimal places where necessary, the values of:
(i) $\cos 60°$ (iii) $\sin 72°$ (v) $\sin 144°$ (vii) $\cos 153°$ (ix) $\sin 234°$
(ii) $\sin 210°$ (iv) $\cos 18°$ (vi) $\cos 36°$ (viii) $\sin 27°$ (x) $\cos 306°$

(b) Find from the graphs two values of x between 0° and 360° for which:
(i) $\sin x = 0{\cdot}5$ (iii) $\sin x = 0{\cdot}9$ (v) $\sin x = 0{\cdot}8$ (vii) $\sin x = -0{\cdot}4$
(ii) $\cos x = -0{\cdot}5$ (iv) $\cos x = 0{\cdot}6$ (vi) $\cos x = -0{\cdot}8$ (viii) $\cos x = -0{\cdot}3$

(c) Find two values of x between 0° and 360° for which $\sin x = \cos x$.

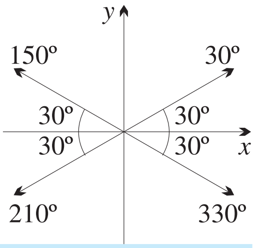
5 D The Quadrant, the Related Angle and the Sign

Symmetry in the x-axis and the y-axis is an essential aspect of trigonometric functions. This section uses that symmetry to express the values of the trigonometric functions of any angle in terms of trigonometric functions of acute angles.



The diagram shows the conventional anticlockwise numbering of the four quadrants of the coordinate plane. Acute angles are in the first quadrant and obtuse angles are in the second quadrant.

The Quadrant and the Related Acute Angle: The diagram to the right shows the four rays corresponding to the four angles 30°, 150°, 210°and 330°. These four rays lie in each of the four quadrants of the plane, and they all make the same acute angle, 30°, with the x-axis. The four rays are thus the reflections of each other in the two axes.

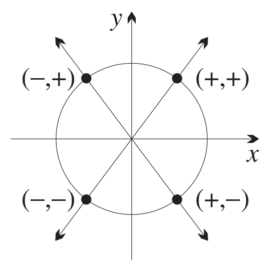


11 QUADRANT AND RELATED ACUTE ANGLE: Let θ be any angle.
- The *quadrant* of θ is the quadrant (1, 2, 3 or 4) in which the ray lies.
- The *related acute angle* of θ is the acute angle between the ray and the x-axis.

Each of the four angles above has the same related acute angle, 30°. Notice that θ and its related angle are only the same when θ is an acute angle.

The Signs of the Trigonometric Functions: The signs of the trigonometric functions depend only on the signs of x and y. (The radius r is always positive.) The signs of x and y depend in turn only on the quadrant in which the ray lies. Thus we can easily compute the signs of the trigonometric functions from the accompanying diagram and the definitions.

quadrant	1st	2nd	3rd	4th
x	+	−	−	+
y	+	+	−	−
r	+	+	+	+
$\sin\theta$	+	+	−	−
$\cos\theta$	+	−	−	+
$\tan\theta$	+	−	+	−
$\operatorname{cosec}\theta$	+	+	−	−
$\sec\theta$	+	−	−	+
$\cot\theta$	+	−	+	−



In NSW, these results are usually remembered by the phrase:

12 SIGNS OF THE TRIGONOMETRIC FUNCTIONS: '<u>A</u>ll <u>S</u>tations <u>T</u>o <u>C</u>entral'

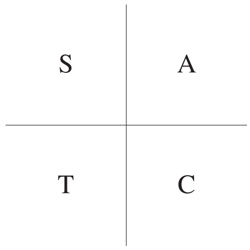
indicating that the four letters A, S, T and C are placed successively in the four quadrants as shown. The significance of the letters is:

A means *all* six functions are positive,

S means only *sine* (and cosecant) are positive,

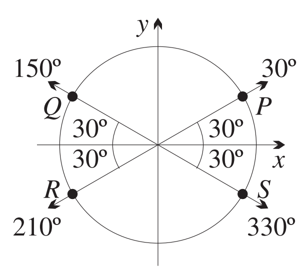
T means only *tangent* (and cotangent) are positive,

C means only *cosine* (and secant) are positive.



Study each of the graphs constructed in the previous exercise to see how the table of signs above, and the ASTC rule, agree with your observations about when the graph is above the x-axis and when it is below.

The Angle and the Related Acute Angle: In the diagram to the right, a circle of radius r has been added to the earlier diagram showing the four angles 30°, 150°, 210° and 330°.



The four points P, Q, R and S where the rays meet the circle are all reflections of each other in the x-axis and y-axis. Because of this symmetry, the coordinates of these four points are identical, apart from their signs.

Hence the trigonometric functions of these angles will all be the same too, except that the signs may be different.

13 THE ANGLE AND THE RELATED ACUTE ANGLE:
- The trigonometric functions of any angle θ are the same as the trigonometric functions of its related acute angle, apart from a possible change of sign.
- The sign is best found using the ASTC diagram.

Evaluating the Trigonometric Functions of Any Angle: This gives a straightforward way of evaluating the trigonometric functions of any angle.

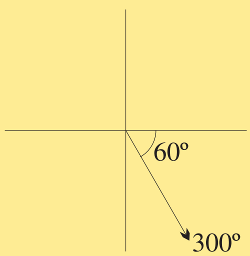
14 TRIGONOMETRIC FUNCTIONS OF ANY ANGLE: Draw a quadrant diagram, then:
- place the ray in the correct quadrant, and use the ASTC rule to work out the sign of the answer,
- find the related acute angle, and work out the value of the trigonometric function at this related angle.

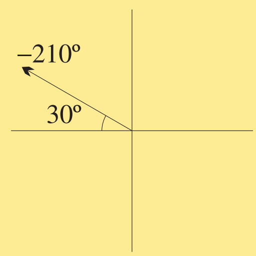
WORKED EXERCISE:

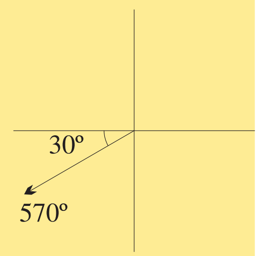
Find the exact value of:

(a) $\tan 300°$ (b) $\sin(-210°)$ (c) $\cos 570°$

SOLUTION:







(a) $300°$ is in quadrant 4, the related angle is $60°$, so $\tan 300° = -\tan 60°$
$= -\sqrt{3}\,.$

(b) $-210°$ is in quadrant 2, the related angle is $30°$, so $\sin(-210°) = +\sin 30°$
$= \frac{1}{2}.$

(c) $570°$ is in quadrant 3, the related angle is $30°$, so $\cos 570° = -\cos 30°$
$= -\dfrac{\sqrt{3}}{2}\,.$

NOTE: The calculator will give approximate values of the trigonometric functions without any need to find the related acute angle. It will *not* give exact values, however, when these values involve surds.

General Angles with Pronumerals: This quadrant-diagram method can be used to generate formulae for expressions such as $\sin(180° + A)$ or $\cot(360° - A)$. The trick is to deal with A on the quadrant diagram *as if it were acute.*

15 SOME FORMULAE WITH GENERAL ANGLES:

$\sin(180° - A) = \sin A$ $\quad\sin(180° + A) = -\sin A$ $\quad\sin(360° - A) = -\sin A$

$\cos(180° - A) = -\cos A$ $\quad\cos(180° + A) = -\cos A$ $\quad\cos(360° - A) = \cos A$

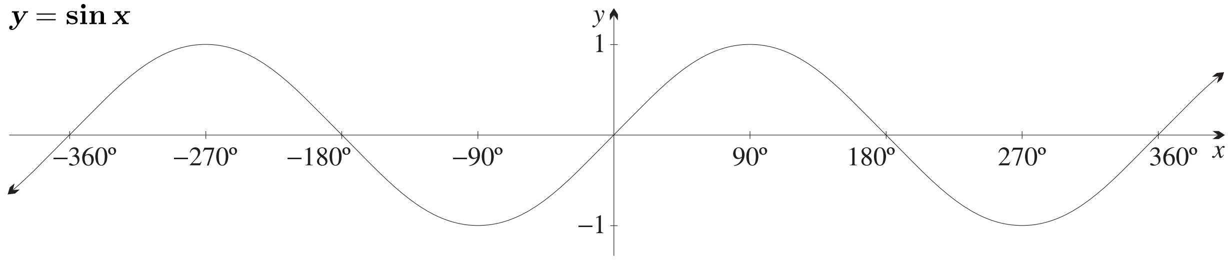
$\tan(180° - A) = -\tan A$ $\quad\tan(180° + A) = \tan A$ $\quad\tan(360° - A) = -\tan A$

Some people prefer to learn this list of identities to evaluate trigonometric functions, but this seems unnecessary when the quadrant-diagram method is so clear.

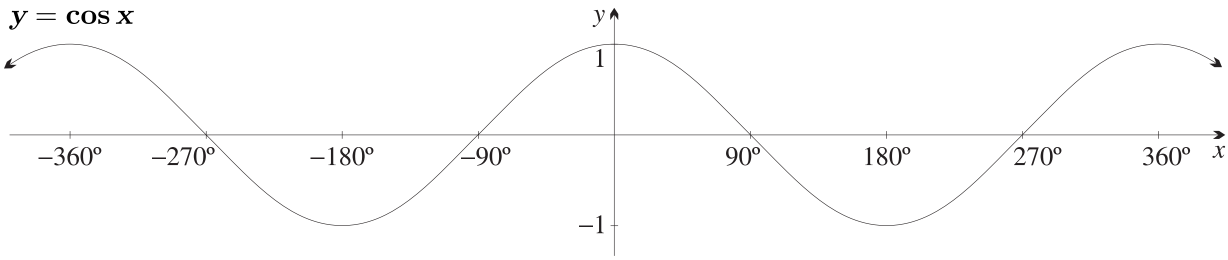
The Graphs of the Six Trigonometric Functions: The diagrams on the next page show the graphs of the six trigonometric functions over the domain $-450° \le x \le 450°$. With this extended domain, it becomes clear how the graphs are built up by infinite repetition of a simple element.

The sine and cosine graphs are waves, and they are the basis of all mathematics that deals with waves. The later trigonometry in this course will mostly deal with these wave properties.

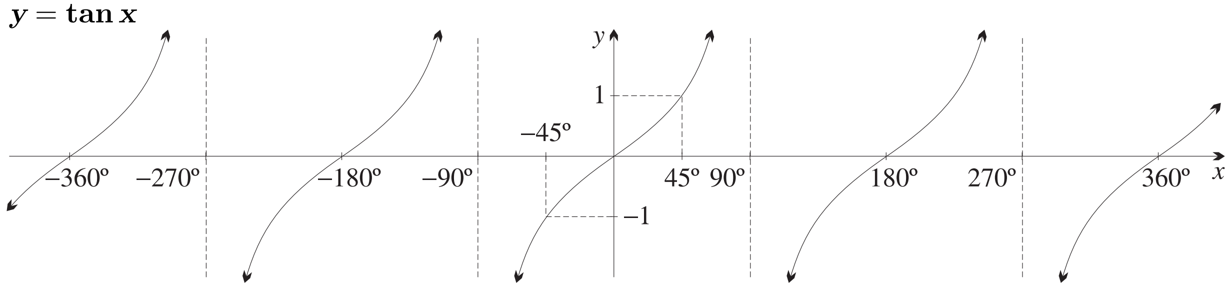
$y = \sin x$



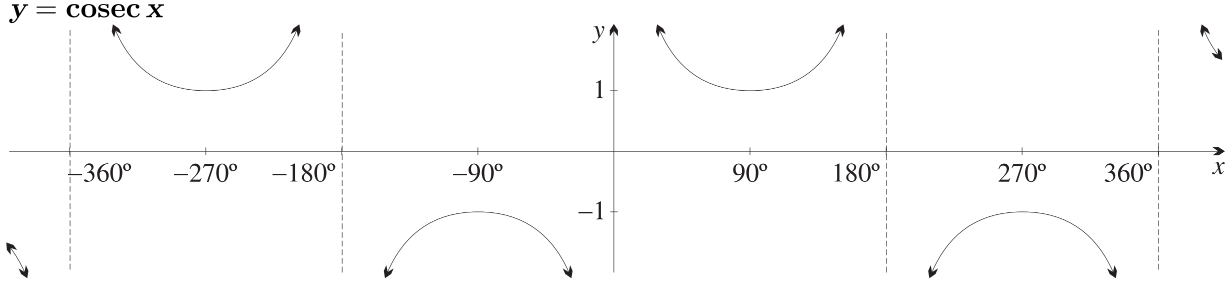
$y = \cos x$



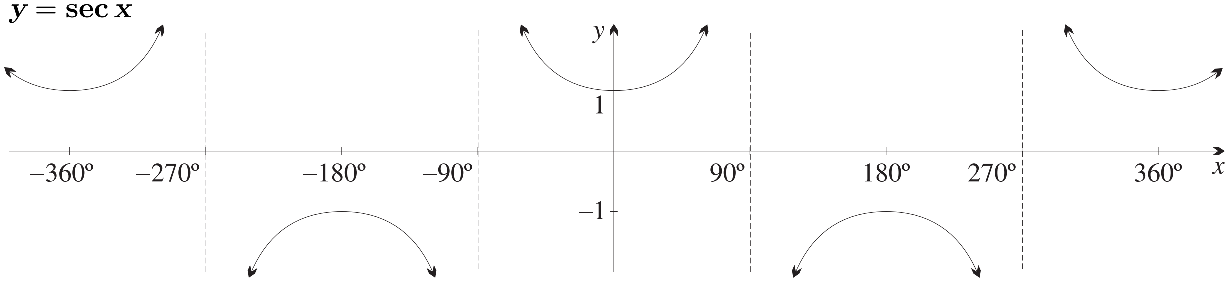
$y = \tan x$



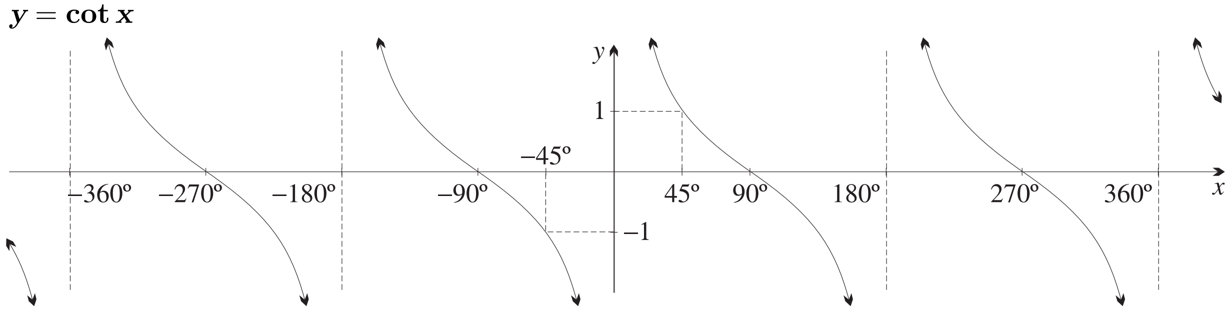
$y = \operatorname{cosec} x$



$y = \sec x$



$y = \cot x$



Exercise 5D

1. Use the ASTC rule to determine the sign (+ or −) of each trigonometric ratio.

(a) $\sin 20°$ (e) $\tan 250°$ (i) $\sin 340°$ (m) $\tan 170°$
(b) $\cos 50°$ (f) $\sin 310°$ (j) $\cos 350°$ (n) $\sin 110°$
(c) $\cos 100°$ (g) $\sin 200°$ (k) $\tan 290°$ (o) $\tan 80°$
(d) $\tan 140°$ (h) $\cos 280°$ (l) $\cos 190°$ (p) $\cos 170°$

2. Find the related acute angle of each angle.

(a) $10°$ (c) $310°$ (e) $80°$ (g) $290°$ (i) $350°$
(b) $150°$ (d) $200°$ (f) $250°$ (h) $100°$ (j) $160°$

3. Write each trigonometric ratio as the ratio of an acute angle with the correct sign attached.

(a) $\tan 130°$ (d) $\tan 260°$ (g) $\cos 185°$ (j) $\sin 85°$
(b) $\cos 310°$ (e) $\cos 170°$ (h) $\sin 125°$ (k) $\cos 95°$
(c) $\sin 220°$ (f) $\sin 320°$ (i) $\tan 325°$ (l) $\tan 205°$

4. Use the trigonometric graphs to find the values (if they exist) of these trigonometric ratios of boundary angles.

(a) $\sin 90°$ (d) $\tan 0°$ (g) $\sin 270°$ (j) $\cos 270°$
(b) $\cos 180°$ (e) $\sin 90°$ (h) $\tan 270°$ (k) $\tan 90°$
(c) $\cos 90°$ (f) $\cos 0°$ (i) $\sin 180°$ (l) $\tan 180°$

5. Find the exact value of:

(a) $\sin 60°$ (d) $\sin 300°$ (g) $\cos 225°$ (j) $\tan 150°$
(b) $\sin 120°$ (e) $\cos 45°$ (h) $\cos 315°$ (k) $\tan 210°$
(c) $\sin 240°$ (f) $\cos 135°$ (i) $\tan 30°$ (l) $\tan 330°$

6. Find the exact value of:

(a) $\cos 120°$ (d) $\sin 135°$ (g) $\tan 315°$ (j) $\cos 150°$
(b) $\tan 225°$ (e) $\tan 240°$ (h) $\cos 300°$ (k) $\sin 210°$
(c) $\sin 330°$ (f) $\cos 210°$ (i) $\sin 225°$ (l) $\tan 300°$

DEVELOPMENT

7. Find the exact value of:

(a) $\operatorname{cosec} 150°$ (c) $\cot 120°$ (e) $\sec 330°$
(b) $\sec 225°$ (d) $\cot 210°$ (f) $\operatorname{cosec} 300°$

8. Use the trigonometric graphs to find (if they exist):

(a) $\sec 0°$ (c) $\sec 90°$ (e) $\cot 90°$
(b) $\operatorname{cosec} 270°$ (d) $\operatorname{cosec} 180°$ (f) $\cot 180°$

9. Find the related acute angle of each angle.

(a) $-60°$ (c) $-150°$ (e) $430°$ (g) $590°$
(b) $-200°$ (d) $-300°$ (f) $530°$ (h) $680°$

10. Find the exact value of:

(a) $\cos(-60°)$ (d) $\sin(-315°)$ (g) $\tan 420°$ (j) $\sin 690°$
(b) $\sin(-120°)$ (e) $\tan(-210°)$ (h) $\cos 510°$ (k) $\cos 600°$
(c) $\tan(-120°)$ (f) $\cos(-225°)$ (i) $\sin 495°$ (l) $\tan 585°$

11. Given that $\sin 25° \doteqdot 0{\cdot}42$ and $\cos 25° \doteqdot 0{\cdot}91$, write down approximate values, without using a calculator, for:

(a) $\sin 155°$
(b) $\cos 205°$
(c) $\cos 335°$
(d) $\sin 335°$
(e) $\sin 205° - \cos 155°$
(f) $\cos 385° - \sin 515°$

12. Given that $\tan 35° \doteqdot 0{\cdot}70$ and $\sec 35° \doteqdot 1{\cdot}22$, write down approximate values, without using a calculator, for:

(a) $\tan 145°$
(b) $\sec 215°$
(c) $\tan 325°$
(d) $\tan 215° + \sec 145°$
(e) $\sec 325° + \tan 395°$
(f) $\sec(-145)° - \tan(-215)°$

CHALLENGE

13. Show by substitution into LHS and RHS that each trigonometric identity is satisfied by the given values of the angles.

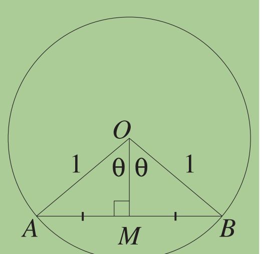
(a) Show that $\sin 2\theta = 2\sin\theta\cos\theta$, when $\theta = 150°$.
(b) Show that $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$, when $\theta = 135°$.
(c) Show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, when $\theta = 225°$.
(d) Show that $\sin(A+B) = \sin A\cos B + \cos A\sin B$, when $A = 300°$ and $B = 240°$.
(e) Show that $\tan(A-B) = \dfrac{\tan A - \tan B}{1+\tan A\tan B}$, when $A = 330°$ and $B = 210°$.

14. Write as a trigonometric ratio of A, with the correct sign attached:

(a) $\sin(-A)$
(b) $\cos(-A)$
(c) $\tan(-A)$
(d) $\sec(-A)$
(e) $\sin(180° - A)$
(f) $\sin(360° - A)$
(g) $\cos(180° - A)$
(h) $\tan(180° + A)$
(i) $\sec(180° + A)$
(j) $\operatorname{cosec}(360° - A)$
(k) $\cot(180° - A)$
(l) $\sec(360° - A)$

15. [An earlier definition of the sine function]
The function $\sin\theta$ was earlier defined as the length of the 'semichord' that subtends an angle θ at the centre of a circle of radius 1.

Let AB be a chord of a circle with centre O and radius 1.
Let the chord AB subtend an angle 2θ at the centre O.
Prove that $\sin\theta = \frac{1}{2}AB$.

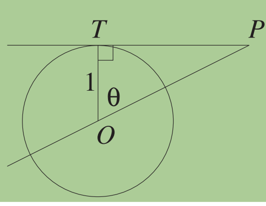


16. [The origin of the words 'secant' and 'tangent']
The word 'tangent' comes from the Latin *tangens*, meaning 'touching' — a tangent is a line touching the circle at one point.

The word 'secant' comes from the Latin *secans*, meaning 'cutting' — a secant is a line that cuts the circle at two points.

Let P be a point outside a circle of radius 1.
Let a tangent from P touch the circle at T.
Let PT subtend an angle θ at the centre O.

(a) Show that $PT = \tan\theta$.
(b) Show that $PO = \sec\theta$.



5 E Given One Trigonometric Function, Find Another

When the exact value of one trigonometric function is known for an angle, the exact values of the other trigonometric functions can easily be found using the circle diagram and Pythagoras' theorem.

16

GIVEN ONE TRIGONOMETRIC FUNCTION, FIND ANOTHER:
- Place a ray or rays on a circle diagram in the quadrants allowed in the question.
- Complete the triangle and use Pythagoras' theorem to find whichever of x, y and r is missing.

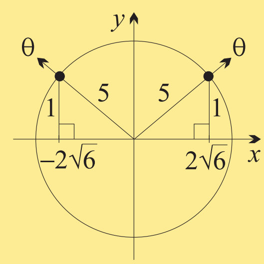
WORKED EXERCISE:

It is known that $\sin\theta = \frac{1}{5}$.

(a) Find the possible values of $\cos\theta$.

(b) Find $\cos\theta$ if it is also known that $\tan\theta$ is negative.

SOLUTION:



(a) The angle must be in quadrant 1 or 2, since $\sin\theta$ is positive.

Since $\sin\theta = \dfrac{y}{r} = \dfrac{1}{5}$, we can take $y = 1$ and $r = 5$,

so by Pythagoras' theorem, $x = \sqrt{24}$ or $-\sqrt{24}$,

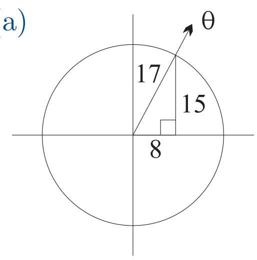
so $\cos\theta = \dfrac{2\sqrt{6}}{5}$ or $-\dfrac{2\sqrt{6}}{5}$.

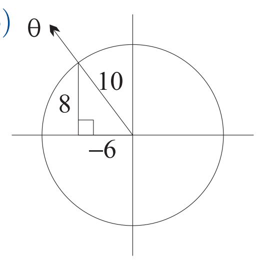
(b) Since $\tan\theta$ is negative, θ can only be in quadrant 2,

so $\cos\theta = -\dfrac{2\sqrt{6}}{5}$.

Exercise 5E

1. Write down the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$ in each part.

(a)


(b)


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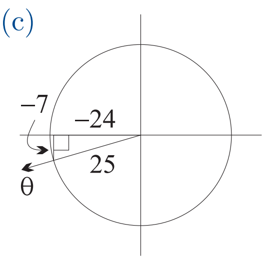

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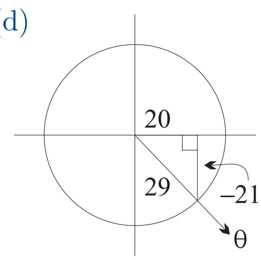
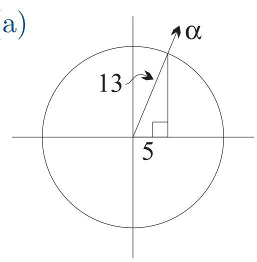


2. In each part use Pythagoras' theorem to find whichever of x, y or r is unknown. Then write down the values of $\sin\alpha$, $\cos\alpha$ and $\tan\alpha$.

(a)


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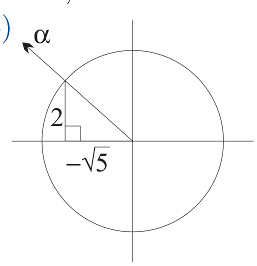

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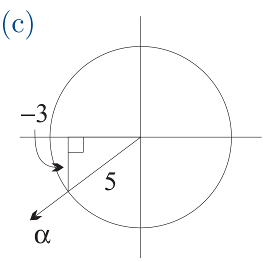
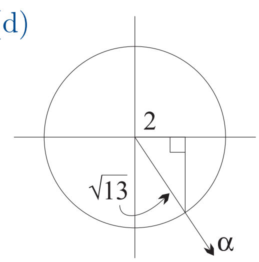
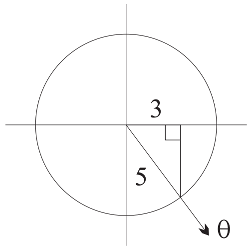


(d)


3. (a)

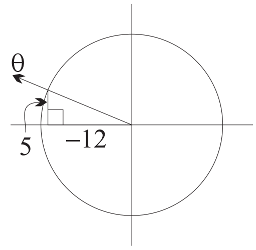


Let $\cos\theta = \frac{3}{5}$, where $270° < \theta < 360°$.

(i) Find $\sin\theta$.

(ii) Find $\tan\theta$.

(b)

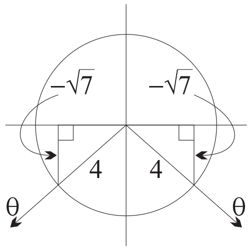


Let $\tan\theta = -\frac{5}{12}$, where θ is obtuse.

(i) Find $\sin\theta$.

(ii) Find $\cos\theta$.

4. (a)

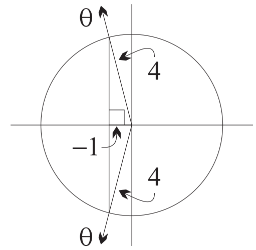


Suppose that $\sin\theta = -\frac{\sqrt{7}}{4}$.

(i) Find the possible values of $\cos\theta$.

(ii) Find the possible values of $\tan\theta$.

(b)



Suppose that $\cos\theta = -\frac{1}{4}$.

(i) Find the possible values of $\sin\theta$.

(ii) Find the possible values of $\tan\theta$.

DEVELOPMENT

5. If $\cos\alpha = \frac{4}{5}$ and $\sin\alpha < 0$, find $\tan\alpha$.

6. If $\tan\theta = -\frac{8}{15}$ and $\sin\theta > 0$, find $\cos\theta$.

7. If $\tan\alpha = \frac{1}{3}$, find the possible values of $\sin\alpha$.

8. If $\cos\theta = \frac{2}{\sqrt{5}}$, find the possible values of $\sin\theta$.

9. If $\sin A = -\frac{1}{3}$ and $\tan A < 0$, find $\sec A$.

10. If $\operatorname{cosec} B = \frac{7}{3}$ and $\cos B < 0$, find $\tan B$.

11. If $\sec C = -\frac{\sqrt{7}}{\sqrt{3}}$, find the possible values of $\cot C$.

12. If $\cot D = \frac{\sqrt{11}}{5}$, find the possible values of $\operatorname{cosec} D$.

CHALLENGE

13. Given that $\sin\theta = \dfrac{p}{q}$, with θ obtuse and p and q both positive, find $\cos\theta$ and $\tan\theta$.

14. If $\tan\alpha = k$, where $k > 0$, find the possible values of $\sin\alpha$ and $\sec\alpha$.

15. (a) Prove the algebraic identity $(1 - t^2)^2 + (2t)^2 = (1 + t^2)^2$.

(b) If $\cos x = \dfrac{1 - t^2}{1 + t^2}$ and x is acute, find expressions for $\sin x$ and $\tan x$.

5 F Trigonometric Identities

Working with the trigonometric functions requires knowledge of a number of formulae called *trigonometric identities*, which relate trigonometric functions to each other. This section introduces eleven trigonometric identities in four groups:

- the three *reciprocal identities*,
- the two *ratio identities*,
- the three *Pythagorean identities*,
- the three *identities concerning complementary angles*.

The Three Reciprocal Identities: It follows immediately from the definitions of the trigonometric functions in terms of x, y and r that:

17 THE RECIPROCAL IDENTITIES: For any angle θ:

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} \quad (\text{provided that } \sin\theta \neq 0)$$

$$\sec\theta = \frac{1}{\cos\theta} \quad (\text{provided that } \cos\theta \neq 0)$$

$$\cot\theta = \frac{1}{\tan\theta} \quad (\text{provided that } \tan\theta \neq 0 \text{ and } \cot\theta \neq 0)$$

NOTE: One cannot use the calculator to find $\cot 90^\circ$ or $\cot 270^\circ$ by first finding $\tan 90^\circ$ or $\tan 270^\circ$, because both are undefined. We already know, however, that

$$\cot 90^\circ = \cot 270^\circ = 0$$

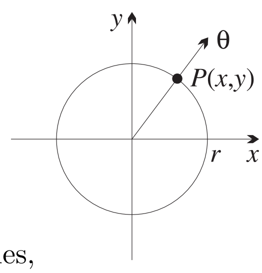
The Two Ratio Identities: Again using the definitions of the trigonometric functions:

18 THE RATIO IDENTITIES: For any angle θ:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad (\text{provided that } \cos\theta \neq 0)$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} \quad (\text{provided that } \sin\theta \neq 0)$$

The Three Pythagorean Identities: Since the point $P(x, y)$ lies on the circle with centre O and radius r, its coordinates satisfy



$$x^2 + y^2 = r^2.$$

Dividing through by r^2 gives $$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1,$$

then by the definitions, $$\sin^2\theta + \cos^2\theta = 1.$$

Dividing through by $\cos^2\theta$ and using the ratio and reciprocal identities,

$$\tan^2\theta + 1 = \sec^2\theta, \quad \text{provided that } \cos\theta \neq 0.$$

Dividing through instead by $\sin^2\theta$ gives $1 + \cot^2\theta = \operatorname{cosec}^2\theta$, provided that $\sin\theta \neq 0$.

These identities are called the *Pythagorean identities* because they rely on the circle equation $x^2 + y^2 = r^2$, which is a restatement of Pythagoras' theorem.

19 THE PYTHAGOREAN IDENTITIES: For any angle θ:

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta \quad (\text{provided that } \cos\theta \neq 0)$$

$$\cot^2\theta + 1 = \operatorname{cosec}^2\theta \quad (\text{provided that } \sin\theta \neq 0)$$

The Three Identities for Complementary Angles: The angles θ and $90^\circ - \theta$ are called *complementary angles*. Three trigonometric identities relate the values of the trigonometric functions at an angle θ and the complementary angle $90^\circ - \theta$.

20 THE COMPLEMENTARY IDENTITIES: For any angle θ:

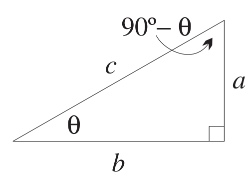
$$\cos(90^\circ - \theta) = \sin\theta$$
$$\cot(90^\circ - \theta) = \tan\theta \quad \text{(provided that } \tan\theta \text{ is defined)}$$
$$\operatorname{cosec}(90^\circ - \theta) = \sec\theta \quad \text{(provided that } \sec\theta \text{ is defined)}$$

For example, $\cos 20^\circ = \sin 70^\circ$, $\operatorname{cosec} 20^\circ = \sec 70^\circ$, $\cot 20^\circ = \tan 70^\circ$.

PROOF:

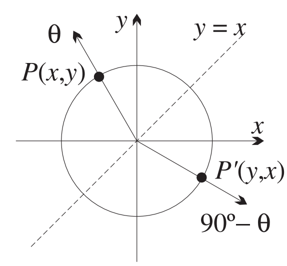
A. [Acute angles] The triangle to the right shows that when a right-angled triangle is viewed from $90^\circ - \theta$ instead of from θ, then the opposite side and the adjacent side are exchanged. Hence

$$\cos(90^\circ - \theta) = \frac{a}{c} = \sin\theta,$$
$$\cot(90^\circ - \theta) = \frac{a}{b} = \tan\theta,$$
$$\operatorname{cosec}(90^\circ - \theta) = \frac{c}{b} = \sec\theta.$$



B. [General angles] For general angles, we take the full circle diagram, and reflect it in the diagonal line $y = x$. Let P' be the image of P under this reflection.

1. The image OP' of the ray OP corresponds to the angle $90^\circ - \theta$.
2. The image P' of $P(x, y)$ has coordinates $P'(y, x)$, because reflection in the line $y = x$ reverses the coordinates of each point.



Applying the definitions of the trigonometric functions to the angle $90^\circ - \theta$:

$$\cos(90^\circ - \theta) = \frac{y}{r} = \sin\theta,$$
$$\cot(90^\circ - \theta) = \frac{y}{x} = \tan\theta, \text{ provided that } x \neq 0,$$
$$\operatorname{cosec}(90^\circ - \theta) = \frac{r}{x} = \sec\theta, \text{ provided that } x \neq 0.$$

Cosine, Cosecant and Cotangent: The complementary identities are the origin of the names 'cosine', 'cosecant' and 'cotangent', because the prefix 'co-' has the same meaning as the prefix 'com-' of 'complementary' angle.

21 COSINE, COSECANT AND COTANGENT:

- <u>co</u>sine θ = sine (<u>com</u>plement of θ).
- <u>co</u>tangent θ = tangent (<u>com</u>plement of θ).
- <u>co</u>secant θ = secant (<u>com</u>plement of θ).

Proving Identities: An *identity* is a statement that needs to be proven true for all values of θ for which both sides are defined. It is quite different from an *equation*, which needs to be solved and to have its solutions listed.

22 PROVING TRIGONOMETRIC IDENTITIES:
Work separately on the LHS and the RHS until they are the same.

WORKED EXERCISE:
Prove that $(1-\cos\theta)(1+\cos\theta)=\sin^2\theta$.

SOLUTION:

$\text{LHS} = 1-\cos^2\theta$ (Use the difference of squares identity, from algebra.)
$\quad = \sin^2\theta$ (Use the Pythagorean identities, Box 19.)
$\quad = \text{RHS}$

WORKED EXERCISE:
Prove that $\sin A\sec A=\tan A$.

SOLUTION:

$\text{LHS} = \sin A\times\dfrac{1}{\cos A}$ (Use the reciprocal identities, Box 17.)
$\quad = \tan A$ (Use the ratio identities, Box 18.)
$\quad = \text{RHS}$

WORKED EXERCISE:
Prove that $\dfrac{1}{\sin^2\theta}+\dfrac{1}{\cos^2\theta}=\sec^2\theta\,\text{cosec}^2\,\theta$.

SOLUTION:

$\text{LHS} = \dfrac{1}{\sin^2\theta}+\dfrac{1}{\cos^2\theta}$
$\quad = \dfrac{\cos^2\theta+\sin^2\theta}{\sin^2\theta\cos^2\theta}$ (Use a common denominator.)
$\quad = \dfrac{1}{\sin^2\theta\cos^2\theta}$ (Use the Pythagorean identities, Box 19.)
$\quad = \sec^2\theta\,\text{cosec}^2\,\theta$ (Use the reciprocal identities, Box 17.)
$\quad = \text{RHS}$

Exercise 5F

1. Use your calculator to verify that:
(a) $\sin 16^\circ=\cos 74^\circ$
(b) $\tan 63^\circ=\cot 27^\circ$
(c) $\sec 7^\circ=\text{cosec}\,83^\circ$
(d) $\sin^2 23^\circ+\cos^2 23^\circ=1$
(e) $1+\tan^2 55^\circ=\sec^2 55^\circ$
(f) $\text{cosec}^2\,32^\circ-1=\cot^2 32^\circ$

2. Simplify: (a) $\dfrac{1}{\sin\theta}$ (b) $\dfrac{1}{\tan\alpha}$ (c) $\dfrac{\sin\beta}{\cos\beta}$ (d) $\dfrac{\cos\phi}{\sin\phi}$

3. Simplify: (a) $\sin\alpha\,\text{cosec}\,\alpha$ (b) $\cot\beta\tan\beta$ (c) $\cos\theta\sec\theta$

4. Prove: (a) $\tan\theta\cos\theta=\sin\theta$ (b) $\cot\alpha\sin\alpha=\cos\alpha$ (c) $\sin\beta\sec\beta=\tan\beta$

5. Use the reciprocal and ratio identities to prove:
(a) $\cos A \operatorname{cosec} A = \cot A$ (b) $\operatorname{cosec} x \cos x \tan x = 1$ (c) $\sin y \cot y \sec y = 1$

6. Use the reciprocal and ratio identities to simplify:
(a) $\dfrac{\cos\alpha}{\sec\alpha}$ (b) $\dfrac{\sin\alpha}{\operatorname{cosec}\alpha}$ (c) $\dfrac{\tan A}{\sec A}$ (d) $\dfrac{\cot A}{\operatorname{cosec} A}$

7. Use the reciprocal and ratio identities to simplify:
(a) $\dfrac{1}{\sec^2\theta}$ (b) $\dfrac{\sin^2\beta}{\cos^2\beta}$ (c) $\dfrac{\cos^2 A}{\sin^2 A}$ (d) $\sin^2\alpha \operatorname{cosec}^2\alpha$

8. Use the complementary identities to simplify:
(a) $\sin(90^\circ - \theta)$ (b) $\sec(90^\circ - \alpha)$ (c) $\dfrac{1}{\cot(90^\circ - \beta)}$ (d) $\dfrac{\cos(90^\circ - \phi)}{\sin(90^\circ - \phi)}$

9. Use the Pythagorean identities to simplify:
(a) $\sin^2\alpha + \cos^2\alpha$ (b) $1 - \cos^2\beta$ (c) $1 + \tan^2\phi$ (d) $\sec^2 x - \tan^2 x$

10. Use the Pythagorean identities to simplify:
(a) $1 - \sin^2\beta$ (b) $1 + \cot^2\phi$ (c) $\operatorname{cosec}^2 A - 1$ (d) $\cot^2\theta - \operatorname{cosec}^2\theta$

DEVELOPMENT

11. Prove the identities:
(a) $\operatorname{cosec}\theta \sec\theta = \dfrac{1}{\sin\theta\cos\theta}$
(b) $\tan\theta \operatorname{cosec}\theta = \dfrac{1}{\cos\theta}$
(c) $\cot\theta\sec\theta = \dfrac{1}{\sin\theta}$
(d) $\operatorname{cosec}\theta + \sec\theta = \dfrac{\sin\theta + \cos\theta}{\sin\theta\cos\theta}$
(e) $\operatorname{cosec}^2\theta + \sec^2\theta = \dfrac{1}{\sin^2\theta\cos^2\theta}$
(f) $\tan\theta + \cot\theta = \dfrac{1}{\sin\theta\cos\theta}$
(g) $\operatorname{cosec}\theta - \sin\theta = \dfrac{\cos^2\theta}{\sin\theta}$
(h) $\sec\theta - \cos\theta = \dfrac{\sin^2\theta}{\cos\theta}$

12. Prove the identities:
(a) $(1 - \sin\theta)(1 + \sin\theta) = \cos^2\theta$
(b) $(1 + \tan^2\alpha)\cos^2\alpha = 1$
(c) $(\sin A + \cos A)^2 = 1 + 2\sin A\cos A$
(d) $\cos^2 x - \sin^2 x = 1 - 2\sin^2 x$
(e) $\tan^2\phi\cos^2\phi + \cot^2\phi\sin^2\phi = 1$
(f) $3\cos^2\theta - 2 = 1 - 3\sin^2\theta$
(g) $2\tan^2 A - 1 = 2\sec^2 A - 3$
(h) $1 - \tan^2\alpha + \sec^2\alpha = 2$
(i) $\cos^4 x + \cos^2 x\sin^2 x = \cos^2 x$
(j) $\cot\theta(\sec^2\theta - 1) = \tan\theta$

CHALLENGE

13. Prove the identities:
(a) $\sin\theta\cos\theta\operatorname{cosec}^2\theta = \cot\theta$
(b) $(\cos\phi + \cot\phi)\sec\phi = 1 + \operatorname{cosec}\phi$
(c) $\dfrac{\sin\alpha}{\cos\alpha} + \dfrac{\cos\alpha}{\sin\alpha} = \sec\alpha\operatorname{cosec}\alpha$
(d) $\dfrac{1 + \tan^2 x}{1 + \cot^2 x} = \tan^2 x$
(e) $\sin^4 A - \cos^4 A = \sin^2 A - \cos^2 A$
(f) $\dfrac{1}{1 + \sin\theta} + \dfrac{1}{1 - \sin\theta} = 2\sec^2\theta$
(g) $\sin\beta + \cot\beta\cos\beta = \operatorname{cosec}\beta$
(h) $\dfrac{1}{\sec\phi - \tan\phi} - \dfrac{1}{\sec\phi + \tan\phi} = 2\tan\phi$
(i) $\dfrac{1 + \cot x}{1 + \tan x} = \cot x$
(j) $\dfrac{\cos\alpha}{1 + \sin\alpha} = \sec\alpha(1 - \sin\alpha)$

5 G Trigonometric Equations

This piece of work is absolutely vital, because so many problems in later work end up with a trigonometric equation that has to be solved.

There are many small details and qualifications in the methods, and the subject needs a great deal of careful study.

Pay Attention to the Domain: To begin with a warning, before any other details:

23 THE DOMAIN:
Always pay attention to the domain in which the angle can lie.

Equations Involving Boundary Angles: Boundary angles are a special case because they do not lie in any quadrant.

24 THE BOUNDARY ANGLES:
If the solutions are boundary angles, read the solutions off a sketch of the graph.

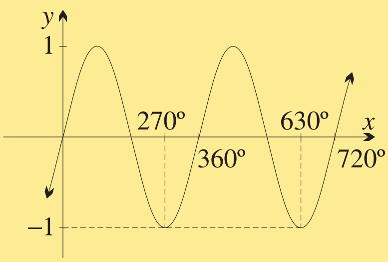
WORKED EXERCISE:

(a) Solve $\sin x = -1$, for $0^\circ \le x \le 720^\circ$.
(b) Solve $\sin x = 0$, for $0^\circ \le x \le 720^\circ$.

SOLUTION:

(a) The graph of $y = \sin x$ is drawn to the right. Examine where the curve touches the line $y = -1$, and read off the x-coordinates of these points.
The solution is $x = 270^\circ$ or 630°.

(b) Examine where the graph crosses the x-axis.
The solution is $x = 0^\circ, 180^\circ, 360^\circ, 540^\circ$ or 720°.



The Standard Method — Quadrants and the Related Acute Angle: Trigonometric equations eventually come down to something like

$$\sin x = -\tfrac{1}{2}, \text{ where } -180^\circ \le x \le 180^\circ.$$

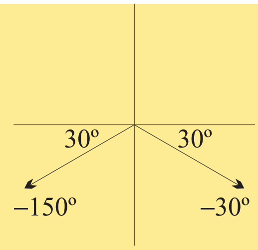
Provided that the angle is not a boundary angle, the method is:

25 THE QUADRANTS-AND-RELATED-ANGLE METHOD:

1. Draw a quadrant diagram, then draw a ray in each quadrant that the angle could be in.
2. Find the related acute angle (only work with positive numbers here):
   (a) using special angles, or
   (b) using the calculator to find an approximation.
   Never enter a negative number into the calculator at this point.
3. Mark the angles on the ends of the rays, taking account of any restrictions on x, and write a conclusion.

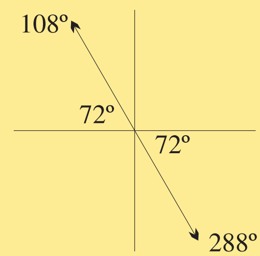
WORKED EXERCISE:
Solve the equation $\sin x = -\frac{1}{2}$, for $-180^\circ \le x \le 180^\circ$.

SOLUTION:
Here $\sin x = -\frac{1}{2}$, where $-180^\circ \le x \le 180^\circ$.
Since $\sin x$ is negative, x is in quadrant 3 or 4.
The sine of the related acute angle is $+\frac{1}{2}$,
so the related angle is 30°.
Hence $x = -150^\circ$ or -30°.



WORKED EXERCISE:
Solve the equation $\tan x = -3$, for $0^\circ \le x \le 360^\circ$, correct to the nearest degree.

SOLUTION:
Here $\tan x = -3$, where $0^\circ \le x \le 360^\circ$.
Since $\tan x$ is negative, x is in quadrant 2 or 4.
The tangent of the related acute angle is $+3$,
so the related angle is about 72°.
Hence $x \doteqdot 108^\circ$ or 288°.



NOTE: When using the calculator, *never enter a negative number and take an inverse trigonometric function of it.*

In the example above, the calculator was used to find the *related acute angle* whose tan was 3, which is $71^\circ 34'$, correct to the nearest minute. The positive number 3 was entered, not -3.

The Three Reciprocal Functions: The calculator doesn't have specific keys for secant, cosecant and cotangent. These function should be converted to sine, cosine and tangent as quickly as possible.

26 THE RECIPROCAL FUNCTIONS:
Take reciprocals to convert each of the three reciprocal functions secant, cosecant and cotangent to the three more common functions.

WORKED EXERCISE:
(a) Solve $\operatorname{cosec} x = -2$, for $-180^\circ \le x \le 180^\circ$.
(b) Solve $\sec x = 0{\cdot}7$, for $-180^\circ \le x \le 180^\circ$.

SOLUTION:
(a) Taking reciprocals of both sides gives
$\sin x = -\frac{1}{2}$,
which was solved in the previous worked exercise,
so $x = -150^\circ$ or -30°.

(b) Taking reciprocals of both sides gives
$\cos x = \frac{10}{7}$,
which has no solutions, because $\cos\theta$ can never be greater than 1.

Equations with Compound Angles: In some equations, the angle is a function of x rather than simply x itself. For example,

$$\tan 2x = \sqrt{3}, \quad \text{where } 0° \le x \le 360°, \qquad \text{or}$$

$$\sin(x - 250°) = \frac{\sqrt{3}}{2}, \quad \text{where } 0° \le x \le 360°.$$

These equations are really trigonometric equations in the *compound angles* $2x$ and $(x - 250°)$ respectively. The secret lies in solving for the *compound angle*, and in *first calculating the domain for that compound angle*.

EQUATIONS WITH COMPOUND ANGLES:

27

1. Let u be the compound angle.
2. Find the restrictions on u from the given restrictions on x.
3. Solve the trigonometric equation for u.
4. Hence solve for x.

WORKED EXERCISE:

Solve $\tan 2x = \sqrt{3}$, where $0° \le x \le 360°$.

SOLUTION:

Let $u = 2x$.

Then $\tan u = \sqrt{3}$.

The restriction on x is $0° \le x \le 360°$

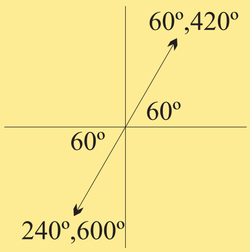
$\boxed{\times 2}$ $0° \le 2x \le 720°$,

and replacing $2x$ by u, $0° \le u \le 720°$.

(The restriction on u is the key step here.)

Hence from the diagram, $u = 60°,\ 240°,\ 420°$ or $600°$.

Since $x = \frac{1}{2}u$, $x = 30°,\ 120°,\ 210°$ or $300°$.



WORKED EXERCISE:

Solve $\sin(x - 250°) = \dfrac{\sqrt{3}}{2}$, where $0° \le x \le 360°$.

SOLUTION:

Let $u = x - 250°$.

Then $\sin u = \dfrac{\sqrt{3}}{2}$.

The restriction on x is $0° \le x \le 360°$

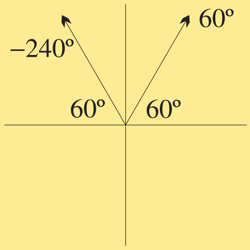
$\boxed{-\,250°}$ $-250° \le x - 250° \le 110°$

and replacing $x - 250°$ by u, $-250° \le u \le 110°$.

(Again, the restriction on u is the key step here.)

Hence from the diagram, $u = -240°$ or $60°$.

Since $x = u + 250°$, $x = 10°$ or $310°$.



Equations Requiring Algebraic Substitutions: Some equations involve powers or reciprocals of a trigonometric function. For example,

$$5\sin^2 x = \sin x,\ \text{ for } 0° \le x \le 360°, \qquad \text{or}$$

$$\frac{4}{\cos x} - \cos x = 0,\ \text{ for } -180° \le x \le 180°.$$

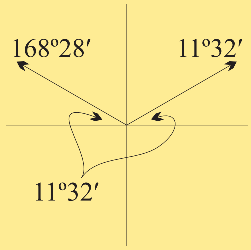
Start with a substitution, 'Let $u = \ldots$', so that the algebra can be done without being confused by the trigonometric notation.

28 ALGEBRAIC SUBSTITUTION:
1. Substitute u to obtain a purely algebraic equation.
2. Solve the algebraic equation — it may have more than one solution.
3. Solve each of the resulting trigonometric equations.

WORKED EXERCISE:

Solve $5\sin^2 x = \sin x$, for $0° \le x \le 360°$, correct to the nearest minute where appropriate.

SOLUTION:



Let $u = \sin x$.

Then $5u^2 = u$ (This is a purely algebraic equation.)

$\boxed{-u}$ $5u^2 - u = 0.$

Factoring, $u(5u-1) = 0$

$u = 0$ or $u = \frac{1}{5}$,

so $\sin x = 0$ or $\sin x = \frac{1}{5}$.

Using the graph of $y = \sin x$ to solve $\sin x = 0$ gives

$x = 0°,\ 180°$ or $360°$,

and using the quadrants-diagram method to solve $\sin x = \frac{1}{5}$ gives

$x \doteqdot 11°32'$ or $168°28'$.

There are thus five solutions altogether.

WORKED EXERCISE:

Solve $\dfrac{4}{\cos x} - \cos x = 0$, for $-180° \le x \le 180°$.

SOLUTION:

Let $u = \cos x$.

Then $\dfrac{4}{u} - u = 0$ (This is a purely algebraic equation.)

$\boxed{\times u}$ $4 - u^2 = 0$

$u^2 = 4$

$u = 2$ or $u = -2$,

so $\cos x = 2$ or $\cos x = -2$.

Neither equation has a solution, because $\cos x$ lies between -1 and 1,
so there are no solutions.

Equations with More than One Trigonometric Function: Some trigonometric equations involve more than one trigonometric function. For example,

$$\sec^2 x + \tan x = 1, \quad \text{where } 180^\circ \le x \le 360^\circ.$$

29 EQUATIONS WITH MORE THAN ONE TRIGONOMETRIC FUNCTION:
- Use trigonometric identities to produce an equation in only one trigonometric function, then proceed by substitution as before.
- If all else fails, reduce everything to sines and cosines, and hope for the best!

WORKED EXERCISE:

Solve $\sec^2 x + \tan x = 1$, where $180^\circ \le x \le 360^\circ$.

SOLUTION:

Using the Pythagorean identity $\sec^2 x = 1 + \tan^2 x$, the equation becomes

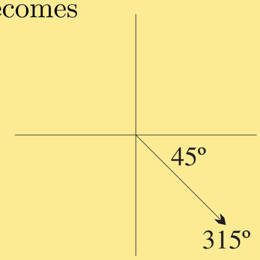
$$1 + \tan^2 x + \tan x = 1, \quad \text{where } 180^\circ \le x \le 360^\circ$$

$\boxed{-1}$ $$\tan x(\tan x + 1) = 0,$$

so $$\tan x = 0 \text{ or } \tan x = -1.$$

Using the graph of $y = \tan x$ to solve $\tan x = 0$,
and the quadrants-diagram method to solve $\tan x = -1$,

$$x = 180^\circ,\ 360^\circ \text{ or } 315^\circ.$$



WORKED EXERCISE:

Solve $\sin^2 x - 3\sin x\cos x + 2\cos^2 x = 0$, for $0^\circ \le x \le 180^\circ$.
[HINT: Dividing through by $\cos^2 x$ will produce an equation in $\tan x$.]

SOLUTION:

Given $$\sin^2 x - 3\sin x\cos x + 2\cos^2 x = 0, \text{ for } 0^\circ \le x \le 180^\circ.$$

$\boxed{\div \cos^2 x}$ $$\tan^2 x - 3\tan x + 2 = 0. \quad \text{(This is an equation in } \tan x \text{ alone.)}$$

Let $$u = \tan x.$$

Then $$u^2 - 3u + 2 = 0$$

$$(u-2)(u-1) = 0$$

$$u = 2 \text{ or } u = 1,$$

$$\tan x = 2 \text{ or } \tan x = 1.$$

Thus within the domain $0^\circ \le x \le 180^\circ$, $x \doteqdot 63^\circ 26'$ or $x = 45^\circ$.

Exercise 5G

1. Solve each equation for $0^\circ \le \theta \le 360^\circ$. (Each related angle is 30°, 45° or 60°.)

(a) $\sin\theta = \frac{\sqrt{3}}{2}$ (c) $\tan\theta = 1$ (e) $\cos\theta = -\frac{1}{\sqrt{2}}$ (g) $\sin\theta = -\frac{1}{2}$

(b) $\sin\theta = \frac{1}{2}$ (d) $\tan\theta = \sqrt{3}$ (f) $\tan\theta = -\sqrt{3}$ (h) $\cos\theta = -\frac{\sqrt{3}}{2}$

2. Solve each equation for $0^\circ \le \theta \le 360^\circ$. (The trigonometric graphs are helpful here.)

(a) $\sin\theta = 1$ (c) $\cos\theta = 0$ (e) $\tan\theta = 0$

(b) $\cos\theta = 1$ (d) $\cos\theta = -1$ (f) $\sin\theta = -1$

3. Solve each equation for $0^\circ \le x \le 360^\circ$. Use your calculator to find the related angle in each case, and give solutions correct to the nearest degree.
(a) $\cos x = \frac{3}{7}$ (b) $\sin x = 0{\cdot}1234$ (c) $\tan x = 7$ (d) $\sin x = -\frac{2}{3}$ (e) $\tan x = -\frac{20}{9}$ (f) $\cos x = -0{\cdot}77$

DEVELOPMENT

4. Solve for $0^\circ \le \alpha \le 360^\circ$. Give solutions correct to the nearest minute where necessary.
(a) $\sin\alpha = 0{\cdot}1$ (b) $\cos\alpha = -0{\cdot}1$ (c) $\tan\alpha = -1$ (d) $\operatorname{cosec}\alpha = -1$ (e) $\sin\alpha = 3$ (f) $\sec\alpha = -2$ (g) $\sqrt{3}\tan\alpha + 1 = 0$ (h) $\cot\alpha = 3$

5. Solve for $-180^\circ \le x \le 180^\circ$. Give solutions correct to the nearest minute where necessary.
(a) $\tan x = -0{\cdot}3$ (b) $\cos x = 0$ (c) $\sec x = \sqrt{2}$ (d) $\sin x = -0{\cdot}7$

6. Solve each equation for $0^\circ \le \theta \le 720^\circ$.
(a) $2\cos\theta - 1 = 0$ (b) $\cot\theta = 0$ (c) $\operatorname{cosec}\theta + 2 = 0$ (d) $\tan\theta = \sqrt{2} - 1$

7. Solve each equation for $0^\circ \le \theta \le 360^\circ$.
(a) $\sin^2\theta = 1$ (b) $\tan^2\theta = 1$ (c) $\cos^2\theta = \frac{1}{4}$ (d) $\cos^2\theta = \frac{3}{4}$

8. Solve each equation for $0^\circ \le x \le 360^\circ$. (Let u be the compound angle.)
(a) $\sin 2x = \frac{1}{2}$ (b) $\tan 2x = \sqrt{3}$ (c) $\cos 2x = -\frac{1}{\sqrt{2}}$ (d) $\sin 2x = -1$

9. Solve each equation for $0^\circ \le \alpha \le 360^\circ$. (Let u be the compound angle.)
(a) $\tan(\alpha - 45^\circ) = \frac{1}{\sqrt{3}}$
(b) $\sin(\alpha + 30^\circ) = -\frac{\sqrt{3}}{2}$
(c) $\cos(\alpha + 60^\circ) = 1$
(d) $\cos(\alpha - 75^\circ) = -\frac{1}{\sqrt{2}}$

10. Solve each equation for $0^\circ \le \theta \le 360^\circ$.
(a) $\sin\theta = \cos\theta$
(b) $\sqrt{3}\sin\theta + \cos\theta = 0$
(c) $4\sin\theta = 3\operatorname{cosec}\theta$
(d) $\sec\theta - 2\cos\theta = 0$

CHALLENGE

11. Solve for $0^\circ \le \theta \le 360^\circ$, giving solutions correct to the nearest minute where necessary.
(a) $\cos^2\theta - \cos\theta = 0$ (Let $u = \cos\theta$.)
(b) $\cot^2\theta = \sqrt{3}\cot\theta$ (Let $u = \cot\theta$.)
(c) $2\sin\theta\cos\theta = \sin\theta$
(d) $\tan^2\theta - \tan\theta - 2 = 0$
(e) $2\sin^2\theta - \sin\theta = 1$
(f) $\sec^2\theta + 2\sec\theta = 8$
(g) $3\cos^2\theta + 5\cos\theta = 2$
(h) $4\operatorname{cosec}^2\theta - 4\operatorname{cosec}\theta - 15 = 0$
(i) $4\sin^3\theta = 3\sin\theta$

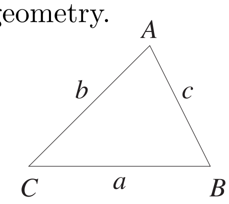
12. Solve for $0^\circ \le x \le 360^\circ$, giving solutions correct to the nearest minute where necessary.
(a) $2\sin^2 x + \cos x = 2$
(b) $\sec^2 x - 2\tan x - 4 = 0$
(c) $8\cos^2 x = 2\sin x + 7$
(d) $6\tan^2 x = 5\sec x$
(e) $6\operatorname{cosec}^2 x = \cot x + 8$

13. Solve for $0^\circ \le \alpha \le 360^\circ$, giving solutions correct to the nearest minute where necessary.
(a) $3\sin\alpha = \operatorname{cosec}\alpha + 2$
(b) $3\tan\alpha - 2\cot\alpha = 5$

14. Solve for $0^\circ \le x \le 360^\circ$ by first dividing through by $\cos^2 x$.
(a) $\sin^2 x + \sin x\cos x = 0$
(b) $\sin^2 x - 5\sin x\cos x + 6\cos^2 x = 0$

5 H The Sine Rule and the Area Formula

The last three sections of this chapter review the sine rule, the area formula and the cosine rule. These rules extend trigonometry to non-right-angled triangles, and are closely connected to the standard congruence tests of Euclidean geometry.



Statement of the Sine Rule: The usual statement of the sine rule uses the convention that each side of a triangle has the lower-case letter of the opposite vertex, as shown in the diagram to the right.

THEOREM — THE SINE RULE:

30 In any triangle ABC,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}\,.$$

'The ratio of each side to the sine of the opposite angle is constant.'

The sine rule is easily proven by dropping an altitude from one of the vertices. The details of the proof are given in the appendix to this chapter.

Using the Sine Rule to Find a Side — The AAS Congruence Situation: When using the sine rule to find a side, one side and two angles must be known. This is the situation described by the AAS congruence test from geometry, and so we know that there will only be one solution.

USING THE SINE RULE TO FIND A SIDE: In the AAS congruence situation:

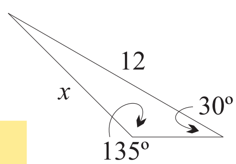
31

$$\frac{\text{unknown side}}{\text{sine of its opposite angle}} = \frac{\text{known side}}{\text{sine of its opposite angle}}\,.$$

Always place the unknown side at the top left of the equation.

WORKED EXERCISE:

Find the side x in the triangle drawn to the right.



SOLUTION:

Using the sine rule, and placing the unknown at the top left,

$$\frac{x}{\sin 30^\circ} = \frac{12}{\sin 135^\circ}$$

$\boxed{\times \sin 30^\circ}$ $$x = \frac{12 \sin 30^\circ}{\sin 135^\circ}\,.$$

Using special angles, $\sin 30^\circ = \frac{1}{2}$,

and $\sin 135^\circ = +\sin 45^\circ$ (Sine is positive for obtuse angles.)

$= \frac{1}{\sqrt{2}}\,.$ (The related acute angle is 45°.)

Hence $$x = 12 \times \frac{1}{2} \times \frac{\sqrt{2}}{1}$$
$$= 6\sqrt{2}\,.$$

The Area Formula: The well-known area formula, area $= \frac{1}{2} \times$ base $\times$ height, can be generalised to a formula involving two sides and the included angle.

32 THEOREM — THE AREA FORMULA:
In any triangle ABC,

$$\text{area } \triangle ABC = \tfrac{1}{2}bc\sin A\,.$$

'The area of a triangle is half the product of any two sides times the sine of the included angle.'

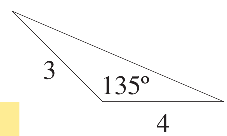
The proof of the area formula uses the same methods as the proof of the sine rule, and is given in the appendix to this chapter.

Using the Area Formula — The SAS Congruence Situation: The area formula requires the SAS situation in which two sides and the included angle are known.

33 USING THE AREA FORMULA: In the SAS congruence situation:

$$\text{area} = (\text{half the product of two sides}) \times (\text{sine of the included angle})\,.$$

WORKED EXERCISE:
Find the area of the triangle drawn to the right.



SOLUTION:
Using the formula, area $= \dfrac{1}{2} \times 3 \times 4 \times \sin 135°$

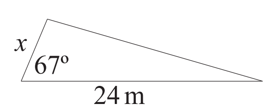
Since $\sin 135° = \dfrac{1}{\sqrt{2}}$, (See the previous worked exercise.)

$$\begin{aligned} \text{area} &= 6 \times \frac{1}{\sqrt{2}} \\ &= 6 \times \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \quad \text{(Rationalise the denominator.)} \\ &= 3\sqrt{2} \text{ square units.} \end{aligned}$$

Using the Area Formula to Find a Side or an Angle: Substituting into the area formula when the area is known may allow an unknown side or angle to be found.

When finding an angle, the formula will always give a single answer for $\sin\theta$. There will be two solutions for θ, however, one acute and one obtuse.

WORKED EXERCISE:
Find x, correct to four significant figures, given that the triangle to the right has area $72\,\text{m}^2$.



SOLUTION:
Substituting into the area formula,

$$72 = \tfrac{1}{2} \times 24 \times x \times \sin 67°$$

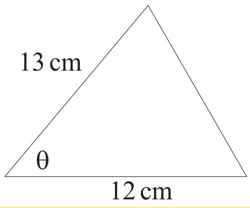
$$72 = 12 \times x \times \sin 67°$$

$\boxed{\div\, 12\sin 67°}$ $\quad x = \dfrac{6}{\sin 67°}$

$\qquad \doteqdot 6{\cdot}518$ metres.

WORKED EXERCISE:

Find θ, correct to the nearest minute, given that the triangle to the right has area $60\,\text{cm}^2$.



SOLUTION:

Substituting into the area formula,

$$60 = \tfrac{1}{2} \times 13 \times 12 \times \sin\theta$$

$$60 = 6 \times 13 \sin\theta$$

$\boxed{\div (6 \times 13)}$ $\sin\theta = \frac{10}{13}$.

Hence $\theta \doteqdot 50°17'$ or $129°43'$. (The second angle is the supplement of the first.)

Using the Sine Rule to Find an Angle — The Ambiguous ASS Situation: The SAS congruence test requires that the angle be included between the two sides. When two sides and a non-included angle are known, the resulting triangle may not be determined up to congruence, and two triangles may be possible. This situation may be referred to as 'the ambiguous ASS situation'.

When the sine rule is applied in the ambiguous ASS situation, there is only one answer for the sine of an angle. There may be two possible solutions for the angle itself, however, one acute and one obtuse.

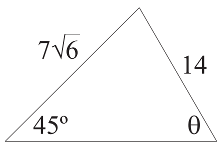
34 USING THE SINE RULE TO FIND AN ANGLE: In the ambiguous ASS situation, where two sides and a non-included angle of the triangle are known,

$$\frac{\text{sine of unknown angle}}{\text{its opposite side}} = \frac{\text{sine of known angle}}{\text{its opposite side}}.$$

Always check the angle sum to see whether both answers are possible.

WORKED EXERCISE:

Find the angle θ in the triangle drawn to the right.



SOLUTION:

$$\frac{\sin\theta}{7\sqrt{6}} = \frac{\sin 45°}{14} \quad \text{(Always place the unknown at the top left.)}$$

$$\sin\theta = 7 \times \sqrt{6} \times \frac{1}{14} \times \frac{1}{\sqrt{2}}, \quad \text{since } \sin 45° = \frac{1}{\sqrt{2}},$$

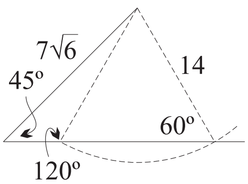
$$\sin\theta = \frac{\sqrt{3}}{2},$$

$$\theta = 60° \text{ or } 120°.$$

NOTE: There are *two* angles whose sine is $\dfrac{\sqrt{3}}{2}$, one of them acute and the other obtuse. Moreover,

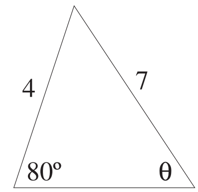
$$120° + 45° = 165°,$$

leaving just 15° for the third angle in the obtuse case, so it all seems to work. Opposite is the ruler-and-compasses construction of the triangle, showing how two different triangles can be produced from the same given ASS measurements.



In many examples, however, the obtuse angle solution can be excluded because the angle sum of a triangle cannot exceed 180°.

WORKED EXERCISE:
Find the angle θ in the triangle drawn to the right, and show that there is only one solution.



SOLUTION:

$$\frac{\sin\theta}{4} = \frac{\sin 80°}{7} \quad \text{(Always place the unknown at the top left.)}$$

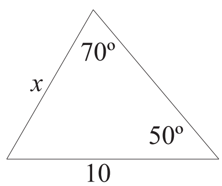
$$\sin\theta = \frac{4\sin 80°}{7}$$

$$\theta \doteqdot 34°15' \text{ or } 145°45'$$

But $\theta \doteqdot 145°45'$ is impossible, because the angle sum would then exceed 180°,
so $\theta \doteqdot 34°15'$.

Exercise 5H

1. Find x in each triangle, correct to one decimal place.

(a)


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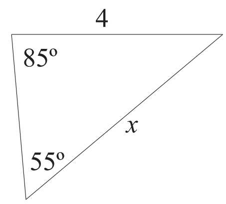

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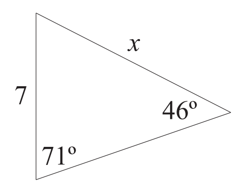

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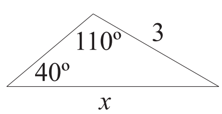

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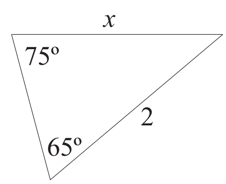

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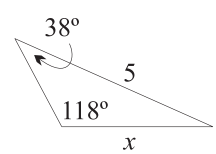
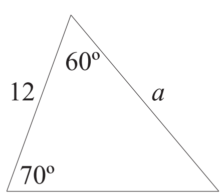


2. Find the value of the pronumeral in each triangle, correct to two decimal places.

(a)


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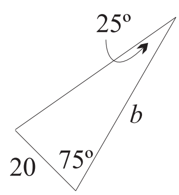

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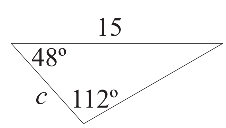
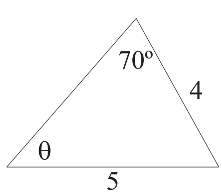


3. Find θ in each triangle, correct to the nearest degree.

(a)


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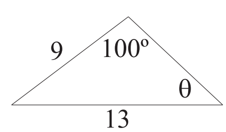

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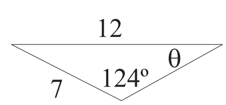

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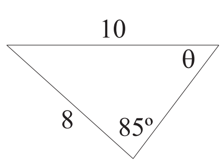

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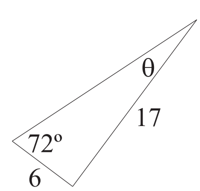

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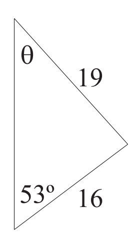
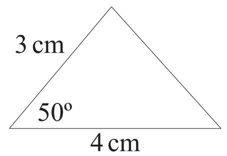


4. Find the area of each triangle, correct to the nearest square centimetre.

(a)



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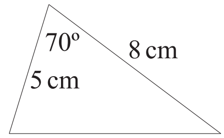

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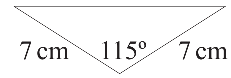
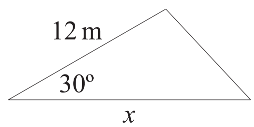


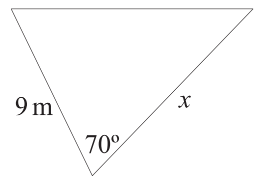
DEVELOPMENT

5. Substitute into the area formula to find the side length x, given that each triangle has area $48\,\text{m}^2$. Give your answers in exact form, or correct to the nearest centimetre.

(a)



(b)



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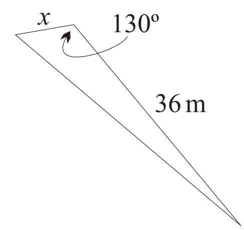
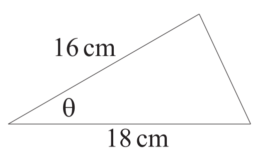


6. Substitute into the area formula to find the angle θ, given that each triangle has area $72\,\text{cm}^2$. Give answers correct to the nearest minute, where appropriate.

(a)



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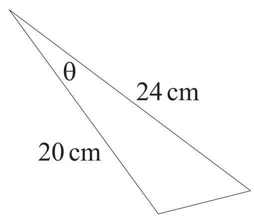

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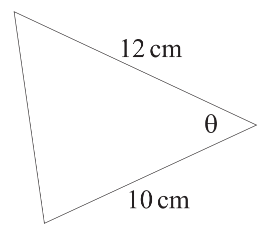
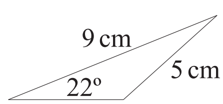
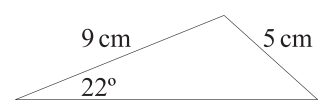
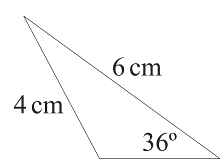


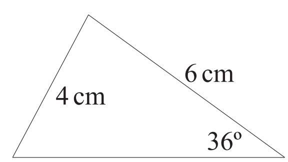
7. There are two triangles that have sides 9 cm and 5 cm, and in which the angle opposite the 5 cm side is 22°. Find, in each case, the size of the angle opposite the 9 cm side, correct to the nearest degree.





8. Two triangles are shown, with sides 6 cm and 4 cm, in which the angle opposite the 4 cm side is 36°. Find, in each case, the angle opposite the 6 cm side, correct to the nearest degree.





9. (a) Sketch $\triangle ABC$ in which $A = 43°$, $B = 101°$ and $a = 7{\cdot}5$ cm.
(b) Find b and c, in cm correct to two decimal places.

10. (a) Sketch $\triangle XYZ$ in which $y = 32$ cm, $Y = 58°$ and $Z = 52°$.
(b) Find the perimeter of $\triangle XYZ$, correct to the nearest centimetre.

11. Sketch $\triangle ABC$ in which $a = 2{\cdot}8$ cm, $b = 2{\cdot}7$ cm and $A = 52°21'$.
(a) Find B, correct to the nearest minute.
(b) Hence find C, correct to the nearest minute.
(c) Hence find the area of $\triangle ABC$ in cm^2, correct to two decimal places.

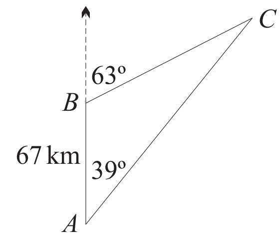
12. Sketch $\triangle PQR$ in which $p = 7$ cm, $q = 15$ cm and $\angle P = 25°50'$.
(a) Find the two possible sizes of $\angle Q$, correct to the nearest minute.
(b) For each possible size of $\angle Q$, find r in cm, correct to one decimal place.

13. (a) Sketch $\triangle ABC$ in which $A = 40°$, $a = 7{\cdot}6$ and $b = 10{\cdot}5$.
(b) Find all the unknown sides (correct to one decimal place) and angles (correct to the nearest minute) of $\triangle ABC$. Note that there are TWO possible triangles.

14. A travelling salesman drove from town A to town B, then to town C, and finally directly home to town A.

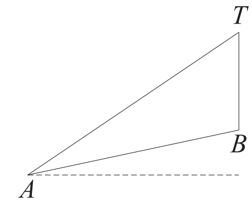
Town B is 67 km north of town A, and the bearings of town C from towns A and B are 039°T and 063°T respectively.

Find, correct to the nearest kilometre, how far the salesman drove.



15. Melissa is standing at A on a path that leads to the base B of a vertical flagpole.

The path is inclined at 12° to the horizontal and the angle of elevation of the top T of the flagpole from A is 34°.
(a) Explain why $\angle TAB = 22°$ and $\angle ABT = 102°$.
(b) Given that $AB = 20$ metres, find the height of the flagpole, correct to the nearest metre.

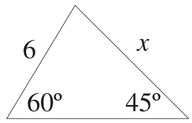


16. (a) Sketch $\triangle ABC$ in which $\sin A = \frac{1}{4}$, $\sin B = \frac{2}{3}$ and $a = 12$.
(b) Find the value of b.

17. (a) Sketch $\triangle PQR$ in which $p = 25$, $q = 21$ and $\sin Q = \frac{3}{5}$.
(b) Find the value of $\sin P$.

18. Find the exact value of x in each diagram.

(a)



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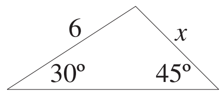

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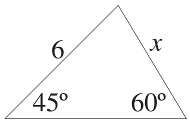

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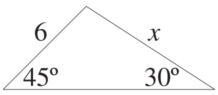
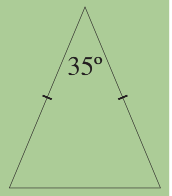


CHALLENGE

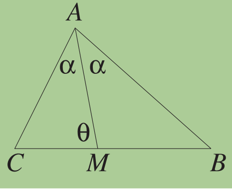
19. The diagram to the right shows an isosceles triangle in which the apex angle is 35°. Its area is 35 cm^2.

Find the length of the equal sides, correct to the nearest millimetre.



20. In a triangle ABC, let the bisector of angle A meet the opposite side BC at M.

Let $\alpha = \angle CAM = \angle BAM$, and let $\theta = \angle CMA$.
(a) Explain why $\sin \angle BMA = \sin \theta$.
(b) Hence show that $AC : AB = MC : MB$.



5 I The Cosine Rule

The cosine rule is a generalisation of Pythagoras' theorem to non-right-angled triangles. It gives a formula for the square of any side in terms of the squares of the other two sides and the cosine of the opposite angle.

35 THEOREM — THE COSINE RULE:
In any triangle ABC,

$$a^2 = b^2 + c^2 - 2bc\cos A.$$

'The square of any side of a triangle equals:
the sum of the squares of the other two sides, minus
twice the product of those sides and the cosine of their included angle.'

The proof is based on Pythagoras' theorem, and again begins with the construction of an altitude. The details are in the appendix, but the following points need to be understood when solving problems using the cosine rule.

36 THE COSINE RULE AND PYTHAGORAS' THEOREM:

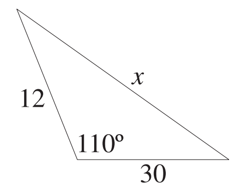
- When $\angle A = 90^\circ$, then $\cos A = 0$ and the cosine rule is Pythagoras' theorem.
- The last term is thus a correction to Pythagoras' theorem when $\angle A \neq 90^\circ$.
- When $\angle A < 90^\circ$, then $\cos A$ is positive, so $a^2 < b^2 + c^2$.
  When $\angle A > 90^\circ$, then $\cos A$ is negative, so $a^2 > b^2 + c^2$.

Using the Cosine Rule to Find a Side — The SAS Situation: For the cosine rule to be applied to find a side, the other two sides and their included angle must be known. This is the SAS congruence situation.

37 USING THE COSINE RULE TO FIND A SIDE: In the SAS congruence situation:

(square of any side) = (sum of squares of other two sides)
− (twice the product of those sides) × (cosine of their included angle).

WORKED EXERCISE:
Find x in the triangle drawn to the right.



SOLUTION:
Applying the cosine rule to the triangle,

$$\begin{aligned} x^2 &= 12^2 + 30^2 - 2 \times 12 \times 30 \times \cos 110^\circ \\ &= 144 + 900 - 720\cos 110^\circ \\ &= 1044 - 720\cos 110^\circ, \end{aligned}$$

and since $\cos 110^\circ = -\cos 70^\circ$, (Cosine is negative in the second quadrant.)

$x^2 = 1044 + 720\cos 70^\circ$ (Until this point, all calculations have been exact.)

Using the calculator to approximate x^2, and then to take the square root,

$x \doteqdot 35{\cdot}92$.

Using the Cosine Rule to Find an Angle — The SSS Situation: To use the cosine rule to find an angle, all three sides need to be known, which is the SSS congruence test. Finding the angle is done most straightforwardly by substituting into the usual form of the cosine rule:

38 USING THE COSINE RULE TO FIND AN ANGLE: In the SSS congruence situation:

'Substitute into the cosine rule and solve for $\cos\theta$.'

There is an alternative approach. Solving the cosine rule for $\cos A$ gives a formula for $\cos A$. Some readers may prefer to remember and apply this second form of the cosine rule.

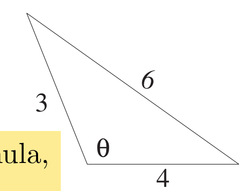
39 THE COSINE RULE WITH $\cos A$ AS SUBJECT:

In any triangle ABC,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}\,.$$

Notice that $\cos\theta$ is positive when θ is acute, and is negative when θ is obtuse. Hence there is only ever one solution for the unknown angle, unlike the situation for the sine rule, where there are often two possible angles.

WORKED EXERCISE:
Find θ in the triangle drawn to the right.



SOLUTION:

Substituting into the cosine rule,

$$6^2 = 3^2 + 4^2 - 2 \times 3 \times 4 \times \cos\theta$$
$$24\cos\theta = -11$$
$$\cos\theta = -\frac{11}{24}$$
$$\theta \doteqdot 117^\circ 17'.$$

OR

Using the boxed formula,

$$\cos\theta = \frac{3^2 + 4^2 - 6^2}{2 \times 3 \times 4}$$
$$= \frac{-11}{24}$$
$$\theta \doteqdot 117^\circ 17'.$$

Exercise 5I

1. Find x in each triangle, correct to one decimal place.

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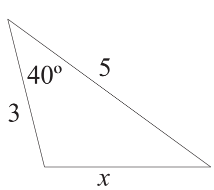

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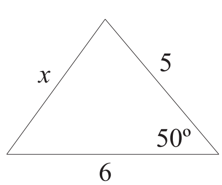

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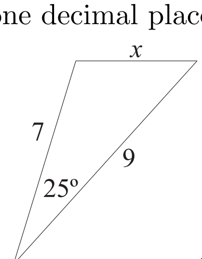

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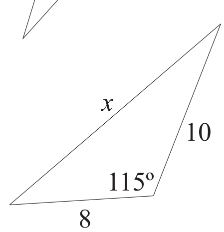

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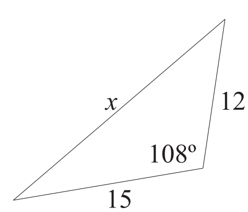

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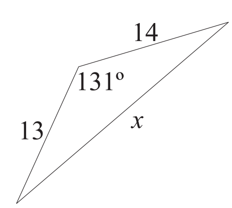
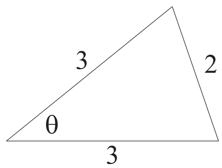


2. Find θ in each triangle, correct to the nearest degree.

(a)


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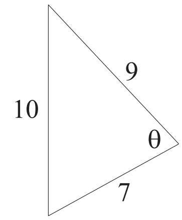

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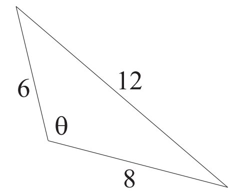

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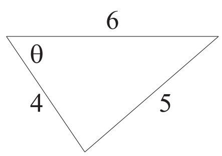

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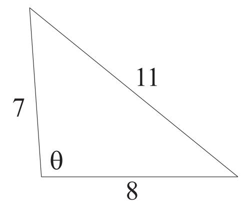

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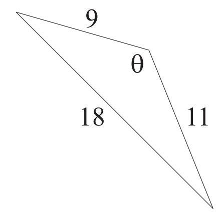
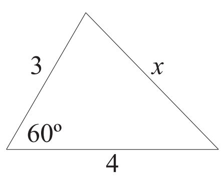


3. Using the fact that $\cos 60^\circ = \frac{1}{2}$ and $\cos 120^\circ = -\frac{1}{2}$, find x as a surd in each triangle.

(a)


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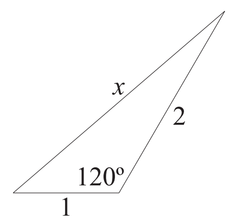
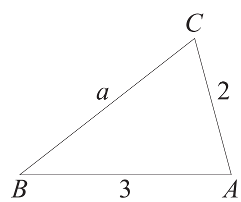
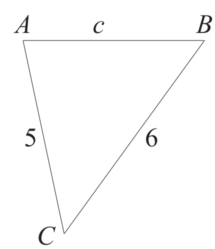
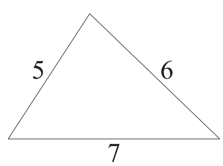


4. (a) If $\cos A = \frac{1}{4}$, find the exact value of a.

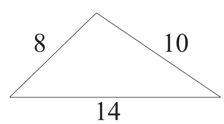


(b) If $\cos C = \frac{2}{3}$, find the exact value of c.

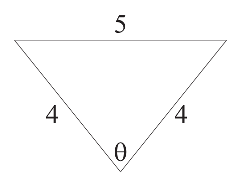


5. (a)


Find the smallest angle of the triangle, correct to the nearest minute.

(b)


Find the largest angle of the triangle, correct to the nearest minute.

(c)


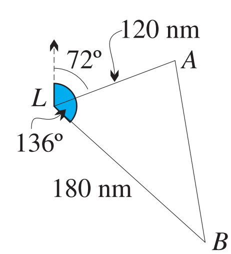
Find the value of $\cos\theta$.

DEVELOPMENT

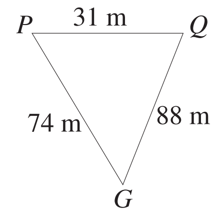
6. There are three landmarks, P, Q and R. It is known that R is 8·7 km from P and 9·3 km from Q, and that $\angle PRQ = 79^\circ 32'$. Draw a diagram and find the distance between P and Q, in kilometres correct to one decimal place.

7. In the diagram to the right, ship A is 120 nautical miles from a lighthouse L on a bearing of 072°T, while ship B is 180 nautical miles from L on a bearing of 136°T.

Calculate the distance between the two ships, correct to the nearest nautical mile.

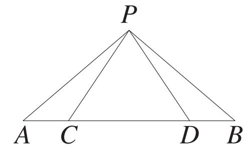


8. A golfer at G wishes to hit a shot between two trees P and Q, as shown in the diagram opposite. The trees are 31 metres apart, and the golfer is 74 metres from P and 88 metres from P. Find the angle within which the golfer must play the shot, correct to the nearest degree.



9. Roof trusses AP, BP, CP and DP are nailed to a horizontal beam AB, as shown in the diagram opposite. Given that $AP = BP = 7{\cdot}2$ metres, $CP = DP = 5{\cdot}5$ metres, $AB = 10{\cdot}6$ metres and $CD = 7{\cdot}4$ metres, find, correct to the nearest minute:

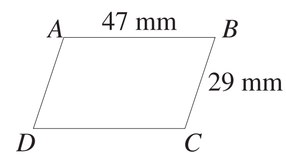
(a) $\angle APB$ (b) $\angle CPD$



10. A parallelogram $ABCD$ has sides $AB = DC = 47$ mm and $AD = BC = 29$ mm. The longer diagonal BD is 60 mm. Giving your answers correct to the nearest minute:

(a) use the cosine rule to find the size of $\angle BCD$,

(b) use co-interior angles to find the size of $\angle ABC$.



11. In $\triangle ABC$, $a = 31$ units, $b = 24$ units and $\cos C = \frac{59}{62}$.

(a) Show that $c = 11$ units. (b) Show that $A = 120°$.

12. The sides of a triangle are in the ratio 5 : 16 : 19. Find the smallest and largest angles of the triangle, correct to the nearest minute where necessary.

13. In $\triangle PQR$, $p = 5\sqrt{3}$ cm, $q = 11$ cm and $R = 150°$. (a) Find r. (b) Find $\cos P$.

14. In $\triangle ABC$, $a = 4$ cm, $b = 5$ cm and $c = 6$ cm. Find $\cos A$, $\cos B$ and $\cos C$, and hence show that $6 \cos A \cos C = \cos B$.

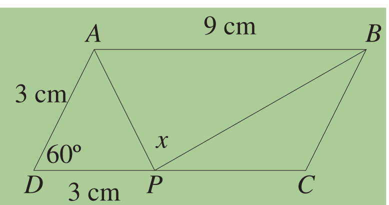
CHALLENGE

15. In a parallelogram $ABCD$, $\angle ADC = 60°$, $AB = 9$ cm and $AD = 3$ cm. The point P lies on DC such that $DP = 3$ cm.

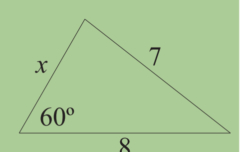
(a) Explain why $\triangle ADP$ is equilateral, and hence find AP.

(b) Use the cosine rule in $\triangle BCP$ to find BP.

(c) Let $\angle APB = x$. Show that $\cos x = -\dfrac{\sqrt{7}}{14}$.



16. Use the cosine rule to find the two possible values of x in the diagram to the right.

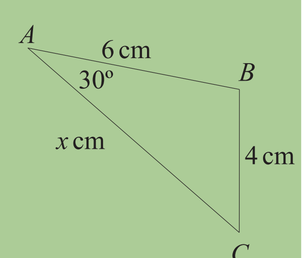


17. The diagram shows $\triangle ABC$ in which $\angle A = 30°$, $AB = 6$ cm and $BC = 4$ cm. Let $AC = x$ cm.

(a) Use the cosine rule to show that $x^2 - 6\sqrt{3}\,x + 20 = 0$.

(b) Use the quadratic formula to show that AC has length $3\sqrt{3} + \sqrt{7}$ cm or $3\sqrt{3} - \sqrt{7}$ cm.

(c) Copy the diagram and indicate on it (approximately) the other possible position of the point C.



18. By substituting into the cosine rule $a^2 = b^2 + c^2 - 2bc \cos A$:

(a) show that if $\angle A = 0°$, then $a = |b - c|$,

(b) show that if $\angle A = 90°$, then $a = \sqrt{b^2 + c^2}$,

(c) show that if $\angle A = 180°$, then $a = b + c$.

5 J Problems Involving General Triangles

A triangle has three lengths and three angles, and most triangle problems involve using three of these six measurements to calculate some of the others. The key to deciding which formula to use is to see which congruence situation applies.

Trigonometry and the Congruence Tests: There are four standard congruence tests — RHS, AAS, SAS and SSS. These tests can also be regarded as theorems about constructing triangles from given data.

If you know three measurements including one length, then apart from the ambiguous ASS situation, any two triangles with these three measurements will be congruent.

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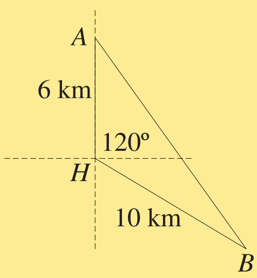
THE SINE, COSINE AND AREA RULES AND THE STANDARD CONGRUENCE TESTS:
In a right-angled triangle, use simple trigonometry and Pythagoras. Otherwise:
AAS: Use the sine rule to find each of the other two sides.
ASS: [The ambiguous situation] Use the sine rule to find the unknown angle opposite a known side. There may be two possible solutions.
SAS: Use the cosine rule to find the third side.
Use the area formula to find the area.
SSS: Use the cosine rule to find any angle.

Problems Requiring Two Steps: Various situations with non-right-angled triangles require two steps for their solution, for example, finding the other two angles in an SAS situation, or finding the area given AAS, ASS or SSS situations.

WORKED EXERCISE:
A boat sails 6 km due north from the harbour H to A, and a second boat sails 10 km from H to B on a bearing of 120°T.
(a) What is the distance AB?
(b) What is the bearing of B from A, correct to the nearest minute?

SOLUTION:
(a) This is an SAS situation,
so we use the cosine rule to find AB:

$$\begin{aligned} AB^2 &= 6^2 + 10^2 - 2 \times 6 \times 10 \times \cos 120° \\ &= 36 + 100 - 120 \times (-\tfrac{1}{2}) \\ &= 196, \\ AB &= 14\,\text{km}. \end{aligned}$$



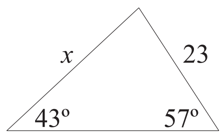
(b) Since AB is now known, this is an SSS situation,
so we use the cosine rule in reverse to find $\angle A$:

$$\begin{aligned} 10^2 &= 14^2 + 6^2 - 2 \times 14 \times 6 \times \cos A \\ 12 \times 14 \times \cos A &= 196 + 36 - 100 \\ \cos A &= \frac{132}{12 \times 14} \\ &= \frac{11}{14}, \\ A &\doteqdot 38°13', \text{ and the bearing of } B \text{ from } A \text{ is about } 141°47'\text{T}. \end{aligned}$$

Exercise 5J

1. Use right-angled triangle trigonometry, the sine rule or the cosine rule to find x in each triangle, correct to one decimal place.

(a)



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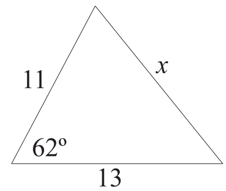

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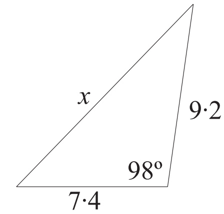

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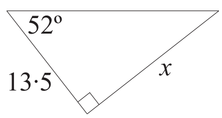

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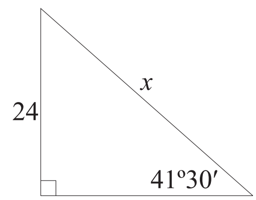

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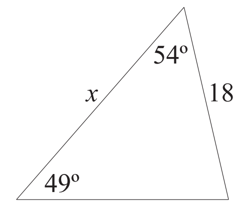
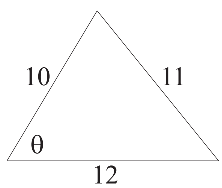


2. Use right-angled triangle trigonometry, the sine rule or the cosine rule to find θ in each triangle, correct to the nearest degree.

(a)



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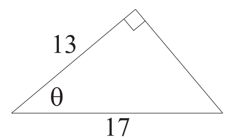

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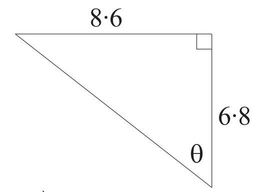

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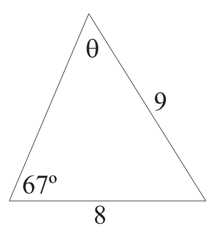

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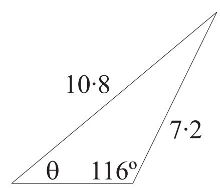

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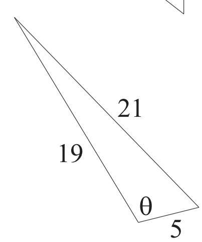
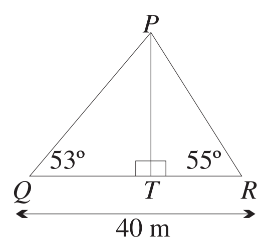


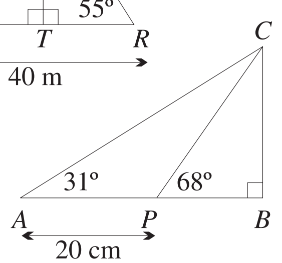
3. In $\triangle PQR$, $\angle Q = 53°$, $\angle R = 55°$ and $QR = 40$ metres. The point T lies on QR such that $PT \perp QR$.



(a) Use the sine rule in $\triangle PQR$ to show that $PQ = \dfrac{40 \sin 55°}{\sin 72°}$.

(b) Use $\triangle PQT$ to find PT, correct to the nearest metre.

4. In $\triangle ABC$, $\angle B = 90°$ and $\angle A = 31°$. The point P lies on AB such that $AP = 20\,\text{cm}$ and $\angle CPB = 68°$.

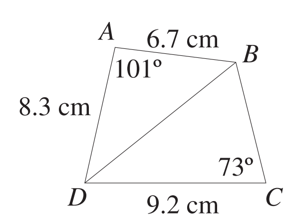


(a) Explain why $\angle ACP = 37°$.

(b) Use the sine rule to show that $PC = \dfrac{20 \sin 31°}{\sin 37°}$.

(c) Hence find PB, correct to the nearest centimetre.

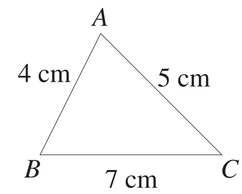
5. In the diagram to the right, $AB = 6{\cdot}7\,\text{cm}$, $AD = 8{\cdot}3\,\text{cm}$ and $DC = 9{\cdot}2\,\text{cm}$. Also, $\angle A = 101°$ and $\angle C = 73°$.



(a) Use the cosine rule to find the diagonal BD, correct to the nearest millimetre.

(b) Hence use the sine rule to find $\angle CBD$, correct to the nearest degree.

6. In $\triangle ABC$, $AB = 4$ cm, $BC = 7$ cm and $CA = 5$ cm.

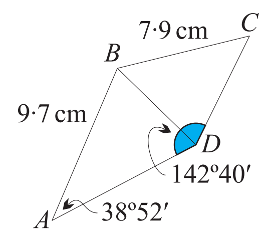


(a) Use the cosine rule to find $\angle ABC$, correct to the nearest minute.

(b) Hence calculate the area of $\triangle ABC$, correct to the nearest square centimetre.

DEVELOPMENT

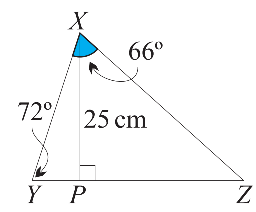
7. In the quadrilateral $ABCD$, $AB = 9{\cdot}7$ cm, $BC = 7{\cdot}9$ cm, $\angle A = 38°52'$ and $\angle D = 142°40'$. The diagonal BD bisects $\angle ADC$.



(a) Use the sine rule to show that $BD = \dfrac{9{\cdot}7 \sin 38°52'}{\sin 71°20'}$.

(b) Hence find the other two angles of the quadrilateral, correct to the nearest minute.

8. In a triangle XYZ, $\angle Y = 72°$ and $\angle YXZ = 66°$. $XP \perp YZ$ and $XP = 25$ cm.



(a) Use the sine ratio in $\triangle PXY$ to show that $XY \doteqdot 26{\cdot}3$ cm.

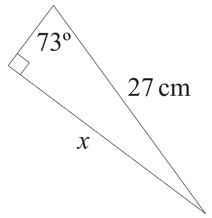
(b) Hence use the sine rule in $\triangle XYZ$ to find YZ, correct to the nearest centimetre.

(c) Check your answer to part (b) by using the tangent ratio in triangles PXY and PXZ to find PY and PZ.

9. A triangle has sides 13 cm, 14 cm and 15 cm. Use the cosine rule to find one of its angles, and hence show that its area is 84 cm^2.

10. [This question is designed to show that the sine and cosine rules work in right-angled triangles, but are NOT the most efficient methods.] In each part find the pronumeral (correct to the nearest cm or to the nearest degree), using either the sine rule or the cosine rule. Then check your answer using right-angled triangle trigonometry.

(a)



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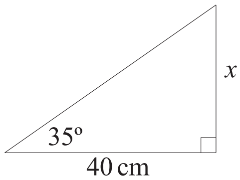
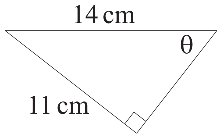


(c)



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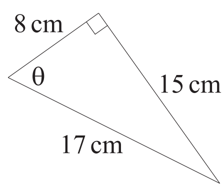


11. Draw a separate sketch of $\triangle ABC$ for each part. In your answers, give lengths and areas correct to four significant figures, and angles correct to the nearest minute.

(a) Find c, given that $a = 12$ cm, $b = 14$ cm and $\angle C = 35°$.

(b) Find b, given that $a = 12$ cm, $\angle A = 47°$ and $\angle B = 80°$.

(c) Find $\angle B$, given that $a = 12$ cm, $b = 24$ and $\angle A = 23°$.

(d) Find $\angle A$, given that $a = 12$ cm, $b = 8$ and $c = 11$.

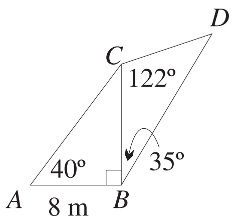
(e) Find the area, given that $a = 12$ cm, $c = 9$ and $\angle B = 28°$.

(f) Find $\angle C$, given that $a = 12$ cm, $b = 7$ and the area is 33 cm^2.

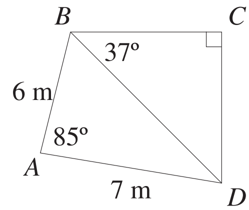
(g) Find c, given that $a = 12$ cm, $\angle B = 65°$ and the area is 60 cm^2.

12. In each diagram, find CD, correct to the nearest centimetre.

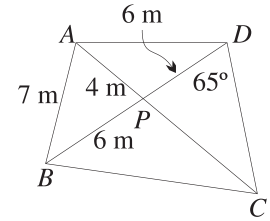
(a)



(b)



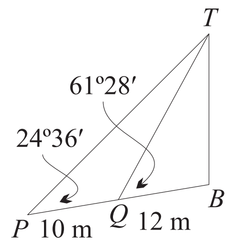
(c)



13. In the diagram opposite, TB is a vertical flagpole at the top of an inclined path PQB.

(a) Show that $TQ = \dfrac{10 \sin 24°36'}{\sin 36°52'}$.

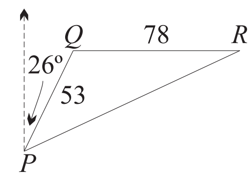
(b) Hence find the height of the flagpole in metres, correct to two decimal places.



14. A ship sails 53 nautical miles from P to Q on a bearing of 026°T. It then sails 78 nautical miles due east from Q to R.

(a) Explain why $\angle PQR = 116°$.

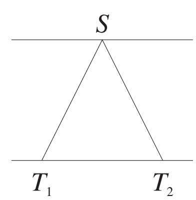
(b) How far apart are P and R, correct to the nearest nautical mile?



15. Two trees T_1 and T_2 on one bank of a river are 86 metres apart. There is a sign S on the opposite bank, and the angles ST_1T_2 and ST_2T_1 are $53°30'$ and $60°45'$ respectively.

(a) Find ST_1.

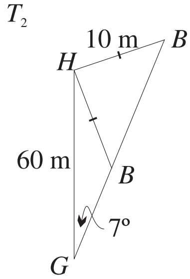
(b) Hence find the width of the river, correct to the nearest metre.



16. A golfer at G played a shot that landed 10 metres from the hole H. The direction of the shot was 7° away from the direct line between G and H.

(a) Find, correct to the nearest minute, the two possible sizes of $\angle GBH$.

(b) Hence find the two possible distances the ball has travelled. (Answer in metres correct to one decimal place.)

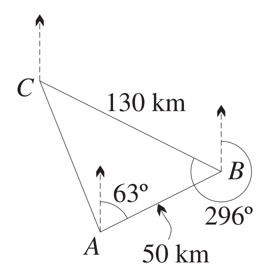


17. A ship sails 50 km from port A to port B on a bearing of 063°T, then sails 130 km from port B to port C on a bearing of 296°T.

(a) Show that $\angle ABC = 53°$.

(b) Find, correct to the nearest km, the distance of port A from port C.

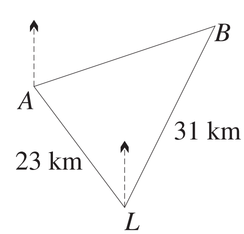
(c) Use the cosine rule to find $\angle ACB$, and hence find the bearing of port A from port C, correct to the nearest degree.



18. Town A is 23 km from landmark L in the direction N56°W, and town B is 31 km from L in the direction N46°E.

(a) Find how far town B is from town A. (Answer correct to the nearest km.)

(b) Find the bearing of town B from town A. (Answer correct to the nearest degree.)



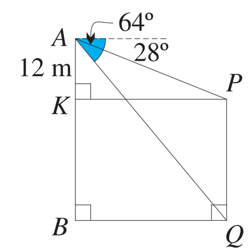
19. Two towers AB and PQ stand on level ground. Tower AB is 12 metres taller than tower PQ. From A, the angles of depression of P and Q are 28° and 64° respectively.

(a) Use $\triangle AKP$ to show that $KP = BQ = 12 \tan 62°$.

(b) Use $\triangle ABQ$ to show that $AB = 12 \tan 62° \tan 64°$.

(c) Hence find the height of the shorter tower, correct to the nearest metre.

(d) Solve the problem again, by using $\triangle AKP$ to find AP, and then using the sine rule in $\triangle APQ$.



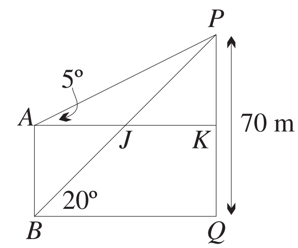
20. Two towers AB and PQ stand on level ground. The angles of elevation of the top of the taller tower from the top and bottom of the shorter tower are 5° and 20° respectively. The height of the taller tower is 70 metres.

(a) Explain why $\angle APJ = 15°$.

(b) Show that $AB = \dfrac{BP \sin 15°}{\sin 95°}$.

(c) Show that $BP = \dfrac{70}{\sin 20°}$.

(d) Hence find the height of the shorter tower, correct to the nearest metre.

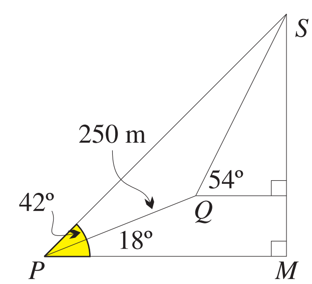


21. The summit S of a mountain is observed from two points, P and Q, 250 metres apart. PQ is inclined at 18° to the horizontal, and the respective angles of elevation of S from P and Q are 42° and 54°.

(a) Explain why $\angle PSQ = 12°$ and $\angle PQS = 144°$.

(b) Show that $SP = \dfrac{250 \sin 144°}{\sin 12°}$.

(c) Hence find the vertical height SM, correct to the nearest metre.

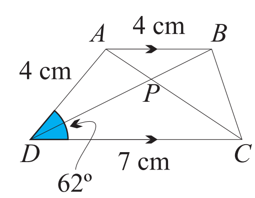


22. In the diagram opposite, $ABCD$ is a trapezium in which $AB \parallel DC$. The diagonals AC and BD meet at P. Also, $AB = AD = 4\,\text{cm}$, $DC = 7\,\text{cm}$ and $\angle ADC = 62°$.

(a) Find $\angle ACD$, correct to the nearest minute.
[HINT: Find AC first.]

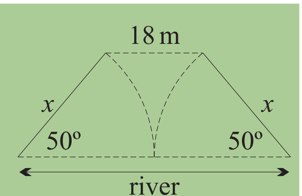
(b) Explain why $\angle PDC = \frac{1}{2}\angle ADC$.

(c) Hence find, correct to the nearest minute, the acute angle between the diagonals of the trapezium.

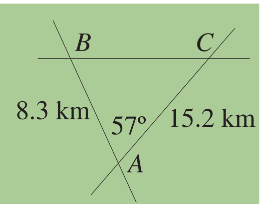


CHALLENGE

23. A bridge spans a river, and the two identical sections of the bridge, each of length x metres, can be raised to allow tall boats to pass. When the two sections are fully raised, they are each inclined at 50° to the horizontal, and there is an 18-metre gap between them, as shown in the diagram. Calculate the width of the river in metres, correct to one decimal place.

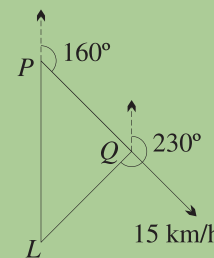


24. The diagram shows three straight roads, AB, BC and CA, where $AB = 8{\cdot}3$ km, $AC = 15{\cdot}2$ km, and the roads AB and AC intersect at $57°$.



Two cars, P_1 and P_2, leave A at the same instant. Car P_1 travels along AB and then BC at 80 km/h while P_2 travels along AC at 50 km/h. Which car reaches C first, and by how many minutes? (Answer correct to one decimal place.)

25. A ship is sailing at 15 km/h on a bearing of 160°T. At 9:00 am it is at P, and lighthouse L is due south. At 9:40 am it is at Q, and the lighthouse is on a bearing of 230°T.

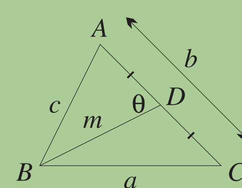


(a) Show that $\angle PQL = 110°$.

(b) Find the distance PL, correct to the nearest kilometre.

(c) Find the time, correct to the nearest minute, at which the lighthouse will be due west of the ship.

26. Let ABC be a triangle and let D be the midpoint of AC. Let $BD = m$ and $\angle ADB = \theta$.



(a) Simplify $\cos(180° - \theta)$.

(b) Show that $\cos\theta = \dfrac{4m^2 + b^2 - 4c^2}{4mb}$, and write down a similar expression for $\cos(180° - \theta)$.

(c) Hence show that $a^2 + c^2 = 2m^2 + \frac{1}{2}b^2$.

5K Chapter Review Exercise

1. Find, correct to four decimal places:

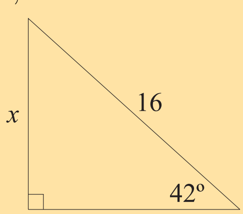
(a) $\cos 73°$ (b) $\tan 42°$ (c) $\sin 38°24'$ (d) $\cos 7°56'$

2. Find the acute angle θ, correct to the nearest minute, given that:

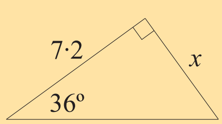
(a) $\sin\theta = 0{\cdot}3$ (b) $\tan\theta = 2{\cdot}36$ (c) $\cos\theta = \frac{1}{4}$ (d) $\tan\theta = 1\frac{1}{3}$

3. Find, correct to two decimal places, the side marked x in each triangle below.

(a)



(b)



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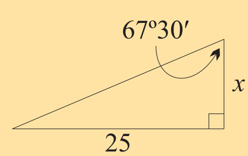

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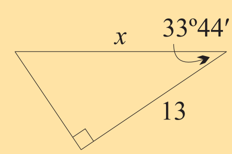
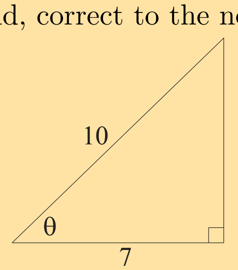
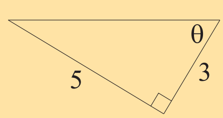


4. Find, correct to the nearest minute, the angle θ in each triangle below.

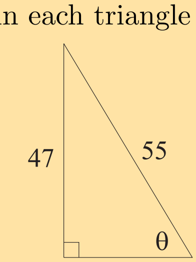
(a)



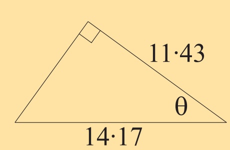
(b)



(c)



(d)



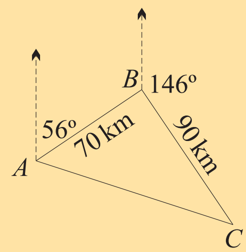
5. Use the special triangles to find the exact values of:

(a) $\tan 60°$ (b) $\sin 45°$ (c) $\cos 30°$ (d) $\cot 45°$ (e) $\sec 60°$ (f) $\operatorname{cosec} 60°$

6. A vertical pole stands on level ground. From a point on the ground 8 metres from its base, the angle of elevation of the top of the pole is 38°. Find the height of the pole, correct to the nearest centimetre.

7. At what angle, correct to the nearest degree, is a 6-metre ladder inclined to the ground if its foot is 2·5 metres out from the wall?

8. A motorist drove 70 km from town A to town B on a bearing of 056°T, and then drove 90 km from town B to town C on a bearing of 146°T.



(a) Explain why $\angle ABC = 90°$.

(b) How far apart are the towns A and C, correct to the nearest kilometre?

(c) Find $\angle BAC$, and hence find the bearing of town C from town A, correct to the nearest degree.

9. Sketch each graph for $0° \le x \le 360°$.

(a) $y = \sin x$ (b) $y = \cos x$ (c) $y = \tan x$

10. Write each trigonometric ratio as the ratio of its related acute angle, with the correct sign attached.

(a) $\cos 125°$ (b) $\sin 312°$ (c) $\tan 244°$ (d) $\sin 173°$

11. Find the exact value of:

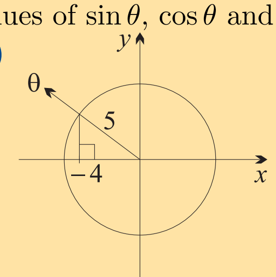
(a) $\tan 240°$ (b) $\sin 315°$ (c) $\cos 330°$ (d) $\tan 150°$

12. Use the graphs of the trigonometric functions to find these values, if they exist.

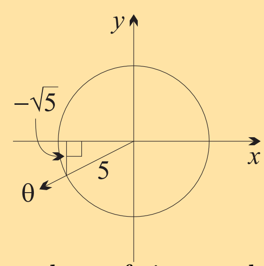
(a) $\sin 180°$ (b) $\cos 180°$ (c) $\tan 90°$ (d) $\sin 270°$

13. Use Pythagoras' theorem to find whichever of x, y or r is unknown. Then write down the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

(a)



(b)



14. (a) If $\tan\alpha = -\frac{9}{40}$ and $270° < \alpha < 360°$, find the values of $\sin\alpha$ and $\cos\alpha$.

(b) If $\sin\beta = \frac{2\sqrt{6}}{7}$ and $90° < \beta < 180°$, find the values of $\cos\beta$ and $\tan\beta$.

15. Simplify:

(a) $\dfrac{1}{\cos\theta}$ (c) $\dfrac{\sin\theta}{\cos\theta}$ (e) $\sec^2\theta - \tan^2\theta$

(b) $\dfrac{1}{\cot\theta}$ (d) $1 - \sin^2\theta$ (f) $\operatorname{cosec}^2\theta - 1$

16. Prove the following trigonometric identities.

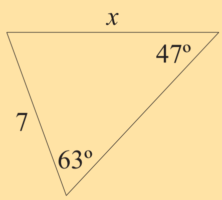
(a) $\cos\theta\sec\theta = 1$ (c) $\dfrac{\cot\theta}{\cos\theta} = \operatorname{cosec}\theta$ (e) $4\sec^2\theta - 3 = 1 + 4\tan^2\theta$

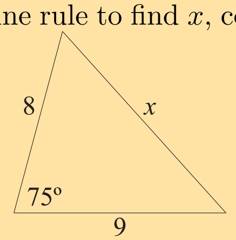
(b) $\tan\theta\operatorname{cosec}\theta = \sec\theta$ (d) $2\cos^2\theta - 1 = 1 - 2\sin^2\theta$ (f) $\cos\theta + \tan\theta\sin\theta = \sec\theta$

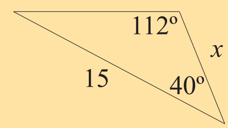
17. Solve each trigonometric equation for $0° \le x \le 360°$.

(a) $\cos x = \frac{1}{2}$
(b) $\sin x = 1$
(c) $\tan x = -1$
(d) $\cos x = 0$
(e) $\sqrt{3}\tan x = 1$
(f) $\tan x = 0$
(g) $\sqrt{2}\sin x + 1 = 0$
(h) $2\cos x + \sqrt{3} = 0$
(i) $\cos^2 x = \frac{1}{2}$
(j) $\cos 2x = \frac{1}{2}$
(k) $\sin x + \cos x = 1$
(l) $\sin^2 x + \sin x = 0$

18. Use the sine rule or the cosine rule to find x, correct to one decimal place, in each triangle.

(a)


(b)


(c)


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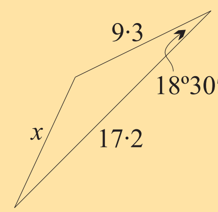
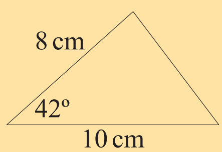
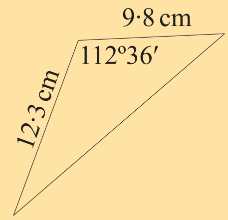
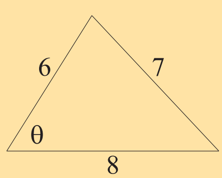


19. Calculate the area of each triangle, correct to the nearest cm^2.

(a)


(b)


20. Use the sine rule or the cosine rule to find θ, correct to the nearest minute, in each triangle.

(a)


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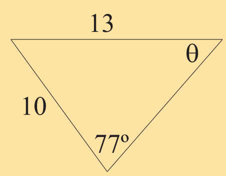
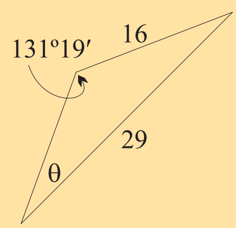


(c)


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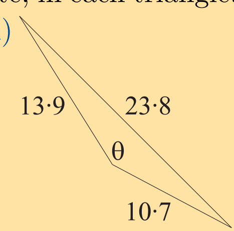
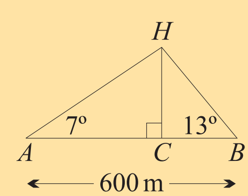


21. A triangle has sides 7 cm, 8 cm and 10 cm. Use the cosine rule to find one of its angles, and hence find the area of the triangle, correct to the nearest cm^2.

22. (a) Find the side a in $\triangle ABC$, where $\angle C = 60°$, $b = 24$ cm and the area is $30\,cm^2$.

(b) Find the size of $\angle B$ in $\triangle ABC$, where $a = 9$ cm, $c = 8$ cm and the area is $18\,cm^2$.

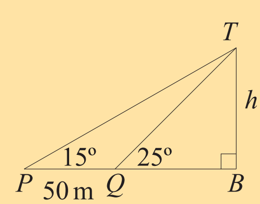
23. A helicopter H is hovering above a straight, horizontal road AB of length 600 metres. The angles of elevation of H from A and B are 7° and 13° respectively. The point C lies on the road directly below H.



(a) Use the sine rule to show that $HB = \dfrac{600\sin 7°}{\sin 160°}$.

(b) Hence find the height CH of the helicopter above the road, correct to the nearest metre.

24. A man is sitting in a boat at P, where the angle of elevation of the top T of a vertical cliff BT is 15°. He then rows 50 metres directly towards the cliff to Q, where the angle of elevation of T is 25°.



(a) Show that $TQ = \dfrac{50\sin 15°}{\sin 10°}$.

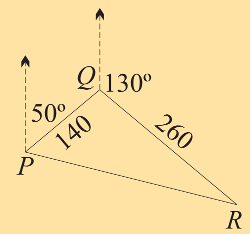
(b) Hence find the height h of the cliff, correct to the nearest tenth of a metre.

25. A ship sailed 140 nautical miles from port P to port Q on a bearing of 050°T. It then sailed 260 nautical miles from port Q to port R on a bearing of 130°T.

(a) Explain why $\angle PQR = 100°$.

(b) Find the distance between ports R and P, correct to the nearest nautical mile.

(c) Find the bearing of port R from port P, correct to the nearest degree.



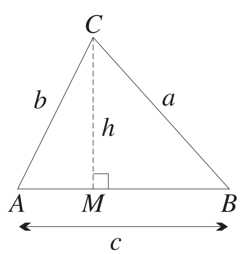
Appendix — Proving the Sine, Cosine and Area Rules

Proof of the Sine Rule: The sine rule says that in any triangle ABC,

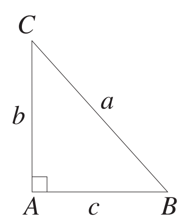
$$\frac{a}{\sin A} = \frac{b}{\sin B}.$$

The proof below begins by constructing an altitude. This breaks the triangle into two right-angled triangles, where previous methods can be used.

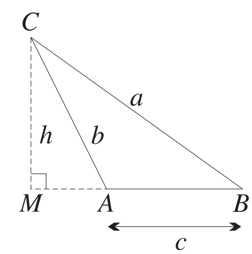
GIVEN: Let ABC be any triangle. There are three cases, depending on whether $\angle A$ is an acute angle, a right angle, or an obtuse angle.



CASE 1: $\angle A$ is acute



CASE 2: $\angle A = 90°$



CASE 3: $\angle A$ is obtuse

AIM: To prove that $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$.

In case 2, $\sin A = \sin 90° = 1$, and $\sin B = \dfrac{b}{a}$, so the result is clear.

CONSTRUCTION: In the remaining cases 1 and 3, construct the altitude from C, meeting AB, produced if necessary, at M. Let h be the length of CM.

PROOF:

CASE 1 — Suppose that $\angle A$ is acute.

In $\triangle ACM$, $\dfrac{h}{b} = \sin A$

$\boxed{\times b}$ $h = b \sin A$.

In $\triangle BCM$, $\dfrac{h}{a} = \sin B$

$\boxed{\times a}$ $h = a \sin B$.

Equating these, $b \sin A = a \sin B$

$$\frac{b}{\sin B} = \frac{a}{\sin A}.$$

CASE 3 — Suppose that $\angle A$ is obtuse.

In $\triangle ACM$, $\dfrac{h}{b} = \sin(180° - A)$,

and since $\sin(180° - A) = \sin A$,

$\boxed{\times b}$ $h = b \sin A$.

In $\triangle BCM$, $\dfrac{h}{a} = \sin B$

$\boxed{\times a}$ $h = a \sin B$.

Equating these, $b \sin A = a \sin B$

$$\frac{b}{\sin B} = \frac{a}{\sin A}.$$

Proof of the Area Formula: The area formula says that in any triangle ABC,

$$\text{area } \triangle ABC = \tfrac{1}{2}bc\sin A.$$

PROOF:

We use the same diagrams as in the proof of the sine rule.
In case 2, $\angle A = 90^\circ$ and $\sin A = 1$, so area $= \frac{1}{2}bc = \frac{1}{2}bc\sin A$, as required.

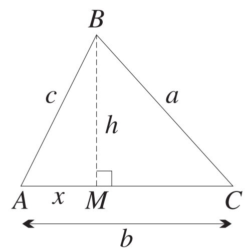
$$\begin{aligned}\text{Otherwise, area} &= \tfrac{1}{2} \times \text{base} \times \text{height} \\ &= \tfrac{1}{2} \times AB \times h \\ &= \tfrac{1}{2} \times c \times b\sin A,\end{aligned}$$

since we proved before that $h = b\sin A$, in both cases.

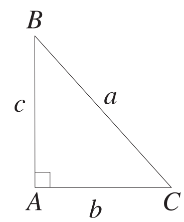
Proof of the Cosine Rule: The cosine rule says that in any triangle ABC,

$$a^2 = b^2 + c^2 - 2bc\cos A.$$

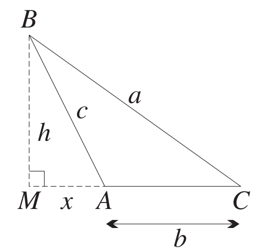
GIVEN: Let ABC be any triangle. Again, there are three cases, depending on whether $\angle A$ is acute, obtuse, or a right angle.



CASE 1: $\angle A$ is acute



CASE 2: $\angle A = 90^\circ$



CASE 3: $\angle A$ is obtuse

AIM: To prove that $a^2 = b^2 + c^2 - 2bc\cos A$.
In case 2, $\cos A = 0$, and this is just Pythagoras' theorem.

CONSTRUCTION: In the remaining cases 1 and 3, construct the altitude from B, meeting AC, produced if necessary, at M. Let $BM = h$ and $AM = x$.

PROOF:

CASE 1 — Suppose that $\angle A$ is acute.
By Pythagoras' theorem in $\triangle BMC$,
$$a^2 = h^2 + (b - x)^2.$$
By Pythagoras' theorem in $\triangle BMA$,
$$h^2 = c^2 - x^2,$$
$$\begin{aligned}\text{so } a^2 &= c^2 - x^2 + (b - x)^2 \\ &= c^2 - x^2 + b^2 - 2bx + x^2 \\ &= b^2 + c^2 - 2bx. \qquad (*)\end{aligned}$$
Using trigonometry in $\triangle ABM$,
$$x = c\cos A,$$
so $a^2 = b^2 + c^2 - 2bc\cos A$.

CASE 3 — Suppose that $\angle A$ is obtuse.
By Pythagoras' theorem in $\triangle BMC$,
$$a^2 = h^2 + (b + x)^2.$$
By Pythagoras' theorem in $\triangle BMA$,
$$h^2 = c^2 - x^2,$$
$$\begin{aligned}\text{so } a^2 &= c^2 - x^2 + (b + x)^2 \\ &= c^2 - x^2 + b^2 + 2bx + x^2 \\ &= b^2 + c^2 + 2bx. \qquad (*)\end{aligned}$$
Using trigonometry in $\triangle ABM$,
$$\begin{aligned}x &= c\cos(180^\circ - A) \\ &= -c\cos A,\end{aligned}$$
so $a^2 = b^2 + c^2 - 2bc\cos A$.

CHAPTER SIX

Coordinate Geometry

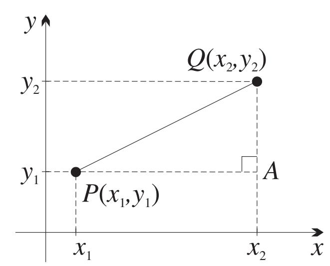
Coordinate geometry is geometry done in the number plane, using algebra.
- Points are represented by pairs of numbers, and lines by linear equations.
- Circles, parabolas and other curves are represented by non-linear equations.

Points, lines and intervals are the main topics of this chapter.

6 A Lengths and Midpoints of Intervals

An *interval* is completely determined by its two endpoints. There are simple formulae for the length of an interval and for the midpoint of an interval.

The Distance Formula: The formula for the length of an interval PQ is just Pythagoras' theorem in different notation.



Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points in the plane.
Construct the right-angled triangle $\triangle PQA$, where $A(x_2, y_1)$ lies level with P and vertically above or below Q.
Then $PA = |x_2 - x_1|$ and $QA = |y_2 - y_1|$,
and so by Pythagoras' theorem in $\triangle PQA$,

$$PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

1 DISTANCE FORMULA: Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points in the plane. Then

$$PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

- First find the square PQ^2 of the distance.
- Then take the square root to find the distance PQ.

WORKED EXERCISE:

Find the lengths of the sides AB and AC of the triangle with vertices $A(1, -2)$, $B(-4, 2)$, and $C(5, -7)$, and hence show that $\triangle ABC$ is isosceles.

SOLUTION:

First, $AB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$
$= \big(-4 - 1\big)^2 + \big(2 - (-2)\big)^2$
$= (-5)^2 + 4^2$
$= 41,$
so $AB = \sqrt{41}$.

Secondly, $AC^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$
$= \big(5 - 1\big)^2 + \big(-7 - (-2)\big)^2$
$= 4^2 + (-5)^2$
$= 41,$
so $AC = \sqrt{41}$.

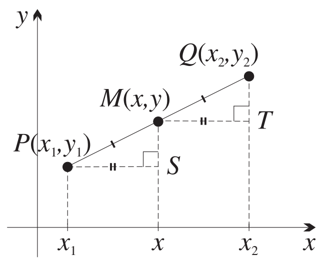
Since the two sides AB and AC are equal, the triangle is isosceles.

The Midpoint Formula: The midpoint of an interval is found by taking the averages of the coordinates of the two points. Congruence is the basis of the proof below.

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points in the plane,
and let $M(x, y)$ be the midpoint of PQ.
Construct $S(x, y_1)$ and $T(x_2, y)$, as shown.
Then $\triangle PMS \equiv \triangle MQT$ (AAS)
and so $PS = MT$ (matching sides of congruent triangles).
That is, $x - x_1 = x_2 - x$

$$2x = x_1 + x_2$$

$$x = \frac{x_1 + x_2}{2},$$ which is the average of x_1 and x_2.



The calculation of the y-coordinate y is similar.

2 MIDPOINT FORMULA:
Let $M(x, y)$ be the midpoint of the interval joining $P(x_1, y_1)$ and $Q(x_2, y_2)$. Then

$$x = \frac{x_1 + x_2}{2} \quad \text{and} \quad y = \frac{y_1 + y_2}{2},$$ (Take the average of the coordinates.)

WORKED EXERCISE:
The interval joining the points $A(3, -1)$ and $B(-7, 5)$ is a diameter of a circle.
(a) Find the centre M of the circle. (b) Find the radius of the circle.

SOLUTION:
(a) The centre of the circle is the midpoint $M(x, y)$ of the interval AB.

Using the midpoint formula, $x = \frac{x_1 + x_2}{2}$ and $y = \frac{y_1 + y_2}{2}$

$= \frac{3 - 7}{2}$ $\quad$ $= \frac{-1 + 5}{2}$

$= -2,$ $\quad$ $= 2,$

so the centre is $M(-2, 2)$.

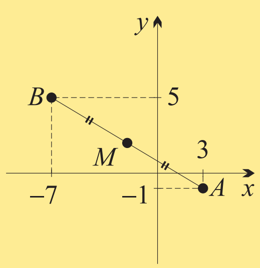
(b) Using the distance formula, $AM^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

$= \big(-2 - 3\big)^2 + \big(2 - (-1)\big)^2$

$= 34,$

$AM = \sqrt{34}.$

Hence the circle has radius $\sqrt{34}$.



Testing for Special Quadrilaterals: Euclidean geometry will be reviewed in the Year 12 volume, but many questions in this chapter ask for proofs that a quadrilateral is of a particular type. The most obvious way is to test the definition itself.

3 DEFINITIONS OF THE SPECIAL QUADRILATERALS:
- A *trapezium* is a quadrilateral with one pair of opposite sides parallel.
- A *parallelogram* is a quadrilateral with both pairs of opposite sides parallel.
- A *rhombus* is a parallelogram with a pair of adjacent sides equal.
- A *rectangle* is a parallelogram with one angle a right angle.
- A *square* is both a rectangle and a rhombus.

There are, however, several further standard tests that the exercises assume. (Tests involving angles are omitted here, being irrelevant in this chapter.)

A QUADRILATERAL IS A PARALLELOGRAM:
- if the opposite sides are equal, or
- if one pair of opposite sides are equal and parallel, or
- if the diagonals bisect each other.

4 A QUADRILATERAL IS A RHOMBUS:
- if all sides are equal, or
- if the diagonals bisect each other at right angles.

A QUADRILATERAL IS A RECTANGLE:
- if the diagonals are equal and bisect each other.

Exercise 6A

NOTE: Diagrams should be drawn wherever possible.

1. Find the midpoint of each interval AB. Use $x = \dfrac{x_1 + x_2}{2}$ and $y = \dfrac{y_1 + y_2}{2}$.

(a) $A(3,5)$ and $B(1,9)$
(b) $A(4,8)$ and $B(6,4)$
(c) $A(-4,7)$ and $B(8,-11)$
(d) $A(-3,6)$ and $B(3,1)$
(e) $A(0,-8)$ and $B(-11,-12)$
(f) $A(4,-7)$ and $B(4,7)$

2. Find the length of each interval. Use $AB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$, then find AB.

(a) $A(1,4)$, $B(5,1)$
(b) $A(-2,7)$, $B(3,-5)$
(c) $A(-5,-2)$, $B(3,4)$
(d) $A(3,6)$, $B(5,4)$
(e) $A(-4,-1)$, $B(4,3)$
(f) $A(5,-12)$, $B(0,0)$

3. (a) Find the midpoint M of the interval joining $P(-2,1)$ and $Q(4,9)$.
(b) Find the lengths PM and MQ, and verify that $PM = MQ$.

4. (a) Find the length of each side of the triangle formed by $P(0,3)$, $Q(1,7)$ and $R(5,8)$.
(b) Hence show that $\triangle PQR$ is isosceles.

5. (a) Find the length of each side of $\triangle ABC$, where $A = (0,5)$, $B = (3,-2)$ and $C = (-3,4)$.
(b) Find the midpoint of each side of this triangle ABC.

DEVELOPMENT

6. (a) A circle with centre $O(0,0)$ passes through $A(5,12)$. What is its radius?
(b) A circle with centre $B(4,5)$ passes through the origin. What is its radius?
(c) Find the centre of the circle with diameter CD, where $C = (2,1)$ and $D = (8,-7)$.
(d) Find the radius of the circle with diameter CD in part (c) above.
(e) Show that $E(-12,-5)$ lies on the circle with centre the origin and radius 13.

7. The interval joining $A(2,-5)$ and $E(-6,-1)$ is divided into four equal subintervals by the three points B, C and D.
(a) Find the coordinates of C by taking the midpoint of AE.
(b) Find the coordinates of B and D by taking the midpoints of AC and CE.

8. (a) Find the midpoint of the interval joining $A(4, 9)$ and $C(-2, 3)$.
(b) Find the midpoint of the interval joining $B(0, 4)$ and $D(2, 8)$.
(c) What can you conclude about the diagonals of the quadrilateral $ABCD$?
(d) What sort of quadrilateral is $ABCD$? [HINT: See Box 4 above.]

9. The points $A(3, 1)$, $B(10, 2)$, $C(5, 7)$ and $D(-2, 6)$ are the vertices of a quadrilateral.
(a) Find the lengths of all four sides.
(b) What sort of quadrilateral is $ABCD$? [HINT: See Box 4 above.]

10. (a) Find the side lengths of the triangle with vertices $X(0, -4)$, $Y(4, 2)$ and $Z(-2, 6)$.
(b) Show that $\triangle XYZ$ is a right-angled isosceles triangle by showing that its side lengths satisfy Pythagoras' theorem.
(c) Hence find the area of $\triangle XYZ$.

11. The quadrilateral $ABCD$ has vertices at the points $A(1, 0)$, $B(3, 1)$, $C(4, 3)$ and $D(2, 2)$. [HINT: You should look at Boxes 3 and 4 in the notes above to answer this question.]
(a) Show that the intervals AC and BD bisect each other, by finding the midpoint of each and showing that these midpoints coincide.
(b) What can you conclude from part (a) about what type of quadrilateral $ABCD$ is?
(c) Show that $AB = AD$. What can you now conclude about the quadrilateral $ABCD$?

12. (a) Find the distance of each point $A(1, 4)$, $B(2, \sqrt{13}\,)$, $C(3, 2\sqrt{2}\,)$ and $D(4, 1)$ from the origin O. Hence explain why the four points lie on a circle with centre the origin.
(b) What are the radius, diameter, circumference and area of this circle?

13. The point $M(3, 7)$ is the midpoint of the interval joining $A(1, 12)$ and $B(x_2, y_2)$. Find the coordinates x_2 and y_2 of B by substituting into the formulae

$$x = \frac{x_1 + x_2}{2} \qquad \text{and} \qquad y = \frac{y_1 + y_2}{2}\,.$$

14. Solve each of these problems using the same methods as in the previous question.
(a) If $A(-1, 2)$ is the midpoint of $S(x, y)$ and $T(3, 6)$, find the coordinates of S.
(b) The midpoint of the interval PQ is $M(2, -7)$. Find the coordinates of P if:
(i) $Q = (0, 0)$ (ii) $Q = (5, 3)$ (iii) $Q = (-3, -7)$
(c) Find B, if AB is a diameter of a circle with centre $Q(4, 5)$ and $A = (8, 3)$.
(d) Given that $P(4, 7)$ is one vertex of the square $PQRS$, and that the centre of the square is $M(8, -1)$, find the coordinates of the opposite vertex R.

15. (a) Write down any two points A and B whose midpoint is $M(4, 6)$.
(b) Write down any two points C and D that are 10 units apart.

CHALLENGE

16. Each set of three points given below forms a triangle of one of these types:

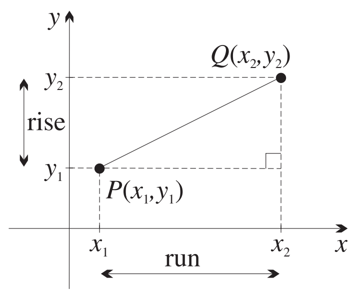
A. isosceles, B. equilateral, C. right-angled, D. none of these.

Find the side lengths of each triangle below and hence determine its type.
(a) $A(-1, 0)$, $B(1, 0)$, $C(0, \sqrt{3})$
(b) $P(-1, 1)$, $Q(0, -1)$, $R(3, 3)$
(c) $D(1, 1)$, $E(2, -2)$, $F(-3, 0)$
(d) $X(-3, -1)$, $Y(0, 0)$, $Z(-2, 2)$

17. As was discussed in Section 3F, the circle with centre (h, k) and radius r has equation $(x - h)^2 + (y - k)^2 = r^2$. By identifying the centre and radius, find the equations of:
(a) the circle with centre $(5, -2)$ and passing through $(-1, 1)$,
(b) the circle with $K(5, 7)$ and $L(-9, -3)$ as endpoints of a diameter.

6 B Gradients of Intervals and Lines

Gradient is the key idea that will be used in the next section to bring lines and their equations into coordinate geometry.



The Gradient of an Interval and the Gradient of a Line:

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points in the plane.
The *gradient* of the interval PQ is a measure of its steepness, as someone walks along the interval from P to Q.
Their *rise* is the vertical difference $y_2 - y_1$,
and their *run* is the horizontal difference $x_2 - x_1$.
The *gradient* of the interval PQ is defined to be the ratio of the rise and the run:

$$\text{gradient of } PQ = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}.$$

5 THE GRADIENT OF AN INTERVAL:
Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be any two points. Then

$$\text{gradient of } PQ = \frac{\text{rise}}{\text{run}}$$

OR

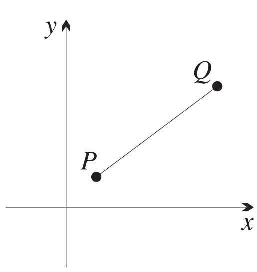
$$\text{gradient of } PQ = \frac{y_2 - y_1}{x_2 - x_1}.$$

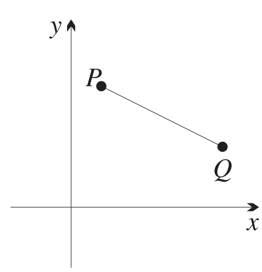
- Horizontal intervals have gradient zero, because the rise is zero.
- Vertical intervals don't have a gradient, because the run is always zero and so the fraction is undefined.

THE GRADIENT OF A LINE:
The *gradient of a line* is then determined by taking any two points A and B on the line and finding the gradient of the interval AB.

Positive and Negative Gradients: If the rise and the run have the same sign, then the gradient is *positive*, as in the first diagram below. In this case the interval slopes *upwards* as one moves from left to right.

If the rise and run have opposite signs, then the gradient is *negative*, as in the second diagram. The interval slopes *downwards* as one moves from left to right.





If the points P and Q are interchanged, then the rise and the run both change signs, but the gradient remains the same.

WORKED EXERCISE:
Find the gradients of the sides of $\triangle XYZ$, where $X = (2,5)$, $Y = (5,-2)$ and $Z = (-3,4)$.

SOLUTION:

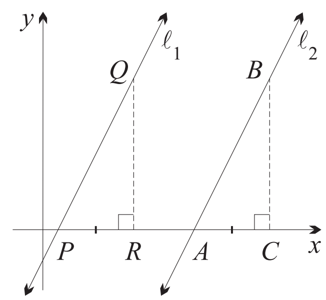
$$\text{gradient of } XY = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2-5}{5-2} = \frac{-7}{3} = -\tfrac{7}{3}$$

$$\text{gradient of } YZ = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-(-2)}{-3-5} = \frac{4+2}{-8} = -\tfrac{3}{4}$$

$$\text{gradient of } ZX = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-4}{2-(-3)} = \frac{1}{2+3} = \tfrac{1}{5}$$

A Condition for Two Lines to be Parallel:

Any two vertical lines are parallel. Otherwise, the condition for lines to be parallel is:



6 **PARALLEL LINES:** Two lines ℓ_1 and ℓ_2 with gradients m_1 and m_2 are parallel if and only if

$$m_1 = m_2.$$

This can easily be seen by looking at the congruent triangles in the diagram to the right.

WORKED EXERCISE:
Given the four points $A(3,6)$, $B(7,-2)$, $C(4,-5)$ and $D(-1,5)$, show that the quadrilateral $ABCD$ is a trapezium with $AB \parallel CD$.

SOLUTION:

$$\text{gradient of } AB = \frac{-2-6}{7-3} = \frac{-8}{4} = -2,$$

$$\text{gradient of } CD = \frac{5-(-5)}{-1-4} = \frac{10}{-5} = -2.$$

Hence $AB \parallel CD$ because their gradients are equal, so $ABCD$ is a trapezium.

Testing for Collinear Points:

Three points are called *collinear* if they all lie on one line.

7 **TESTING FOR COLLINEAR POINTS:** To test whether three given points A, B and C are collinear, find the gradients of AB and BC.
If these gradients are equal, then the three points must be collinear, because then AB and BC are parallel lines passing through a common point B.

WORKED EXERCISE:
Test whether the three points $A(-2,5)$, $B(1,3)$ and $C(7,-1)$ are collinear.

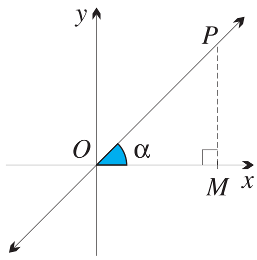
SOLUTION:

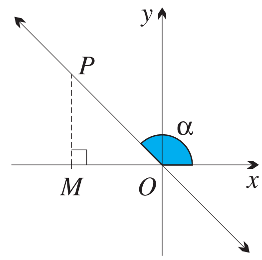
$$\text{gradient of } AB = \frac{3-5}{1+2} = -\tfrac{2}{3},$$

$$\text{gradient of } BC = \frac{-1-3}{7-1} = -\tfrac{2}{3}.$$

Since the gradients are equal, the points A, B and C are collinear.

Gradient and the Angle of Inclination: The *angle of inclination* of a line is the angle between the upward direction of the line and the positive direction of the x-axis.





The two diagrams above show that lines with positive gradients have acute angles of inclination, and lines with negative gradients have obtuse angles of inclination. They also illustrate the trigonometric relationship between the gradient and the angle of inclination α.

8 ANGLE OF INCLINATION: Let the line ℓ have angle of inclination α. Then

$$\text{gradient of } \ell = \tan\alpha.$$

PROOF:

A. When α is acute, as in the first diagram, then the rise MP and the run OM are just the opposite and adjacent sides of the triangle POM, so

$$\tan\alpha = \frac{MP}{OM} = \text{gradient of } OP.$$

B. When α is obtuse, as in the second diagram, then $\angle POM = 180° - \alpha$, so

$$\tan\alpha = -\tan\angle POM = -\frac{MP}{OM} = \text{gradient of } OP.$$

WORKED EXERCISE:

(a) Given the points $A(-3, 5)$, $B(-6, 0)$ and $O(0, 0)$, find the angles of inclination of the intervals AB and AO.

(b) What sort of triangle is $\triangle ABO$?

SOLUTION:

(a) First, $\text{gradient of } AB = \dfrac{0-5}{-6+3}$

$= \frac{5}{3}$,

and using a calculator to solve $\tan\alpha = \frac{5}{3}$,

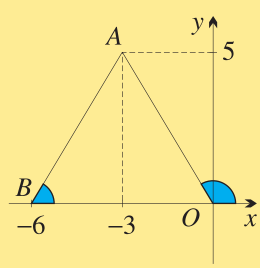
angle of inclination of $AB \doteqdot 59°$.

Secondly, $\text{gradient of } AO = \dfrac{0-5}{0+3}$

$= -\frac{5}{3}$,

and using a calculator to solve $\tan\alpha = -\frac{5}{3}$,

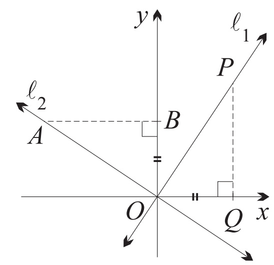
angle of inclination of $AO \doteqdot 121°$.



(b) Hence $\angle AOB = 59°$ (straight angle).

Thus the base angles of $\triangle AOB$ are equal, and so the triangle is isosceles.

A Condition for Lines to be Perpendicular: A vertical and a horizontal line are perpendicular. Otherwise, the condition is:



9 **PERPENDICULAR LINES:** Lines ℓ_1 and ℓ_2 with gradients m_1 and m_2 are perpendicular if and only if

$$m_1 \times m_2 = -1$$

that is, $m_2 = -\dfrac{1}{m_1}$.

PROOF:

Shift each line sideways, without rotating it, so that it passes through the origin.
One line must have positive gradient and the other negative gradient, otherwise one of the angles between them would be acute.
So let ℓ_1 be a line with positive gradient through the origin, and let ℓ_2 be a line with negative gradient through the origin.
Construct the two triangles POQ and AOB as shown in the diagram above, with the run OQ of ℓ_1 equal to the rise OB of ℓ_2.

Then $m_1 \times m_2 = \dfrac{QP}{OQ} \times \left(-\dfrac{OB}{AB}\right) = -\dfrac{QP}{AB}$, since $OQ = OB$.

A. If the lines are perpendicular, then $\angle AOB = \angle POQ$. (adjacent angles at O)
Hence $\triangle AOB \equiv \triangle POQ$ (AAS)
so $QP = AB$ (matching sides of congruent triangles)
and so $m_1 \times m_2 = -1$.

B. Conversely, if $m_1 \times m_2 = -1$, then $QP = AB$.
Hence $\triangle AOB \equiv \triangle POQ$ (SAS)
so $\angle AOB = \angle POQ$, (matching angles of congruent triangles)
and so ℓ_1 and ℓ_2 are perpendicular.

WORKED EXERCISE:
What is the gradient of a line perpendicular to a line with gradient $\frac{2}{3}$?

SOLUTION:
Perpendicular gradient $= -\frac{3}{2}$. (Take the opposite of the reciprocal of $\frac{2}{3}$.)

WORKED EXERCISE:
Show that the diagonals of the quadrilateral $ABCD$ are perpendicular, where the vertices are $A(3,7)$, $B(-1,6)$, $C(-2,-3)$ and $D(11,0)$.

SOLUTION:

Gradient of $AC = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{-3-7}{-2-3} = 2$,

gradient of $BD = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{0-6}{11+1} = -\frac{1}{2}$.

Hence $AC \perp BD$, because the product of the gradients of AC and BD is -1.

WORKED EXERCISE:

The interval joining the points $C(-6, 0)$ and $D(-1, a)$ is perpendicular to a line with gradient 10. Find the value of a.

SOLUTION:

$$\text{The interval } CD \text{ has gradient} = \frac{a-0}{-1+6} = \frac{a}{5}.$$

Since the interval CD is perpendicular to a line with gradient 10,

$$\frac{a}{5} \times \frac{10}{1} = -1 \qquad \text{(The product of the gradients is } -1.)$$

$$a \times 2 = -1$$

$\boxed{\div 2}$ $\qquad a = -\frac{1}{2}.$

Exercise 6B

NOTE: Diagrams should be drawn wherever possible.

1. (a) Write down the gradient of a line parallel to a line with gradient:
(i) 2 (ii) -1 (iii) $\frac{3}{4}$ (iv) $-1\frac{1}{2}$

(b) Find the gradient of a line perpendicular to a line with gradient:
(i) 2 (ii) -1 (iii) $\frac{3}{4}$ (iv) $-1\frac{1}{2}$

2. Use the formula gradient $= \dfrac{y_2 - y_1}{x_2 - x_1}$ to find the gradient of each interval AB below. Then find the gradient of a line perpendicular to it.
(a) $A(1, 4)$, $B(5, 0)$ (c) $A(-5, -2)$, $B(3, 2)$ (e) $A(-1, -2)$, $B(1, 4)$
(b) $A(-2, -7)$, $B(3, 3)$ (d) $A(3, 6)$, $B(5, 5)$ (f) $A(-5, 7)$, $B(15, -7)$

3. The points $A(2, 5)$, $B(4, 11)$, $C(12, 15)$ and $D(10, 9)$ form a quadrilateral.
(a) Find the gradients of AB and DC, and hence show that $AB \parallel DC$.
(b) Find the gradients of BC and AD, and hence show that $BC \parallel AD$.
(c) What type of quadrilateral is $ABCD$? [HINT: Look at the definitions in Box 3 above.]

4. (a) Show that $A(-2, -6)$, $B(0, -5)$, $C(10, -7)$ and $D(8, -8)$ form a parallelogram.
(b) Show that $A(6, 9)$, $B(11, 3)$, $C(0, 0)$ and $D(-5, 6)$ form a parallelogram.
(c) Show that $A(2, 5)$, $B(3, 7)$, $C(-4, -1)$ and $D(-5, 2)$ do not form a parallelogram.

5. Use the formula gradient $= \tan\alpha$ to find the gradient, correct to two decimal places where necessary, of a line with angle of inclination:
(a) $15°$ (b) $135°$ (c) $22\frac{1}{2}°$ (d) $72°$

6. Use the formula gradient $= \tan\alpha$ to find the angle of inclination, correct to the nearest degree where necessary, of a line with gradient:
(a) 1 (b) $-\sqrt{3}$ (c) 4 (d) $\dfrac{1}{\sqrt{3}}$

DEVELOPMENT

7. The quadrilateral $ABCD$ has vertices $A(-1,1)$, $B(3,-1)$, $C(5,3)$ and $D(1,5)$. Use the definitions of the special quadrilaterals in Box 3 above to answer these questions.
(a) Show that the opposite sides are parallel, and hence that $ABCD$ is a parallelogram.
(b) Show that $AB \perp BC$, and hence that $ABCD$ is a rectangle.
(c) Show that $AB = BC$, and hence that $ABCD$ is a square.

8. Use gradients to show that each quadrilateral $ABCD$ below is a parallelogram. Then use the definitions in Box 3 of the notes to show that it is:
(a) a rhombus, for the vertices $A(2,1)$, $B(-1,3)$, $C(1,0)$ and $D(4,-2)$,
(b) a rectangle, for the vertices $A(4,0)$, $B(-2,3)$, $C(-3,1)$ and $D(3,-2)$,
(c) a square, for the vertices $A(3,3)$, $B(-1,2)$, $C(0,-2)$ and $D(4,-1)$.

9. A quadrilateral has vertices $W(2,3)$, $X(-7,5)$, $Y(-1,-3)$ and $Z(5,-1)$.
(a) Show that WZ is parallel to XY, but that WZ and XY have different lengths.
(b) What type of quadrilateral is $WXYZ$? [HINT: Look at Boxes 3 and 4 above.]
(c) Show that the diagonals WY and XZ are perpendicular.

10. Find the gradients of PQ and QR, and hence determine whether P, Q and R are collinear.
(a) $P(-2,7)$, $Q(1,1)$, $R(4,-6)$
(b) $P(-5,-4)$, $Q(-2,-2)$, $R(1,0)$

11. Show that the four points $A(2,5)$, $B(5,6)$, $C(11,8)$ and $D(-16,-1)$ are collinear.

12. The triangle ABC has vertices $A(-1,0)$, $B(3,2)$ and $C(4,0)$. Calculate the gradient of each side and hence show that $\triangle ABC$ is a right-angled triangle.

13. Similarly, show that each triangle below is right-angled. Then find the lengths of the sides enclosing the right angle, and calculate the area of each triangle.
(a) $P(2,-1)$, $Q(3,3)$, $R(-1,4)$
(b) $X(-1,-3)$, $Y(2,4)$, $Z(-3,2)$

14. (a) Write down two points A and B for which the interval AB has gradient 3.
(b) Write down two points A and B for which the interval AB is vertical.
(c) Write down two points A and B for which AB has gradient 2 and midpoint $M(4,6)$.

15. The interval PQ has gradient -3. A second line passes through $A(-2,4)$ and $B(1,k)$.
(a) Find k if AB is parallel to PQ.
(b) Find k if AB is perpendicular to PQ.

16. Find the points A and B where each line below meets the x-axis and y-axis respectively. Hence find the gradient of AB and its angle of inclination α (correct to the nearest degree).
(a) $y = 3x + 6$
(b) $y = -\frac{1}{2}x + 1$
(c) $3x + 4y + 12 = 0$
(d) $\dfrac{x}{3} - \dfrac{y}{2} = 1$
(e) $4x - 5y - 20 = 0$
(f) $\dfrac{x}{2} + \dfrac{y}{5} = 1$

17. The quadrilateral $ABCD$ has vertices $A(1,-4)$, $B(3,2)$, $C(-5,6)$ and $D(-1,-2)$.
(a) Find the midpoints P of AB, Q of BC, R of CD, and S of DA.
(b) Prove that $PQRS$ is a parallelogram by showing that $PQ \parallel RS$ and $PS \parallel QR$.

18. The points $A(1,4)$, $B(5,0)$ and $C(9,8)$ form the vertices of a triangle.
(a) Find the coordinates of the midpoints P and Q of AB and AC respectively.
(b) Show that PQ is parallel to BC and half its length.

19. (a) Show that the points $A(-5,0)$, $B(5,0)$ and $C(3,4)$ all lie on the circle $x^2 + y^2 = 25$.
(b) Explain why AB is a diameter of the circle.
(c) Show that $AC \perp BC$.

CHALLENGE

20. Find the gradient, correct to two decimal places where appropriate, of a line sloping upwards, if its acute angle with the y-axis is:
(a) $15°$ (b) $45°$ (c) $22\frac{1}{2}°$ (d) $72°$

21. Given the points $X(-1, 0)$, $Y(1, a)$ and $Z(a, 2)$, find a if $\angle YXZ = 90°$.

22. For the four points $P(k, 1)$, $Q(-2, -3)$, $R(2, 3)$ and $S(1, k)$, it is known that PQ is parallel to RS. Find the possible values of k.

6 C Equations of Lines

In coordinate geometry, a line is represented by a linear equation in x and y. This section and the next develop various useful forms for the equation of a line.

Horizontal and Vertical Lines: All the points on a vertical line must have the same x-coordinate, but the y-coordinate can take any value.

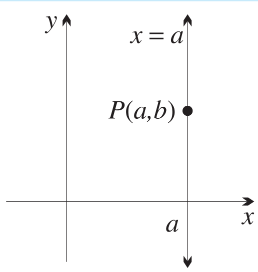
10 VERTICAL LINES: The vertical line through the point $P(a, b)$ has equation

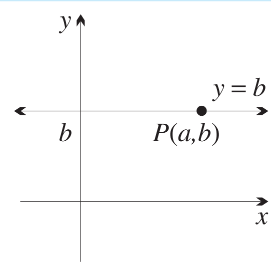
$$x = a.$$

All the points on a horizontal line must have the same y-coordinate, but the x-coordinate can take any value.

11 HORIZONTAL LINES: The horizontal line through the point $P(a, b)$ has equation

$$y = b.$$





Gradient–Intercept Form: There is a simple form of the equation of a line whose gradient and y-intercept are known.

Let ℓ have gradient m and y-intercept b, passing through the point $B(0, b)$.

Let $Q(x, y)$ be any other point in the plane.

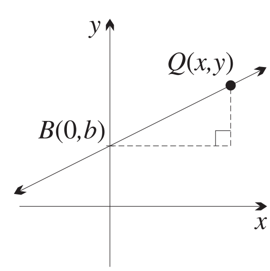
Then the condition for Q to lie on the line ℓ is

gradient of $BQ = m$,

that is, $\dfrac{y-b}{x-0} = m$ (This is the formula for gradient.)

$$y - b = mx.$$

$$y = mx + b.$$



12 GRADIENT–INTERCEPT FORM: The line with gradient m and y-intercept b is

$$y = mx + b.$$

WORKED EXERCISE:

(a) Write down the gradient and the y-intercept of the line $\ell: \; y = 3x - 2$.
(b) Find the equation of the line through $B(0,5)$ parallel to ℓ.
(c) Find the equation of the line through $B(0,5)$ perpendicular to ℓ.

SOLUTION:

(a) The line $\ell: \; y = 3x - 2$ has gradient 3 and y-intercept -2.

(b) The line through $B(0,5)$ parallel to ℓ has gradient 3 and y-intercept 5, so its equation is $y = 3x + 5$.

(c) The line through $B(0,5)$ perpendicular to ℓ has gradient $-\frac{1}{3}$ and y-intercept 5, so its equation is $y = -\frac{1}{3}x + 5$.

General Form: It is often useful to write the equation of a line so that all the terms are on the LHS and only zero is on the RHS. This is called *general form*.

13 GENERAL FORM: The equation of a line is said to be in *general form* if it is

$$ax + by + c = 0, \quad \text{where } a,\ b \text{ and } c \text{ are constants.}$$

- When an equation is given in general form, it should usually be *simplified* by multiplying out all fractions and dividing through by all common factors.

WORKED EXERCISE:

(a) Put the equation of the line $3x + 4y + 5 = 0$ in gradient–intercept form.
(b) Hence write down the gradient and y-intercept of the line $3x + 4y + 5 = 0$.

SOLUTION:

(a) Solving the equation for y, $4y = -3x - 5$

$\boxed{\div 4}$ $y = -\frac{3}{4}x - \frac{5}{4}$, which is gradient–intercept form.

(b) Hence the line has gradient $-\frac{3}{4}$ and y-intercept $-1\frac{1}{4}$.

WORKED EXERCISE:

Find, in general form, the equation of the line passing through $B(0,-2)$ and:
(a) perpendicular to a line ℓ with gradient $\frac{2}{3}$,
(b) having angle of inclination 60°.

SOLUTION:

(a) The line through B perpendicular to ℓ has gradient $-\frac{3}{2}$ and y-intercept -2,

so its equation is $y = -\frac{3}{2}x - 2$ (This is gradient–intercept form.)

$\boxed{\times 2}$ $2y = -3x - 4$

$3x + 2y + 4 = 0.$

(b) The line through B with angle of inclination 60° has gradient $\tan 60^\circ = \sqrt{3}$,

so its equation is $y = x\sqrt{3} - 2$ (This is gradient–intercept form.)

$x\sqrt{3} - y - 2 = 0.$

Exercise 6C

1. Determine, by substitution, whether the point $A(3, -2)$ lies on the line:
(a) $y = 4x - 10$ (b) $8x + 10y - 4 = 0$ (c) $x = 3$

2. Write down the coordinates of any three points on the line $x + 3y = 24$.

3. Write down the equations of the vertical and horizontal lines through:
(a) $(1, 2)$ (b) $(-1, 1)$ (c) $(0, -4)$ (d) $(5, 0)$ (e) $(-2, -3)$

4. Write down the gradient and y-intercept of each line.
(a) $y = 4x - 2$ (b) $y = \frac{1}{5}x - 3$ (c) $y = 2 - x$ (d) $y = -\frac{5}{7}x$

5. Use the formula $y = mx + b$ to write down the equation of the line with gradient -3 and:
(a) y-intercept 5, (b) y-intercept $-\frac{2}{3}$, (c) y-intercept $17\frac{1}{2}$, (d) y-intercept 0.

6. Use the formula $y = mx+b$ to write down the equation of the line with y-intercept -4 and:
(a) gradient 5, (b) gradient $-\frac{2}{3}$, (c) gradient $17\frac{1}{2}$, (d) gradient 0.

7. Use the formula $y = mx + b$ to write down the equation of the line:
(a) with gradient 1 and y-intercept 3,
(b) with gradient -2 and y-intercept 5,
(c) with gradient $\frac{1}{5}$ and y-intercept -1,
(d) with gradient $-\frac{1}{2}$ and y-intercept 3.

8. Solve each equation for y and hence write down its gradient m and y-intercept b.
(a) $x - y + 3 = 0$
(b) $y + x - 2 = 0$
(c) $2x - y = 5$
(d) $x - 3y = 0$
(e) $3x + 4y = 5$
(f) $2y - 3x = -4$

9. Write down the gradient m of each line. Then use the formula gradient $= \tan\alpha$ to find its angle of inclination α, correct to the nearest minute where appropriate.
(a) $y = x + 3$ (b) $y = -x - 16$ (c) $y = 2x$ (d) $y = -\frac{3}{4}x$

DEVELOPMENT

10. Substitute $y = 0$ and $x = 0$ into the equation of each line below to find the points A and B where the line crosses the x-axis and y-axis respectively. Hence sketch the line.
(a) $5x + 3y - 15 = 0$ (b) $2x - y + 6 = 0$ (c) $3x - 5y + 12 = 0$

11. Find the gradient of the line through each pair of given points. Then find its equation, using gradient–intercept form. Give your final answer in general form.
(a) $(0, 4)$, $(2, 8)$ (b) $(0, 0)$, $(1, -1)$ (c) $(-9, -1)$, $(0, -4)$ (d) $(2, 6)$, $(0, 11)$

12. Find the gradient of each line below. Hence find, in gradient–intercept form, the equation of a line passing through $A(0, 3)$ and: (i) parallel to it, (ii) perpendicular to it.
(a) $2x + y + 3 = 0$ (b) $5x - 2y - 1 = 0$ (c) $3x + 4y - 5 = 0$

13. Find the gradients of the four lines in each part. Hence state what sort of special quadrilateral they enclose.
(a) $3x + y + 7 = 0$, $x - 2y - 1 = 0$, $3x + y + 11 = 0$, $x - 2y + 12 = 0$
(b) $4x - 3y + 10 = 0$, $3x + 4y + 7 = 0$, $4x - 3y - 7 = 0$, $3x + 4y + 1 = 0$

14. Find the gradients of the three lines $5x - 7y + 5 = 0$, $2x - 5y + 7 = 0$ and $7x + 5y + 2 = 0$. Hence show that they enclose a right-angled triangle.

15. Draw a sketch of, then find the equations of the sides of:
(a) the rectangle with vertices $P(3, -7)$, $Q(0, -7)$, $R(0, -2)$ and $S(3, -2)$,
(b) the triangle with vertices $F(3, 0)$, $G(-6, 0)$ and $H(0, 12)$.

16. In each part below, the angle of inclination α and the y-intercept A of a line are given. Use the formula gradient $= \tan\alpha$ to find the gradient of each line, then find its equation in general form.

(a) $\alpha = 45°$, $A = (0, 3)$
(b) $\alpha = 60°$, $A = (0, -1)$
(c) $\alpha = 30°$, $A = (0, -2)$
(d) $\alpha = 135°$, $A = (0, 1)$

CHALLENGE

17. A triangle is formed by the x-axis and the lines $5y = 9x$ and $5y + 9x = 45$.
(a) Find (correct to the nearest degree) the angles of inclination of the two lines.
(b) What sort of triangle has been formed?

18. Consider the two lines ℓ_1: $3x - y + 4 = 0$ and ℓ_2: $x + ky + \ell = 0$. Find the value of k if:
(a) ℓ_1 is parallel to ℓ_2,
(b) ℓ_1 is perpendicular to ℓ_2.

6 D Further Equations of Lines

This section introduces another standard form of the equation of a line, called *point–gradient form*. It also deals with lines through two given points, and the point of intersection of two lines.

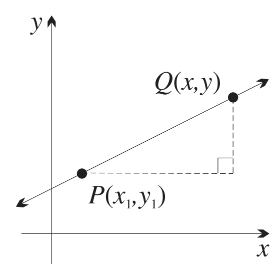
Point–Gradient Form: Point–gradient form gives the equation of a line with gradient m passing through a particular point $P(x_1, y_1)$.

Let $Q(x, y)$ be any other point in the plane.
Then the condition for Q to lie on the line is

gradient of $PQ = m$,

that is, $\dfrac{y - y_1}{x - x_1} = m$ (This is the formula for gradient.)

$y - y_1 = m(x - x_1)$.



14 POINT–GRADIENT FORM: The line with gradient m through the point (x_1, y_1) is

$$y - y_1 = m(x - x_1).$$

WORKED EXERCISE:

(a) Find the equation of the line through $(-2, -5)$ and parallel to $y = 3x + 2$.
(b) Express the answer in gradient–intercept form, and hence find its y-intercept.

SOLUTION:

(a) The line $y = 3x + 2$ has gradient 3.
Hence the required line is $y - y_1 = m(x - x_1)$ (This is point–gradient form.)
$y + 5 = 3(x + 2)$
$y + 5 = 3x + 6$
$y = 3x + 1$. (This is gradient–intercept form.)

(b) Hence the new line has y-intercept 1.

The Line through Two Given Points: Given two distinct points, there is just one line passing through them both. Its equation is best found by a two-step approach.

15 THE LINE THROUGH TWO GIVEN POINTS:

- First find the gradient of the line, using the gradient formula.
- Then find the equation of the line, using point–gradient form.

WORKED EXERCISE:
Find the equation of the line passing through $A(1,5)$ and $B(4,-1)$.

SOLUTION:
First, using the gradient formula,

$$\text{gradient of } AB = \frac{-1-5}{4-1} = -2.$$

Then, using point–gradient form for a line with gradient -2 through $A(1,5)$,
the line AB is

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 5 &= -2(x-1) \\ y - 5 &= -2x + 2 \\ y &= -2x + 7. \end{aligned}$$

NOTE: Using the coordinates of $B(4,-1)$ rather than $A(1,5)$ would give the same equation.

WORKED EXERCISE:
Given the points $A(6,0)$ and $B(0,9)$, find, in general form, the equation of:
(a) the line AB,
(b) the line through A perpendicular to AB.

SOLUTION:
(a) First, gradient of $AB = \dfrac{9-0}{0-6}$
$= -\frac{3}{2}$,

so the line AB is $y = mx + b$ (This is gradient–intercept form.)
$y = -\frac{3}{2}x + 9$ (The line has y-intercept 9.)
$\boxed{\times 2}$ $2y = -3x + 18$
$3x + 2y - 18 = 0.$

(b) The gradient of the line perpendicular to AB is $\frac{2}{3}$,
so the line is $y - y_1 = m(x - x_1)T$ (This is point–gradient form.)
$y - 0 = \frac{2}{3}(x-6)$
$\boxed{\times 3}$ $3y = 2(x-6)$
$3y = 2x - 12$
$2x - 3y - 12 = 0.$

Intersection of Lines — Concurrent Lines: The point where two distinct lines intersect can be found using simultaneous equations, as discussed in Chapter One.

Three distinct lines are called *concurrent* if they all pass through the same point.

16 TESTING FOR CONCURRENT LINES: To test whether three given lines are concurrent:
- First find the point of intersection of two of them.
- Then test, by substitution, whether this point lies on the third line.

WORKED EXERCISE:
Test whether the following three lines are concurrent.

$$\ell_1 : 5x - y - 10 = 0, \qquad \ell_2 : x + y - 8 = 0, \qquad \ell_3 : 2x - 3y + 9 = 0.$$

SOLUTION:

A. Solve ℓ_1 and ℓ_2 simultaneously.
Adding ℓ_1 and ℓ_2, $\quad 6x - 18 = 0$
$x = 3,$
and substituting into ℓ_2, $\quad 3 + y - 8 = 0$
$y = 5$
so the lines ℓ_1 and ℓ_2 intersect at $(3, 5)$.

B. Substituting the point $(3, 5)$ into the third line ℓ_3,
$$\begin{aligned} \text{LHS} &= 6 - 15 + 9 \\ &= 0 \\ &= \text{RHS}, \end{aligned}$$
so the three lines are concurrent, meeting at $(3, 5)$.

Exercise 6D

1. Use point–gradient form $y - y_1 = m(x - x_1)$ to find the equation of the line through the point $P(3, 5)$ with gradient 3. Rearrange your answer into general form.

2. Use point–gradient form $y - y_1 = m(x - x_1)$ to find the equation of the line through the point $P(-2, 7)$ with each given gradient. Rearrange your answer into general form.
(a) gradient 6 (b) gradient -2 (c) gradient $\frac{2}{3}$ (d) gradient $-\frac{7}{2}$

3. Use point–gradient form $y - y_1 = m(x - x_1)$ to find the equation of the line through each given point with gradient $-\frac{3}{5}$. Rearrange your answer into general form.
(a) through $(1, 2)$ (b) through $(6, 0)$ (c) through $(-5, 3)$ (d) through $(0, -4)$

4. Find, in general form, the equation of the line:
(a) through $(1, 1)$ with gradient 2,
(b) with gradient -1 through $(3, 1)$,
(c) with gradient 3 through $(-5, -7)$,
(d) through $(0, 0)$ with gradient -5,
(e) through $(-1, 3)$ with gradient $-\frac{1}{3}$,
(f) with gradient $-\frac{4}{5}$ through $(3, -4)$.

5. Find, in gradient–intercept form, the equation of:
(a) the line through $(2, 5)$ and parallel to $y = 2x + 5$,
(b) the line through $(2, 5)$ and perpendicular to $y = 2x + 5$,
(c) the line through $(5, -7)$ and parallel to $y = -5x$,
(d) the line through $(5, -7)$ and perpendicular to $y = -5x$,
(e) the line through $(-7, 6)$ and parallel to $y = \frac{3}{7}x - 8$,
(f) the line through $(-4, 0)$ and perpendicular to $y = -\frac{2}{5}x$.

6. (a) Find the gradient of the line through the points $A(4, 7)$ and $B(6, 13)$.
(b) Hence use point–gradient form to find, in general form, the equation of the line AB.

7. Find the gradient of the line through each pair of points, and hence find its equation.
(a) $(3, 4)$, $(5, 8)$ (c) $(-4, -1)$, $(6, -6)$ (e) $(-1, 0)$, $(0, 2)$ (g) $(0, -1)$, $(-4, 0)$
(b) $(-1, 3)$, $(1, -1)$ (d) $(5, 6)$, $(-1, 4)$ (f) $(2, 0)$, $(0, 3)$ (h) $(0, -3)$, $(3, 0)$

8. (a) Find the gradient of the line through $A(1, -2)$ and $B(-3, 4)$.
(b) Hence find, in general form, the equation of:
(i) the line AB, (ii) the line through A and perpendicular to AB.

9. Find the equation of the line parallel to $2x - 3y + 1 = 0$ and:
(a) passing through $(2, 2)$, (b) passing through $(3, -1)$.

10. Find the equation of the line perpendicular to $3x + 4y - 3 = 0$ and:
(a) passing through $(-1, -4)$, (b) passing through $(-2, 1)$.

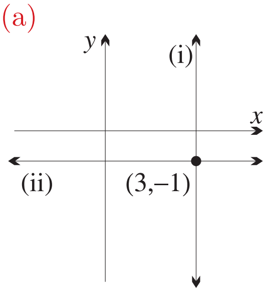
DEVELOPMENT

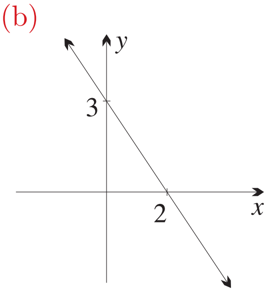
11. (a) Find the point M of intersection of the lines ℓ_1: $x + y = 2$ and ℓ_2: $4x - y = 13$.
(b) Show that M lies on ℓ_3: $2x - 5y = 11$, and hence that ℓ_1, ℓ_2 and ℓ_3 are concurrent.
(c) Use the same method to test whether each set of lines is concurrent.
(i) $2x + y = -1$, $x - 2y = -18$ and $x + 3y = 15$
(ii) $6x - y = 26$, $5x - 4y = 9$ and $x + y = 9$

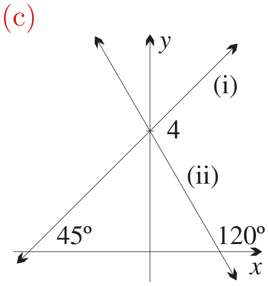
12. Put the equation of each line in gradient–intercept form and hence write down the gradient. Then find, in gradient–intercept form, the equation of the line that is:
(i) parallel to it through $A(3, -1)$, (ii) perpendicular to it through $B(-2, 5)$.
(a) $2x + y + 3 = 0$ (b) $5x - 2y - 1 = 0$ (c) $4x + 3y - 5 = 0$

13. The angle of inclination α and a point A on a line are given below. Use the formula gradient $= \tan\alpha$ to find the gradient of each line, then find its equation in general form.
(a) $\alpha = 45°$, $A = (1, 0)$ (c) $\alpha = 30°$, $A = (4, -3)$
(b) $\alpha = 120°$, $A = (-1, 0)$ (d) $\alpha = 150°$, $A = (-2, -5)$

14. Determine, in general form, the equation of each straight line sketched below.

(a)


(b)


(c)


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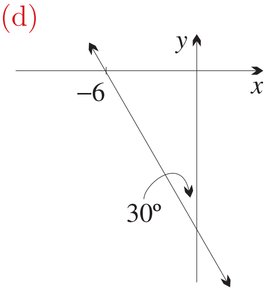
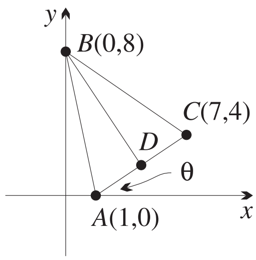


15. Explain why the four lines ℓ_1: $y = x + 1$, ℓ_2: $y = x - 3$, ℓ_3: $y = 3x + 5$ and ℓ_4: $y = 3x - 5$ enclose a parallelogram. Then find the vertices of this parallelogram.

16. Show that the triangle with vertices $A(1, 0)$, $B(6, 5)$ and $C(0, 2)$ is right-angled. Then find the equation of each side.

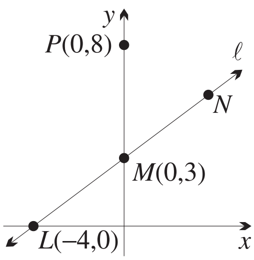
17. The three points $A(1, 0)$, $B(0, 8)$ and $C(7, 4)$ form a triangle. Let θ be the angle between AC and the x-axis.



(a) Find the gradient of the line AC and hence determine θ, correct to the nearest degree.
(b) Derive the equation of AC.
(c) Find the coordinates of the midpoint D of AC.
(d) Show that AC is perpendicular to BD.
(e) What type of triangle is ABC?
(f) Find the area of this triangle.
(g) Write down the coordinates of a point E such that the parallelogram $ABCE$ is a rhombus.

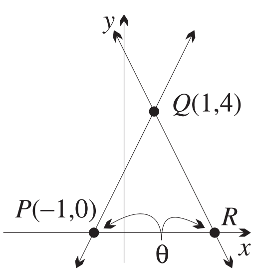
18. (a) On a number plane, plot the points $A(4, 3)$, $B(0, -3)$ and $C(4, 0)$.
(b) Find the equation of BC.
(c) Explain why $OABC$ is a parallelogram.
(d) Find the area of $OABC$ and the length of the diagonal AB.

19. The line ℓ crosses the x- and y-axes at $L(-4, 0)$ and $M(0, 3)$. The point N lies on ℓ and P is the point $P(0, 8)$.



(a) Copy the sketch and find the equation of ℓ.
(b) Find the lengths of ML and MP and hence show that LMP is an isosceles triangle.
(c) If M is the midpoint of LN, find the coordinates of N.
(d) Show that $\angle NPL = 90°$.
(e) Write down the equation of the circle through N, P and L.

20. The vertices of a triangle are $P(-1, 0)$ and $Q(1, 4)$ and R, where R lies on the x-axis and $\angle QPR = \angle QRP = \theta$.



(a) Find the coordinates of the midpoint of PQ.
(b) Find the gradient of PQ and show that $\tan\theta = 2$.
(c) Show that PQ has equation $y = 2x + 2$.
(d) Explain why QR has gradient -2, and hence find its equation.
(e) Find the coordinates of R and hence the area of triangle PQR.
(f) Find the length QR, and hence find the perpendicular distance from P to QR.

CHALLENGE

21. Find k if the lines ℓ_1: $x + 3y + 13 = 0$, ℓ_2: $4x + y - 3 = 0$ and ℓ_3: $kx - y - 10 = 0$ are concurrent. [HINT: Find the point of intersection of ℓ_1 and ℓ_2 and substitute into ℓ_3.]

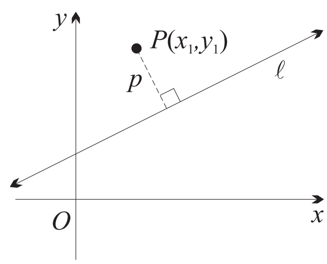
22. Consider the two lines ℓ_1: $3x + 2y + 4 = 0$ and ℓ_2: $6x + \mu y + \lambda = 0$.
(a) Write down the value of μ if: (i) ℓ_1 is parallel to ℓ_2, (ii) ℓ_1 is perpendicular to ℓ_2.
(b) Given that ℓ_1 and ℓ_2 intersect at a point, what condition must be placed on μ?
(c) Given that ℓ_1 is parallel to ℓ_2, write down the value of λ if:
  (i) ℓ_1 is the same line as ℓ_2,
  (ii) the distance between the y-intercepts of the two lines is 2.

6 E Perpendicular Distance

The shortest distance from a point P to a line ℓ is the *perpendicular distance*, which is the distance p in the diagram to the right. There is a simple formula for this perpendicular distance, which is proven in the appendix to this chapter.

17 PERPENDICULAR DISTANCE FORMULA:
The perpendicular distance p from the point (x_1, y_1) to the line $ax + by + c = 0$ is

$$p = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}.$$



Notice that the numerator is the point (x_1, y_1) substituted into $|ax + by + c|$. The line must be rearranged into general form before the formula can be applied.

WORKED EXERCISE:
Find the perpendicular distance from the point $P(-2, 5)$ to the line $y = 2x - 1$.

SOLUTION:
The line in general form is $2x - y - 1 = 0$,

$$\begin{aligned} \text{so perpendicular distance} &= \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}} \\ &= \frac{|-4-5-1|}{\sqrt{2^2 + (-1)^2}} && \text{(Substitute } P(-2,5) \text{ into } 2x - y - 1.) \\ &= \frac{10}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} && \text{(Rationalise the denominator.)} \\ &= \frac{10\sqrt{5}}{5} \\ &= 2\sqrt{5}. \end{aligned}$$

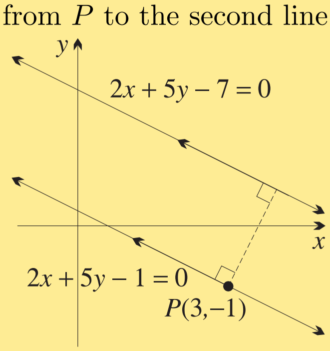
Distance between Parallel Lines: To find the distance between two parallel lines, choose any point on one line and find its perpendicular distance from the other line.

WORKED EXERCISE:
Find the perpendicular distance between the two parallel lines

$$2x + 5y - 1 = 0 \qquad \text{and} \qquad 2x + 5y - 7 = 0.$$

SOLUTION:
Choose any convenient point on the first line, say $P(3, -1)$. (Its coefficients are integers.)
The distance between the lines is the perpendicular distance from P to the second line,

$$\begin{aligned} \text{so perpendicular distance} &= \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}} \\ &= \frac{|6-5-7|}{\sqrt{2^2 + 5^2}} \\ &= \frac{|-6|}{\sqrt{4+25}} \\ &= \frac{6}{\sqrt{29}}. \end{aligned}$$



Circles and the Perpendicular Distance Formula: A line is a tangent to a circle when its perpendicular distance from the centre is equal to the radius. Lines closer to the centre are secants, and lines more distant miss the circle entirely.

WORKED EXERCISE:

(a) Show that ℓ: $3x+4y-20=0$ is a tangent to the circle $(x-7)^2+(y-6)^2=25$.

(b) Find the length of the chord of the circle cut off by the line m: $3x+4y-60=0$.

SOLUTION:

The circle has centre $(7,6)$ and radius 5.

(a) The distance from the line ℓ to the centre is

$$p = \frac{|21+24-20|}{\sqrt{3^2+4^2}}$$

$= 5$, so ℓ is a tangent to the circle.

(b) The distance p from the line m to the centre is

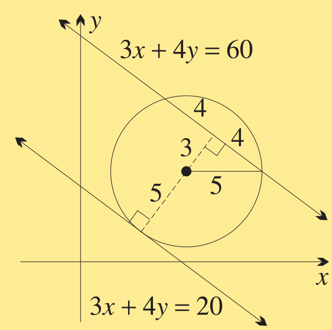
$$p = \frac{|21+24-60|}{\sqrt{3^2+4^2}}$$

$= 3$.

Using Pythagoras' theorem in the circle to the right,

chord length $= 2 \times 4$

$= 8$ units.



WORKED EXERCISE:

For what values of k will the line $5x-12y+k=0$ never intersect the circle with centre $P(-3,1)$ and radius 6?

SOLUTION:

The condition is $\dfrac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}} > 6$ (The perpendicular distance exceeds the radius.)

$$\frac{|-15-12+k|}{\sqrt{5^2+12^2}} > 6$$

$$\frac{|-27+k|}{13} > 6$$

$$|k-27| > 78$$

$$k < -51 \text{ or } k > 105.$$

Exercise 6E

1. Use $p = \dfrac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$ to find the perpendicular distance from each line to the origin.

(a) $x+3y+5=0$ (b) $2x-y+4=0$ (c) $2x+4y-5=0$

2. Use $p = \dfrac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$ to find the perpendicular distance from each point to each line.

(a) $(2,0)$ and $3x+4y-1=0$

(b) $(-2,1)$ and $12x-5y+3=0$

(c) $(-3,2)$ and $4x-y-3=0$

(d) $(-3,-2)$ and $x+3y+4=0$

(e) $(3,-1)$ and $x+2y-1=0$

(f) $(1,3)$ and $2x+4y+1=0$

3. Which of the given points is: (a) closest to, (b) furthest from, the line $6x-8y-9=0$?
$A(1,-1)$ $B(3,2)$ $C(-4,1)$ $D(-3,-3)$

4. Which of the given lines is: (a) closest to, (b) furthest from, the point $(-1,5)$?
ℓ_1: $2x+3y+4=0$ ℓ_2: $x-4y+7=0$ ℓ_3: $3x+y-8=0$

DEVELOPMENT

5. Write down the centre and radius of each circle. Then use the perpendicular distance formula to determine how many times the given line intersects the circle.
(a) $3x-5y+16=0$, $x^2+y^2=5$
(b) $7x+y-10=0$, $x^2+y^2=2$
(c) $3x-y-8=0$, $(x-1)^2+(y-5)^2=10$
(d) $x+2y+3=0$, $(x+2)^2+(y-1)^2=6$

6. Choose any point on the first line to find the distance between the parallel lines in each pair.
(a) $x-3y+5=0$, $x-3y-2=0$
(b) $4x+y-2=0$, $4x+y+8=0$

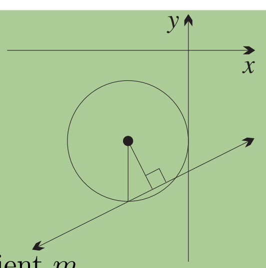
7. (a) The line $y-2x+k=0$ is $2\sqrt{5}$ units from the point $(1,-3)$. Find k.
(b) The line $3x-4y+2=0$ is $\frac{3}{5}$ units from the point $(-1,\ell)$. Find ℓ.

8. Find the range of values that k may take if:
(a) the line $y-x+k=0$ is more than $\frac{1}{\sqrt{2}}$ units from the point $(2,7)$.
(b) the line $x+2y-5=0$ is at most $\sqrt{5}$ units from the point $(k,3)$.

9. The vertices of a triangle are $A(-3,-2)$, $B(3,1)$ and $C(-1,4)$.
(a) Find the equation of the side AB in general form.
(b) How far is C from this line?
(c) Find the length of AB and hence find the area of this triangle.
(d) Similarly, find the area of the triangle with vertices $P(1,-1)$, $Q(-1,5)$ and $R(-3,1)$.

10. Draw on a number plane the triangle ABC with vertices $A(5,0)$, $B(8,4)$ and $C(0,10)$.
(a) Show that the line AB has equation $3y=4x-20$.
(b) Show that the gradient of BC is $-\frac{3}{4}$.
(c) Hence show that AB and BC are perpendicular.
(d) Show that the interval AB has length 5 units.
(e) Show that the triangles AOC and ABC are congruent.
(f) Find the area of quadrilateral $OABC$.
(g) Find the distance from $D(0,8)$ to the line AB.

CHALLENGE



11. (a) Write down the centre and radius of the circle with equation $(x+2)^2+(y+3)^2=4$. Then find the distance from the line $2y-x+8=0$ to the centre.
(b) Use Pythagoras' theorem to determine the length of the chord cut off from the line by the circle.

12. (a) Write down the equation of a line through the origin with gradient m.
(b) Write down the distance from this line to the point $(3,1)$.
(c) If the line is tangent to the circle $(x-3)^2+(y-1)^2=4$, show that m satisfies the equation $5m^2-6m-3=0$.
(d) Find the possible values of m, and hence find the equations of the two tangents.

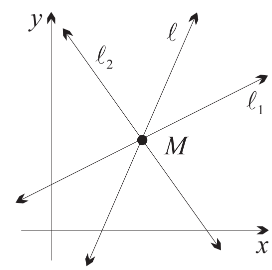
6 F Lines Through the Intersection of Two Given Lines

This section develops an ingenious way of finding the equations of particular lines through the intersection of two given lines without actually solving the two given lines simultaneously to find their point of intersection. The method is another situation where the general form of the equation of the line is used.

The General Form of Such a Line: Suppose that two lines

$$\ell_1\colon a_1x + b_1y + c_1 = 0 \quad \text{and} \quad \ell_2\colon a_2x + b_2y + c_2 = 0$$

intersect at a point M. The set of all the lines through M forms a *family* of lines through M. In the diagram to the right, the line ℓ is one of these lines.



It is proven in the appendix to this chapter that every line passing through M has the form

$$(a_1x + b_1y + c_1) \; + \; k(a_2x + b_2y + c_2) = 0 \qquad (*)$$

for some value of the constant k. Notice that substituting different values of k into $(*)$ will clearly yield different lines — the proof in the appendix simply shows that all these lines pass through M.

18 LINE THROUGH THE INTERSECTION OF TWO GIVEN LINES:
Let two lines $\ell_1\colon a_1x + b_1y + c_1 \; = 0$ and $\ell_2\colon a_2x + b_2y + c_2) \; = 0$ intersect at a point M. Then every line passing through M has the form

$$(a_1x + b_1y + c_1) \; + \; k(a_2x + b_2y + c_2) = 0,$$

where k is a constant.

To find any particular line through M, all that is necessary is to substitute the extra information into the equation above to find the value of k

WORKED EXERCISE:
Find the equation of the line that passes through the intersection M of the lines

$$\ell_1\colon \; x + 2y - 6 = 0 \qquad \text{and} \qquad \ell_2\colon 3x - 2y - 6 = 0$$

and also passes through the point $P(2, -1)$.

SOLUTION:
Using Box 18, any line through M has equation

$$(x + 2y - 6) \; + \; k(3x - 2y - 6) = 0, \text{ for some constant } k. \qquad (1)$$

Substituting the point $P(2, -1)$ into equation (1) gives

$$\begin{aligned} (2 - 2 - 6) \; + \; k(6 + 2 - 6) &= 0 \\ -6 + 2k &= 0 \\ 2k &= 6 \\ k &= 3. \end{aligned}$$

Substituting $k = 3$ back into equation (1), the required line is

$$\begin{aligned} (x + 2y - 6) \; + \; 3(3x - 2y - 6) &= 0 \\ x + 2y - 6 + 9x - 6y - 18 &= 0 \\ 10x - 4y - 24 &= 0 \\ 5x - 2y - 12 &= 0. \end{aligned}$$

WORKED EXERCISE:

Find the equation of the line that passes through the intersection M of the lines

$$\ell_1:\ x + 2y - 6 = 0 \qquad \text{and} \qquad \ell_2: 3x - 2y - 6 = 0$$

(a) and has gradient 5, (b) and is vertical.

SOLUTION:

As in the previous worked exercise, any line through M has equation

$$(x + 2y - 6) + k(3x - 2y - 6) = 0, \text{ for some constant } k. \qquad (1)$$

Equation (1) can be rearranged into general form by collecting terms in x, and in y.

$$\begin{aligned}(x + 2y - 6) + k(3x - 2y - 6) &= 0\\ x + 2y - 6 + 3kx - 2ky - 6k &= 0\\ (1 + 3k)x + (2 - 2k)y - (6 + 6k) &= 0 \qquad (2)\end{aligned}$$

(a) Now making y the subject, $(2 - 2k)y = (-1 - 3k)x + (6 + 6k)$

$$y = \frac{(-1 - 3k)x}{2 - 2k} + \frac{6 + 6k}{2 - 2k}$$

and so the gradient of the line with equation (1) is $\dfrac{-1 - 3k}{2 - 2k}$.

Since the gradient is 5, $\dfrac{-1 - 3k}{2 - 2k} = 5$

$$\begin{aligned}-1 - 3k &= 10 - 10k\\ 7k &= 11\\ k &= \tfrac{11}{7}.\end{aligned}$$

Substituting back into equation (1), the required line is

$$\begin{aligned}(x + 2y - 6) + \tfrac{11}{7}(3x - 2y - 6) &= 0\\ 7(x + 2y - 6) + 11(3x - 2y - 6) &= 0\\ 40x - 8y - 108 &= 0\\ 10x - 2y - 27 &= 0.\end{aligned}$$

(b) Since the line is vertical, the coefficient of y in equation (2) is zero.

that is, $2 - 2k = 0$

$$k = 1.$$

Hence from (2), the required line is $4x - 12 = 0$

$$x = 3.$$

Exercise **6F**

1. (a) Graph the lines $x - y = 0$ and $x + y - 2 = 0$ on grid or graph paper and label them (they intersect at $(1, 1)$).

(b) Simplify the equation $(x - y) + k(x + y - 2) = 0$ for $k = 2$, 1, $\frac{1}{2}$ and 0. Add these lines to your graph, and label each line with its value of k. Observe that each line passes through $(1, 1)$.

(c) Repeat this process for $k = -\frac{1}{2}$, -1 and -2, adding these lines to your graph.

2. The lines $x + 2y + 9 = 0$ and $2x - y + 3 = 0$ intersect at B.

(a) Write down the general equation of a line through B.

(b) Hence find the equation of the line ℓ through B and the origin O.

3. (a) Write down the general form of a line through the point T of intersection of the two lines $2x - 3y + 6 = 0$ and $x + 3y - 15 = 0$.

(b) Hence find the equation of the line through T and:

(i) $(3, 8)$ (ii) $(6, 0)$ (iii) $(-3, 3)$ (iv) $(0, 0)$

4. Find the equation of the line through the intersection of the lines $x - y - 3 = 0$ and $y + 3x - 5 = 0$ and each given point. Do not find the point of intersection of the lines.

(a) $(0, -2)$ (b) $(-1, 5)$ (c) $(3, 0)$

5. The lines $2x + y - 5 = 0$ and $x - y + 2 = 0$ intersect at A.

(a) Write down the general equation of a line through A, and show that it can be written in the form $x(2 + k) + y(1 - k) + (2k - 5) = 0$.

(b) Find the value of k that makes the coefficient of x zero, and hence find the equation of the horizontal line through A.

(c) Find the value of k that makes the coefficient of y zero, and hence find the equation of the vertical line through A.

(d) Hence write down the coordinates of A.

DEVELOPMENT

6. (a) The general form of a line through the intersection M of $x-2y+5 = 0$ and $x+y+2 = 0$ is ℓ: $(x - 2y + 5) + k(x + y + 2) = 0$. Show that the gradient of ℓ is $\dfrac{1 + k}{2 - k}$.

(b) Hence find the equation of the line that passes through M and is:

(i) parallel to $3x + 4y = 5$,

(ii) perpendicular to $2x - 3y = 6$,

(iii) perpendicular to $5y - 2x = 4$,

(iv) parallel to $x - y - 7 = 0$.

7. (a) Show that the three lines ℓ_1: $2x-3y+13 = 0$, ℓ_2: $x+y-1 = 0$ and ℓ_3: $4x+3y-1 = 0$ are concurrent by the following method.

(i) Without finding any points of intersection, find the equation of the line that passes through the intersection of ℓ_1 and ℓ_2 and is parallel to ℓ_3.

(ii) Show that this line is the same line as ℓ_3.

(b) Use the same method as in the previous question to test each family of lines for concurrency.

(i) ℓ_1: $2x - y = 0$, ℓ_2: $x + y = 9$ and ℓ_3: $x - 3y + 15 = 0$

(ii) ℓ_1: $x + 4y + 6 = 0$, ℓ_2: $x + y - 3 = 0$ and ℓ_3: $7x - 3y - 10 = 0$

8. (a) Find the point P of intersection of $x + y - 2 = 0$ and $2x - y - 1 = 0$.

(b) Show that P satisfies the equation $x + y - 2 + k(2x - y - 1) = 0$.

(c) Find the equation of the line through P and $Q(-2, 2)$:

(i) using the coordinates of both P and Q,

(ii) without using the coordinates of P.

The two answers should be the same.

CHALLENGE

9. (a) It is known that the line ℓ: $x + 2y + 10 = 0$ is tangent to the circle $\mathcal{C}$: $x^2 + y^2 = 20$ at the point T. Use the fact that a tangent is perpendicular to the radius at the point of contact to write down the equation of the radius OT of the circle.

(b) Without actually finding the coordinates of T, use the result of part (a) to find the equation of the line through $S(1, -3)$ and the point of contact T.

6 G Coordinate Methods in Geometry

When the French mathematician and philosopher René Descartes introduced coordinate geometry in the 17th century, he intended it as a method of proving all the theorems of Euclidean geometry using algebraic rather than geometric arguments. The questions in this exercise apply coordinate methods to proving some well-known geometric theorems.

Some questions use the words *altitude* and *median*.

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MEDIAN: A *median* of a triangle is the interval from a vertex of the triangle to the midpoint of the opposite side.

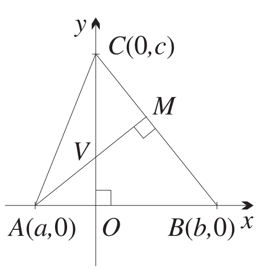
ALTITUDE: An *altitude* of a triangle is the perpendicular from a vertex of the triangle to the opposite side (extended if necessary).

Example — The Three Altitudes of a Triangle are Concurrent: There are three altitudes in a triangle — the theorem below asserts the concurrence of these three altitudes.

The proof begins by placing the triangle carefully on the coordinate plane so that two points lie on the x-axis and the third lies on the y-axis. Any triangle can be placed in this way by rotating and shifting it appropriately, so the final theorem applies to any triangle.

THEOREM: The three altitudes of a triangle are concurrent. (Their point of intersection is called the *orthocentre* of the triangle.)

PROOF: Let the side AB of the triangle ABC lie on the x-axis, and the vertex C lie on the positive side of the y-axis. Let the coordinates of the vertices be $A(a,0)$, $B(b,0)$ and $C(0,c)$, as in the diagram.



First, the altitude through C is the interval CO on the y-axis.

Secondly, let M be the foot of the altitude through A.

Since BC has gradient $-\dfrac{c}{b}$, AM has gradient $\dfrac{b}{c}$ (see Box 9),

so the equation of AM is $y - 0 = \dfrac{b}{c}(x-a)$ (See Box 14.)

$$y = \frac{bx}{c} - \frac{ab}{c}$$

and substituting $x = 0$, the altitude AM meets CO at $V\left(0,\ -\dfrac{ab}{c}\right)$.

Thirdly, let N be the foot of the altitude through B (not shown on the diagram).

Since AC has gradient $-\dfrac{c}{a}$, BN has gradient $\dfrac{a}{c}$ (see Box 9),

so the equation of BN is $y - 0 = \dfrac{a}{c}(x-b)$ (See Box 14.)

$$y = \frac{ax}{c} - \frac{ab}{c}$$

and substituting $x = 0$, the altitude BN meets CO at the same point $V\left(0,\ -\dfrac{ab}{c}\right)$.

Hence the three altitudes are concurrent.

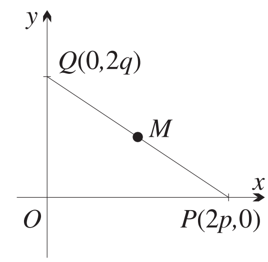
Exercise 6G

NOTE: Diagrams should be drawn wherever possible.

1. (a) On a number plane, plot the points $O(0,0)$, $A(6,0)$, $B(6,6)$ and $C(0,6)$, which form a square.
(i) Find the gradients of the diagonals OB and AC.
(ii) Hence show that the diagonals OB and AC are perpendicular.

(b) THEOREM: *The diagonals of a square are perpendicular.*
Let the vertices of the square be $O(0,0)$, $A(a,0)$, $B(a,a)$ and $C(0,a)$.
(i) Find the gradients of the diagonals OB and AC.
(ii) Hence show that the diagonals OB and AC are perpendicular.

2. (a) The points $O(0,0)$, $P(8,0)$ and $Q(0,10)$ form a right-angled triangle. Let M be the midpoint of PQ.
(i) Find the coordinates of M.
(ii) Find the distances OM, PM and QM, and show that M is equidistant from each of the vertices.
(iii) Explain why a circle with centre M can be drawn through the three vertices O, P and Q



(b) THEOREM: *The midpoint of the hypotenuse of a right-angled triangle is the centre of a circle through all three vertices.*
Prove this theorem for any right-angled triangle by placing its vertices at $O(0,0)$, $P(2p,0)$ and $Q(0,2q)$ and repeating the procedures of part (a).

3. (a) Let $PQRS$ be the quadrilateral with vertices $P(1,0)$, $Q(0,2)$, $R(-3,0)$ and $S(0,-4)$.
(i) Find PQ^2, RS^2, PS^2 and QR^2.
(ii) Show that $PQ^2 + RS^2 = PS^2 + QR^2$.

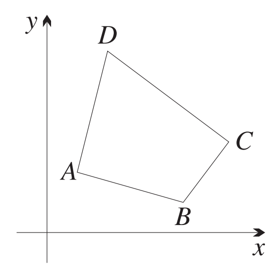
(b) THEOREM: *If the diagonals of a quadrilateral are perpendicular, then the two sums of squares of opposite sides are equal.*
Prove this theorem for any quadrilateral by placing the vertices on the axes, giving them coordinates $P(p,0)$, $Q(0,q)$, $R(-r,0)$ and $S(0,-s)$, and proceeding as in part (a).

4. (a) A triangle has vertices at $A(1,-3)$, $B(3,3)$ and $C(-3,1)$.
(i) Find the coordinates of the midpoint P of AB and the midpoint Q of BC.
(ii) Show that $PQ \parallel AC$ and that $PQ = \frac{1}{2}AC$.

(b) THEOREM: *The interval joining the midpoints of two sides of a triangle is parallel to the base and half its length.*
Prove this theorem for any triangle by placing its vertices at $A(2a,0)$, $B(2b,2c)$ and $C(0,0)$, where $a > 0$, and proceeding as in part (a).

DEVELOPMENT

5. THEOREM: *The midpoints of the sides of a quadrilateral form a parallelogram.*
Let the vertices of the quadrilateral be $A(a_1,a_2)$, $B(b_1,b_2)$, $C(c_1,c_2)$ and $D(d_1,d_2)$, as in the diagram opposite.
(a) Find the midpoints P, Q, R and S of the sides AB, BC, CD and DA respectively. (The figure $PQRS$ is also a quadrilateral.)
(b) Find the midpoints of the diagonals PR and QS.
(c) Explain why this proves that $PQRS$ is a parallelogram.



6. The points $A(a, 0)$ and $Q(q, 0)$ are points on the positive x-axis, and the points $B(0, b)$ and $P(0, p)$ lie on the positive y-axis. Show that $AB^2 - AP^2 = QB^2 - QP^2$.

7. The triangle OBA has its vertices at the origin $O(0, 0)$, $A(3, 0)$ and $B(0, 4)$. The point C lies on AB, and OC is perpendicular to AB. Draw a diagram showing this information.

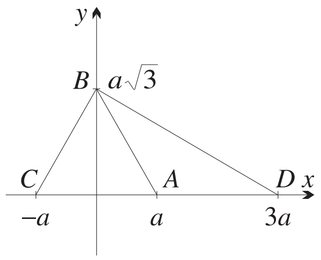
(a) Find the equations of AB and OC and hence find the coordinates of C.

(b) Find the lengths OA, AB, OC, BC and AC.

(c) Thus confirm these important results for a right-angled triangle:

(i) $OC^2 = AC \times BC$ (ii) $OA^2 = AC \times AB$

8. The diagram opposite shows the points A, B, C and D on the number plane.



(a) Show that $\triangle ABC$ is equilateral.

(b) Show that $\triangle ABD$ is isosceles, with $AB = AD$.

(c) Show that $AB^2 = \frac{1}{3}BD^2$.

9. THEOREM: *The diagonals of a parallelogram bisect each other.*
Place three vertices of the parallelogram at $A(0, 0)$, $B(2a, 2b)$ and $D(2c, 2d)$.

(a) Use gradients to show that with $C = (2a + 2c, 2b + 2d)$, the quadrilateral $ABCD$ is a parallelogram.

(b) Find the midpoints of the diagonals AC and BD.

(c) Explain why this proves that the diagonals bisect each other.

CHALLENGE

10. (a) The points $A(1, -2)$, $B(5, 6)$ and $C(-3, 2)$ are the vertices of a triangle, and P, Q and R are the midpoints of BC, CA and AB respectively.

(i) Find the equations of the three medians BQ, CR and AP.

(ii) Find the intersection of BQ and CR, and show that it lies on the third median AP.

(b) THEOREM: *The three medians of a triangle are concurrent. (Their point of intersection is called the* centroid.*)*
Prove that the theorem is true for any triangle by choosing as vertices $A(6a, 6b)$, $B(-6a, -6b)$ and $C(0, 6c)$, and following these steps.

(i) Find the midpoints P, Q and R of BC, CA and AB respectively.

(ii) Show that the median through C is $x = 0$ and find the equations of the other two medians.

(iii) Find the point where the median through C meets the median through A, and show that this point lies on the median through B.

11. THEOREM: *The perpendicular bisectors of the sides of a triangle are concurrent. Their point of intersection (called the* circumcentre*) is the centre of a circle through all three vertices (called the* circumcircle*).*
Prove this theorem for any triangle by placing its vertices at $A(2a, 0)$, $B(-2a, 0)$ and $C(2b, 2c)$, and proceeding as follows.

(a) Find the gradients of AB, BC and CA.

(b) Hence find the equations of the three perpendicular bisectors.

(c) Find the intersection M of any two, and show that it lies on the third.

(d) Explain why the circumcentre must be equidistant from each vertex.

6H Chapter Review Exercise

1. Let $X = (2, 9)$ and $Y = (14, 4)$. Use the standard formulae to find:
(a) the midpoint of XY, (b) the gradient of XY, (c) the length of XY.

2. A triangle has vertices $A(1, 4)$, $B(-3, 1)$ and $C(-2, 0)$.
(a) Find the lengths of all three sides of $\triangle ABC$.
(b) What sort of triangle is $\triangle ABC$?

3. A quadrilateral has vertices $A(2, 5)$, $B(4, 9)$, $C(8, 1)$ and $D(-2, -7)$.
(a) Find the midpoints P of AB, Q of BC, R of CD and S of DA.
(b) Find the gradients of PQ, QR, RS and SP.
(c) What sort of quadrilateral is $PQRS$?

4. A circle has diameter AB, where $A = (2, -5)$ and $B = (-4, 7)$.
(a) Find the centre C and radius r of the circle.
(b) Use the distance formula to test whether $P(6, -1)$ lies on the circle.

5. (a) Find the gradients of the sides of $\triangle LMN$, given $L(3, 9)$, $M(8, -1)$ and $N(-1, 7)$.
(b) Explain why $\triangle LMN$ is a right-angled triangle.

6. (a) Find the gradient of the interval AB, where $A = (3, 0)$ and $B = (5, -2)$.
(b) Find a if $AP \perp AB$, where $P = (a, 5)$.
(c) Find the point $Q(b, c)$ if B is the midpoint of AQ.
(d) Find d if the interval AD has length 5, where $D = (6, d)$.

7. Find, in general form, the equation of the line:
(a) with gradient -2 and y-intercept 5,
(b) with gradient $\frac{2}{3}$ through the point $A(3, 5)$,
(c) through the origin perpendicular to $y = 7x - 5$,
(d) through $B(-5, 7)$ parallel to $y = 4 - 3x$,
(e) with y-intercept -2 and angle of inclination 60°.

8. Put the equation of each line in gradient–intercept form, and hence find its y-intercept b, its gradient m and its angle of inclination α (correct to the nearest minute when necessary).
(a) $5x - 6y - 7 = 0$ (b) $4x + 4y - 3 = 0$

9. Find the gradient of each line AB, then find its equation in general form.
(a) $A(3, 0)$ and $B(4, 8)$ (b) $A(5, -2)$ and $B(7, -7)$

10. (a) Are the points $L(7, 4)$, $M(13, 2)$ and $N(25, -3)$ collinear?
(b) Are the lines $2x + 5y - 29 = 0$, $4x - 2y + 2 = 0$ and $7x - 3y + 1 = 0$ concurrent?

11. (a) Determine whether the lines $8x + 7y + 6 = 0$, $6x - 4y + 3 = 0$ and $2x + 3y + 9 = 0$ enclose a right-angled triangle.
(b) Determine what sort of figure the lines $4x + 8y + 3 = 0$, $5x - 2y + 7 = 0$, $x + 2y - 6 = 0$ and $9x - 3y = 0$ enclose.

12. (a) Find the points where the line $5x + 4y - 30 = 0$ meets the x-axis and y-axis.
(b) Hence find the area of the triangle formed by the line, the x-axis and the y-axis.

13. Consider the line ℓ: $kx + 3y + 12 = 0$, where k is a constant.

(a) Find k if ℓ passes through $(2, 4)$.
(b) Find k if ℓ has x-intercept -36.
(c) Find k if ℓ has gradient 9.
(d) Find k if ℓ is parallel to $6x - y - 7 = 0$.
(e) Find k if ℓ is perpendicular to $6x - y - 7 = 0$.
(f) Explain why ℓ can never have y-intercept 1.

14. A sketch is essential in this question.

(a) Find the gradient, length and midpoint M of the interval joining $A(10, 2)$ and $B(2, 8)$.
(b) Show that the perpendicular bisector of the interval AB has equation $4x - 3y - 9 = 0$.
(c) Find the point C where the perpendicular bisector meets the line $x - y + 2 = 0$.
(d) Use the distance formula to show that C is equidistant from A and B.
(e) Show that $CM = 15$ and hence find the area of $\triangle ABC$.
(f) Let $\theta = \angle ACB$. Use the area formula area $= \frac{1}{2} \times AC \times BC \times \sin\theta$ to find θ, correct to the nearest minute.

15. (a) Find the perpendicular distance from $A(-2, -3)$ to the line $5x + y + 2 = 0$.
(b) Find the distance between the parallel lines $3x + y - 3 = 0$ and $3x + y + 7 = 0$.
[HINT: Choose a point on one line and find its perpendicular distance from the other.]
(c) Find k if ℓ: $3x + 4y + 3 = 0$ is a tangent to the circle with centre $A(k, -5)$ and radius 3.
[HINT: This means that the perpendicular distance from A to ℓ is 3.]

16. A triangle has vertices $R(3, -4)$, $S(-6, 1)$ and $T(-2, -2)$.

(a) Find the gradient of the line ST, and hence find its equation.
(b) Find the perpendicular distance from R to ST.
(c) Find the length of the interval ST and hence find the area of $\triangle RST$.

17. Draw a sketch of the quadrilateral with vertices $J(2, -5)$, $K(-4, 3)$, $L(4, 9)$ and $M(13, -3)$.

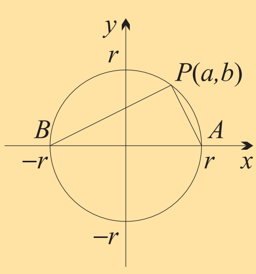
(a) Show that $JKLM$ is a trapezium with $JK \parallel ML$.
(b) Find the lengths of the parallel sides JK and ML.
(c) Find the equation of the line JK.
(d) Find the perpendicular distance from M to JK.
(e) Hence find the area of the trapezium $JKLM$.

18. Let M be the point of intersection of the lines ℓ_1: $3x - 4y - 5 = 0$ and ℓ_2: $4x + y + 7 = 0$. Write down the equation of the general line through M. Hence, without actually finding the coordinates of M, find the equation of:

(a) the line through M and $A(1, 1)$,
(b) the line through M with gradient -3,
(c) the vertical line through M,
(d) the horizontal line through M.

19. THEOREM: *The angle subtended by a diameter of a circle at any point on the circumference of the circle is a right angle.*

Let the circle have centre the origin O and radius r.
Let the diameter AB lie on the x-axis.
Let $A = (r, 0)$ and $B = (-r, 0)$.
Let $P(a, b)$ be any point on the circle.



(a) Find PO^2 and hence explain why $a^2 + b^2 = r^2$.
(b) Find the gradients of AP and BP.
(c) Show that the product of these gradients is $\dfrac{b^2}{a^2 - r^2}$.
(d) Use parts (a) and (c) to show that $AP \perp BP$.

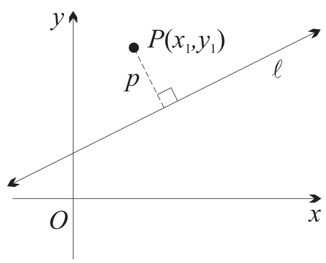
Appendix — The Proofs of Two Results

This appendix contains proofs of two results used in the chapter — the perpendicular distance formula and the equation of the general line through the intersection of two given lines.

The Perpendicular Distance Formula: The formula was stated in Section 6E.

THEOREM: Let p be the perpendicular distance from the point $P(x_1, y_1)$ to the line with equation $ax + by + c = 0$.

Then $p = \dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$.



Perpendicular Distance from the Origin: The first step in the proof is to develop a formula for the perpendicular distance p of a given line ℓ: $ax + by + c = 0$ from the origin.

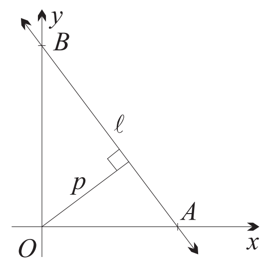
Consider the triangle OAB formed by the line ℓ and the two axes.

We find two different expressions for the area of $\triangle OAB$ and equate them.

First, substituting $y = 0$, and then $x = 0$, shows that the line $ax + by + c = 0$ has x-intercept $-\dfrac{c}{a}$ and y-intercept $-\dfrac{c}{b}$,

and so, using OA as the base of $\triangle OAB$,

$$\begin{aligned}\text{area of } \triangle OAB &= \tfrac{1}{2} \times OA \times OB \\ &= \tfrac{1}{2} \times \left|-\frac{c}{a}\right| \times \left|-\frac{c}{b}\right| \\ &= \tfrac{1}{2}\left|\frac{c^2}{ab}\right|.\end{aligned}$$



Secondly, the side AB is the hypotenuse of $\triangle OAB$,

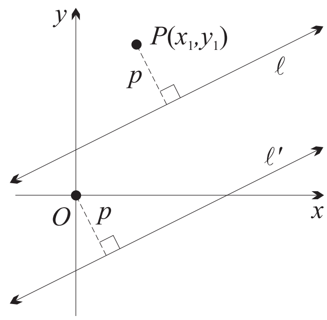
so by Pythagoras' theorem,
$$\begin{aligned}AB^2 &= \frac{c^2}{a^2} + \frac{c^2}{b^2} \\ &= \frac{c^2(a^2 + b^2)}{a^2b^2}, \\ AB &= \left|\frac{c}{ab}\right|\sqrt{a^2 + b^2},\end{aligned}$$

and so, using AB as the base of $\triangle OAB$,

$$\text{area of } \triangle OAB = \tfrac{1}{2}p\left|\frac{c}{ab}\right|\sqrt{a^2 + b^2}\,.$$

Equating the two expressions for the area of $\triangle ABO$,

$$\begin{aligned}\tfrac{1}{2}p\left|\frac{c}{ab}\right|\sqrt{a^2 + b^2} &= \tfrac{1}{2}\left|\frac{c^2}{ab}\right| \\ p\sqrt{a^2 + b^2} &= |c| \\ p &= \frac{|c|}{\sqrt{a^2 + b^2}}\,.\end{aligned}$$



Completion of Proof: Now we can use shifting to find the perpendicular distance p from any point $P(x_1, y_1)$ to the line

$$\ell\colon ax + by + c = 0.$$

The perpendicular distance remains the same if we shift both line and point x_1 units to the left and y_1 units down.

This shift moves the point P to the origin, and moves the line ℓ to the new line ℓ' with equation

$$a(x + x_1) + b(y + y_1) + c = 0 \qquad \text{(Replace } x \text{ by } x + x_1 \text{ and } y \text{ by } y + y_1.\text{)}$$

Rearranging the equation into general form,

$$ax + by + (ax_1 + by_1 + c) = 0.$$

Then, using the formula previously established for the distance from the origin, the perpendicular distance p from the point (x_1, y_1) to the line $ax + by + c = 0$ is

$$p = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}.$$

Lines Through the Intersection of Two Given Lines:

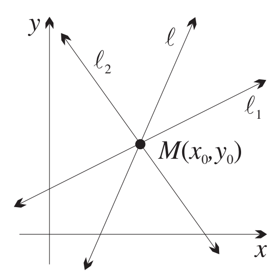
Section 6F gave the general form of the equation of any line through the intersection of two given lines. Here is the proof of that formula.

THEOREM: Let two lines $\ell_1\colon a_1x + b_1y + c_1 = 0$ and $\ell_2\colon a_2x + b_2y + c_2) = 0$ intersect at a point M. Then every line passing through M has the form

$$(a_1x + b_1y + c_1) + k(a_2x + b_2y + c_2) = 0 \qquad (*)$$

bp where k is a constant.

PROOF: For all values of k, equation $(*)$ above is the equation of a line, and different values of k give different lines.



The proof below simply shows that the line ℓ with equation $(*)$ always passes through M, whatever the value of k.

Let the intersection of ℓ_1 and ℓ_2 have coordinates $M(x_0, y_0)$. We know that the first line ℓ_1 passes through M,

so, substituting the coordinates of M into the equation of ℓ_1,

$$a_1x_0 + b_1y_0 + c_1 = 0. \qquad (1)$$

Similarly, we know that the second line ℓ_2 passes through M, so, substituting the coordinates of M into the equation of ℓ_2,

$$a_2x_0 + b_2y_0 + c_2 = 0. \qquad (2)$$

To prove that ℓ passes through M, substitute $M(x_0, y_0)$ into $(*)$:

$$\begin{aligned} \text{LHS} &= (a_1x_0 + b_1y_0 + c_1) + k(a_2x_0 + b_2y_0 + c_2) \\ &= 0 + 0, \quad \text{by the identities (1) and (2)} \\ &= \text{RHS}, \quad \text{as required.} \end{aligned}$$

CHAPTER **SEVEN**

Indices and Logarithms

Suppose that the amount of mould on a piece of cheese is doubling every day. In two days, the mould increases by a factor of $2^2 = 4$, in three days by a factor of $2^3 = 8$, in four days by a factor of $2^4 = 16$, and so on.

Applying mathematics to situations like this requires indices and logarithms and their graphs. Logarithms may be unfamiliar to many readers.

7 A Indices

This section introduces powers like 2^3 and 5^{-2}, whose indices are integers, and develops the various index laws that apply to them.

Power, Base and Index: These three words need to be clearly defined.

1

POWER, BASE AND INDEX:
- An expression a^n is called a *power*.
- The number a is called the *base*.
- The number n is called the *index* or *exponent*.

Thus 2^3 is a *power* with *base 2* and *index 3*.

The words *exponent* and *index* (plural *indices*) mean exactly the same thing.

Powers whose Indices are Positive Integers: Powers are defined differently depending on what sort of number the index is. When the index is a positive integer, the definition is quite straightforward.

2

POWERS WHOSE INDICES ARE POSITIVE INTEGERS: For any number a,

$$a^1 = a, \qquad a^2 = a \times a, \qquad a^3 = a \times a \times a, \qquad \ldots$$

In general, $a^n = \overbrace{a \times a \times a \times \cdots \times a}^{n \text{ factors}}$, for all positive integers n.

For example, $7^1 = 7$ $\qquad 2^3 = 2 \times 2 \times \times 2 = 8$ $\qquad 3^4 = 3 \times 3 \times 3 \times 3 = 81$

Scientific calculators can help to evaluate or approximate powers. The button for powers is labelled $\boxed{x^y}$ or $\boxed{\wedge}$, depending on the calculator. For example,

$$2^{26} = 67\,108\,864 \qquad 3^{50} \doteqdot 7{\cdot}179 \times 10^{23} \qquad \pi^4 \doteqdot 97{\cdot}41$$

Index Laws — Combining Powers with the Same Base: The three index laws in the group below show how to combine powers when the base is fixed.

3

INDEX LAWS — PRODUCTS, QUOTIENTS AND POWERS OF POWERS:

- To multiply powers with the same base, add the indices:
  $a^m \times a^n = a^{m+n}$
- To divide powers with the same base, subtract the indices:
  $a^m \div a^n = a^{m-n}$ $\left(\text{also written as } \dfrac{a^m}{a^n} = a^{m-n}\right)$
- To raise a power to a power, multiply the indices:
  $(a^m)^n = a^{mn}$

Demonstrating the results when $m = 5$ and $n = 3$ should make the general results obvious.

- $$\begin{aligned} a^5 \times a^3 &= (a \times a \times a \times a \times a) \times (a \times a \times a) \\ &= a^8 \\ &= a^{5+3} \end{aligned}$$
- $$\begin{aligned} a^5 \div a^3 &= \frac{a \times a \times a \times a \times a}{a \times a \times a} \\ &= a^2 \\ &= a^{5-3} \end{aligned}$$
- $$\begin{aligned} (a^5)^3 &= a^5 \times a^5 \times a^5 \\ &= a^{5+5+5}, \text{ using the first index law } a^\ell \times a^m \times a^n = a^{\ell+m+n}, \\ &= a^{5\times 3} \end{aligned}$$

WORKED EXERCISE:

Use the index laws above to simplify each expression.

(a) $x^3 \times x^7$ (c) $w^{12} \div w^3$ (e) $(y^6)^9$

(b) $3^x \times 3^{5x}$ (d) $10^{a+b} \div 10^b$ (f) $(2^{3x})^{2y}$

SOLUTION:

(a) $x^3 \times x^7 = x^{10}$ (c) $w^{12} \div w^3 = w^9$ (e) $(y^6)^9 = y^{54}$

(b) $3^x \times 3^{5x} = 3^{6x}$ (d) $10^{a+b} \div 10^b = 10^a$ (f) $(2^{3x})^{2y} = 2^{6xy}$

Index Laws — Powers of Products and Quotients: The index laws in this second group show how to work with powers of products and powers of quotients.

4

INDEX LAWS — POWERS OF PRODUCTS AND QUOTIENTS:

- The power of a product is the product of the powers: $(ab)^n = a^n \times b^n$
- The power of a quotient is the quotient of the powers: $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

Demonstrating the results when $n = 3$ should make the general results obvious.

- $$\begin{aligned} (ab)^3 &= ab \times ab \times ab \\ &= a \times a \times a \times b \times b \times b \\ &= a^3 b^3 \end{aligned}$$
- $$\begin{aligned} \left(\frac{a}{b}\right)^3 &= \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} \\ &= \frac{a^3}{b^3} \end{aligned}$$

WORKED EXERCISE:

Expand the brackets in each expression.

(a) $\left(10a^5\right)^3$ (b) $(3x^2y)^3$ (c) $\left(\dfrac{a^3}{3b}\right)^4$

SOLUTION:

(a) $\left(10a^5\right)^3 = 1000\,a^{15}$ (b) $(3x^2y)^3 = 27x^6y^3$ (c) $\left(\dfrac{a^3}{3b}\right)^4 = \dfrac{a^{12}}{81b^4}$

Zero and Negative Indices: The index laws were demonstrated only when the indices were all positive integers. Powers with negative indices are defined so that these laws are valid for negative indices as well. Suppose that a is any non-zero number.

We know that $a^3 \div a^3 = 1$.

Using the index laws, $a^3 \div a^3 = a^0$.

Hence we shall define $a^0 = 1$.

We know that $a^2 \div a^3 = \dfrac{1}{a}$.

Using the index laws, $a^2 \div a^3 = a^{-1}$.

Hence we shall define $a^{-1} = \dfrac{1}{a}$.

Similarly, we shall define $a^{-2} = \dfrac{1}{a^2}$ and $a^{-3} = \dfrac{1}{a^3}$, and so on.

5 ZERO AND NEGATIVE INDICES: Let a be any non-zero number.

- Define $a^0 = 1$.
- Define $a^{-1} = \dfrac{1}{a}$, define $a^{-2} = \dfrac{1}{a^2}$, define $a^{-3} = \dfrac{1}{a^3}$, ...

  In general, define $a^{-n} = \dfrac{1}{a^n}$, for all positive integers n.

WORKED EXERCISE:

Write each expression using fractions instead of negative indices, then simplify it.

(a) 12^{-1} (b) 7^{-2} (c) $(\tfrac{2}{3})^{-1}$ (d) $(\tfrac{2}{3})^{-4}$

SOLUTION:

(a) $12^{-1} = \tfrac{1}{12}$

(b) $7^{-2} = \dfrac{1}{7^2} = \tfrac{1}{49}$

(c) $(\tfrac{2}{3})^{-1} = \tfrac{3}{2}$

(d) $(\tfrac{2}{3})^{-4} = \left(\tfrac{3}{2}\right)^4 = \tfrac{81}{16}$

WORKED EXERCISE:

Write each expression using negative indices instead of fractions.

(a) $\dfrac{1}{2^{20}}$ (b) $\dfrac{1}{x^3}$ (c) $\dfrac{5}{a^7}$ (d) $\dfrac{w^2}{v^4}$

SOLUTION:

(a) $\dfrac{1}{2^{20}} = 2^{-20}$

(b) $\dfrac{1}{x^3} = x^{-3}$

(c) $\dfrac{5}{a^7} = 5 \times \dfrac{1}{a^7} = 5a^{-7}$

(d) $\dfrac{w^2}{v^4} = w^2 \times \dfrac{1}{v^4} = w^2v^{-4}$

WORKED EXERCISE:
Use the index laws, extended to negative indices, to simplify these expressions.
(a) $x^2y^{-1} \times x^{-2}y^5$ (b) $12a^2b^2 \div 6a^3b$ (c) $(2^n)^{-3}$

SOLUTION:
(a) $x^2y^{-1} \times x^{-2}y^5 = y^4$ (b) $12a^2b^2 \div 6a^3b = 2a^{-1}b$ (c) $(2^n)^{-3} = 2^{-3n}$

Exercise 7A

NOTE: Do not use a calculator in this exercise at all.

1. **Preparation**: The answers to this question will be needed for all the exercises in this chapter and the next. Keep the results in a place where you can refer to them easily.
(a) Write down all the powers of 2 from $2^0 = 1$ up to $2^{12} = 4096$.
(b) In the same way, write down:
(i) the powers of 3 up to $3^6 = 729$,
(ii) the powers of 5 up to $5^5 = 3125$,
(iii) the powers of 6 up to $6^3 = 216$,
(iv) the powers of 7 up to 7^3,
(v) the powers of 10 up to 10^6,
(vi) the powers of 20 up to 20^6.
(c) From your list of powers of 2, read off: (i) the powers of 4, (ii) the powers of 8.
(d) From your list of powers of 3, read off the powers of 9.
(e) From your list of powers of 5, read off the powers of 25.

2. Simplify:
(a) 2^3 (b) 2^6 (c) 3^4 (d) 9^3 (e) $(\frac{2}{3})^2$ (f) $(\frac{2}{3})^3$ (g) $(\frac{3}{10})^4$ (h) $(\frac{4}{7})^2$ (i) $(\frac{5}{9})^1$ (j) 1^1

3. Simplify:
(a) 3^0 (b) 7^0 (c) 5^{-1} (d) 11^{-1} (e) 6^{-2} (f) 10^{-2} (g) 3^{-3} (h) 5^{-3} (i) 2^{-5} (j) 10^{-6}

4. Simplify:
(a) $(\frac{1}{5})^{-1}$ (b) $(\frac{1}{11})^{-1}$ (c) $(\frac{2}{7})^{-1}$ (d) $(\frac{7}{2})^{-1}$ (e) $(\frac{3}{4})^{-1}$ (f) $(\frac{10}{23})^{-1}$ (g) $(10)^{-1}$ (h) $(0{\cdot}1)^{-1}$ (i) $(0{\cdot}01)^{-1}$ (j) $(0{\cdot}02)^{-1}$

5. Simplify:
(a) 5^{-2} (b) $(\frac{1}{5})^{-2}$ (c) $(\frac{1}{5})^{-3}$ (d) $(\frac{1}{2})^{-4}$ (e) $(\frac{1}{10})^{-6}$ (f) $(\frac{2}{3})^{-2}$ (g) $(\frac{2}{3})^{-4}$ (h) $(\frac{3}{2})^{-4}$ (i) $(\frac{2}{5})^{-2}$ (j) $(\frac{3}{7})^0$

6. Simplify, leaving your answers in index form:
(a) $2^9 \times 2^5$ (b) $a^8 \times a^7$ (c) $7^2 \times 7^{-10}$ (d) $x^7 \times x^{-5}$ (e) $9^6 \times 9^{-6}$ (f) $a^5 \times a^{-5}$ (g) 5×5^{-4} (h) $8^4 \times 8 \times 8^{-4}$

7. Simplify, leaving your answers in index form:
(a) $7^8 \div 7^3$ (b) $a^5 \div a^7$ (c) $x^{10} \div x^{-2}$ (d) $x^{-10} \div x^2$ (e) $2^8 \div 2^{-8}$ (f) $2^8 \div 2^8$ (g) $y^{12} \div y$ (h) $y \div y^{12}$

8. Simplify, leaving your answers in index form:
(a) $x^5 \times x^5 \times x^5$ (b) $(x^5)^3$ (c) $(z^2)^7$ (d) $a^{-2} \times a^{-2} \times a^{-2}$ (e) $(a^{-2})^3$ (f) $(5^4)^{-7}$ (g) $(y^{-2})^{-5}$ (h) $(2^{-4})^{-4}$

9. Write down the solutions of these index equations.

(a) $3^x = 9$ (b) $2^x = 16$ (c) $5^x = 125$ (d) $2^x = 64$ (e) $7^x = \frac{1}{7}$ (f) $5^x = \frac{1}{5}$ (g) $9^x = \frac{1}{81}$ (h) $2^x = \frac{1}{8}$ (i) $(\frac{1}{3})^x = 3$ (j) $(\frac{5}{6})^x = \frac{6}{5}$ (k) $9^x = 1$ (l) $12^x = 1$

10. Expand the brackets in each expression.

(a) $(3x)^2$ (b) $(5a)^3$ (c) $(2c)^6$ (d) $(3st)^4$ (e) $(7xyz)^2$ (f) $\left(\dfrac{1}{x}\right)^5$ (g) $\left(\dfrac{3}{x}\right)^2$ (h) $\left(\dfrac{y}{5}\right)^2$ (i) $\left(\dfrac{7a}{5}\right)^2$ (j) $\left(\dfrac{3x}{2y}\right)^3$

DEVELOPMENT

11. Write each expression as a fraction without negative indices.

(a) 9^{-1} (b) x^{-1} (c) b^{-2} (d) $-a^{-4}$ (e) $(7x)^{-1}$ (f) $7x^{-1}$ (g) $-9x^{-1}$ (h) $(3a)^{-2}$ (i) $3a^{-2}$ (j) $4x^{-3}$

12. Write each fraction in index form.

(a) $\dfrac{1}{x}$ (b) $-\dfrac{1}{x^2}$ (c) $-\dfrac{12}{x}$ (d) $\dfrac{9}{x^2}$ (e) $-\dfrac{1}{x^3}$ (f) $\dfrac{12}{x^5}$ (g) $\dfrac{7}{x^3}$ (h) $-\dfrac{6}{x}$ (i) $\dfrac{1}{6x}$ (j) $-\dfrac{1}{4x^2}$

13. First change each mixed numeral or decimal to a fraction, then simplify the expression.

(a) $(1\frac{1}{2})^{-1}$ (b) $(2\frac{1}{3})^{-1}$ (c) $(2\frac{2}{3})^{-1}$ (d) $(2\frac{1}{2})^{-2}$ (e) $(3\frac{1}{3})^{-3}$ (f) $(6\frac{2}{3})^{-2}$ (g) $0{\cdot}2^{-1}$ (h) $2{\cdot}4^{-1}$ (i) $2{\cdot}25^{-1}$ (j) $2{\cdot}5^{-2}$ (k) $2{\cdot}5^{-3}$ (l) $0{\cdot}05^{-2}$

14. Write down the solutions of these index equations.

(a) $(\frac{5}{8})^x = \frac{25}{64}$ (b) $(\frac{5}{8})^x = \frac{8}{5}$ (c) $(\frac{3}{5})^x = \frac{25}{9}$ (d) $(\frac{3}{2})^x = \frac{8}{27}$ (e) $x^2 = \frac{100}{169}$ (f) $x^{-2} = \frac{1}{4}$ (g) $x^{-3} = 27$ (h) $x^{-2} = \frac{64}{81}$

15. Simplify each expression, giving the answer without negative indices.

(a) $x^2y \times x^4y^3$ (b) $x^{-5}y^3 \times x^3y^{-2}$ (c) $3ax^{-2} \times 7a^2x$ (d) $\frac{1}{15}s^{-2}t^3 \times 5st^{-5}$ (e) $56x^2y^6 \div 8xy^8$ (f) $35a^{-2}b^4 \div 28a^4b^{-6}$ (g) $(s^2y^{-3})^3$ (h) $(5c^{-2}d^3)^{-1}$ (i) $(3x^2y^3)^3 \times (xy^4)^2$ (j) $(2a^{-2}y^3)^{-2} \times (2ay^{-3})^3$ (k) $(5st^2)^3 \div (5s^{-1}t^3)^2$ (l) $(10x^2y^{-2})^3 \div (2x^{-1}y^3)^2$

16. Simplify these expressions.

(a) $2^x \times 2^3$ (b) $3^x \times 3$ (c) $7^{4x} \times 7^{-5x}$ (d) $5^{2x} \div 5^3$ (e) $(10^{2x})^3$ (f) $(5^{4x})^{-2}$ (g) $(6^x)^4 \times (6^{2x})^5$ (h) $(2^x)^3 \div 2^4$

CHALLENGE

17. Expand and simplify, answering without using negative indices:

(a) $(x + x^{-1})^2$ (b) $(x - x^{-1})^2$ (c) $(x^2 - x^{-2})^2$

18. Write each expression as a single power.

(a) 2×2^x (b) $2^x + 2^x$ (c) 3×3^x (d) $3^x + 3^x + 3^x$ (e) 4×2^x (f) 32×2^x (g) 125×5^x (h) 81×3^x (i) $\frac{1}{2} \times 2^x$ (j) $\frac{1}{9} \times 3^x$

19. Explain why if $3^{3x-1} = 9$, then $3x - 1 = 2$, and so $x = 1$. Similarly, solve:

(a) $5^{x+3} = 25$ (b) $3^{2x-8} = 81$ (c) $4^{7-x} = \frac{1}{4}$ (d) $4^{5x+2} = \frac{1}{64}$ (e) $(\frac{1}{3})^{x+1} = 27$ (f) $(\frac{1}{5})^{x+1} = (\frac{1}{125})^{x-1}$

7 B Fractional Indices

In this section, the definition of powers is extended to powers like $4^{\frac{1}{2}}$ and $27^{-\frac{2}{3}}$, where the index is a positive or negative fraction. Again, the definitions will be made so that the index laws work for all indices.

Fractional Indices with Numerator 1: Suppose that a is any positive number.

We know that $\left(\sqrt{a}\right)^2 = a$.
Using the index laws, $\left(a^{\frac{1}{2}}\right)^2 = a$.
Hence we shall define $a^{\frac{1}{2}} = \sqrt{a}$.

We know that $\left(\sqrt[3]{a}\right)^3 = a$.
Using the index laws, $\left(a^{\frac{1}{3}}\right)^3 = a$.
Hence we shall define $a^{\frac{1}{3}} = \sqrt[3]{a}$.

6 FRACTIONAL INDICES WITH NUMERATOR 1: Let a be any positive number.

Define $a^{\frac{1}{2}} = \sqrt{a}$, define $a^{\frac{1}{3}} = \sqrt[3]{a}$, define $a^{\frac{1}{4}} = \sqrt[4]{a}$, ...

In general, define $a^{\frac{1}{n}} = \sqrt[n]{a}$, for all positive integers n, where in every case, the *positive root* of a is to be taken.

WORKED EXERCISE:
Write each expression using surd notation, then simplify it.

(a) $64^{\frac{1}{2}}$ (b) $27^{\frac{1}{3}}$ (c) $10\,000^{\frac{1}{4}}$ (d) $32^{\frac{1}{5}}$

SOLUTION:
(a) $64^{\frac{1}{2}} = \sqrt{64} = 8$
(b) $27^{\frac{1}{3}} = \sqrt[3]{27} = 3$
(c) $10\,000^{\frac{1}{4}} = \sqrt[4]{10\,000} = 10$
(d) $32^{\frac{1}{5}} = \sqrt[5]{32} = 2$

WORKED EXERCISE:
Write each expression using a fractional index.

(a) $\sqrt{2}$ (b) $\sqrt[3]{15}$ (c) $\sqrt{x}$ (d) $\sqrt[5]{x}$

SOLUTION:
(a) $\sqrt{2} = 2^{\frac{1}{2}}$ (b) $\sqrt[3]{15} = 15^{\frac{1}{3}}$ (c) $\sqrt{x} = x^{\frac{1}{2}}$ (d) $\sqrt[5]{x} = x^{\frac{1}{5}}$

General Fractional Indices:
If the index laws are to apply to a power like $a^{\frac{2}{3}}$, then we must be able to write

$$\left(a^{\frac{1}{3}}\right)^2 = a^{\frac{2}{3}} \quad \text{(To raise a power to a power, multiply the indices.)}$$

Hence we shall define $a^{\frac{2}{3}}$ as $\left(a^{\frac{1}{3}}\right)^2$, that is, as $\left(\sqrt[3]{a}\right)^2$.

7 GENERAL FRACTIONAL INDICES:
Let a be any positive number, and m and n be positive integers.

- Define $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$. (This is the same as $\sqrt[n]{a^m}$.)
- Define $a^{-\frac{m}{n}} = \dfrac{1}{a^{\frac{m}{n}}}$.

METHOD WITH FRACTIONAL INDICES: Deal with complicated indices in this order:
1. If the index is negative, take the reciprocal.
2. If the index has a denominator, take the root.
3. Lastly, take the power indicated by the numerator of the index.

WORKED EXERCISE:
Simplify each power. First take the root indicated by the denominator.

(a) $125^{\frac{2}{3}}$ (b) $100^{\frac{3}{2}}$ (c) $\left(\frac{16}{81}\right)^{\frac{3}{4}}$

SOLUTION:

(a) $125^{\frac{2}{3}} = 5^2 = 25$

(b) $100^{\frac{3}{2}} = 10^3 = 1000$

(c) $\left(\frac{16}{81}\right)^{\frac{3}{4}} = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$

WORKED EXERCISE:
Simplify each power. First take the reciprocal as indicated by the negative index.

(a) $25^{-\frac{3}{2}}$ (b) $1000^{-\frac{2}{3}}$ (c) $\left(\frac{8}{27}\right)^{-\frac{4}{3}}$

SOLUTION:

(a) $25^{-\frac{3}{2}} = \left(\frac{1}{25}\right)^{\frac{3}{2}} = \left(\frac{1}{5}\right)^3 = \frac{1}{125}$

(b) $1000^{-\frac{2}{3}} = \left(\frac{1}{1000}\right)^{\frac{2}{3}} = \left(\frac{1}{10}\right)^2 = \frac{1}{100}$

(c) $\left(\frac{8}{27}\right)^{-\frac{4}{3}} = \left(\frac{27}{8}\right)^{\frac{4}{3}} = \left(\frac{3}{2}\right)^4 = \frac{81}{16}$

WORKED EXERCISE:
Write each expression using a fractional index.

(a) $\sqrt{x^3}$ (b) $\dfrac{1}{\sqrt{x^3}}$ (c) $\sqrt[3]{x^2}$

SOLUTION:

(a) $\sqrt{x^3} = x^{\frac{3}{2}}$ (b) $\dfrac{1}{\sqrt{x^3}} = x^{-\frac{3}{2}}$ (c) $\sqrt[3]{x^2} = x^{\frac{2}{3}}$

Exercise 7B

NOTE: Do not use a calculator in this exercise at all, except in question 4. Make sure that you can refer easily to the the list of powers of 2, 3, 5, ... from the previous exercise.

1. Simplify these powers, giving each answer as a whole number.
(a) $25^{\frac{1}{2}}$ (b) $36^{\frac{1}{2}}$ (c) $100^{\frac{1}{2}}$ (d) $27^{\frac{1}{3}}$ (e) $64^{\frac{1}{3}}$ (f) $1000^{\frac{1}{3}}$ (g) $81^{\frac{1}{4}}$ (h) $32^{\frac{1}{5}}$ (i) $1\,000\,000^{\frac{1}{6}}$ (j) $1\,000\,000^{\frac{1}{2}}$

2. Simplify these powers. First take the root indicated by the denominator.
(a) $25^{\frac{3}{2}}$ (b) $9^{\frac{3}{2}}$ (c) $27^{\frac{2}{3}}$ (d) $8^{\frac{2}{3}}$ (e) $16^{\frac{3}{4}}$ (f) $81^{\frac{3}{4}}$ (g) $27^{\frac{4}{3}}$ (h) $4^{\frac{5}{2}}$ (i) $32^{\frac{3}{5}}$ (j) $64^{\frac{2}{3}}$

3. Simplify these powers, giving each answer as a fraction.
(a) $\left(\frac{1}{49}\right)^{\frac{1}{2}}$ (b) $\left(\frac{1}{8}\right)^{\frac{1}{3}}$ (c) $\left(\frac{25}{49}\right)^{\frac{1}{2}}$ (d) $\left(\frac{27}{8}\right)^{\frac{1}{3}}$ (e) $\left(\frac{1}{4}\right)^{\frac{3}{2}}$ (f) $\left(\frac{1}{25}\right)^{\frac{3}{2}}$ (g) $\left(\frac{4}{9}\right)^{\frac{3}{2}}$ (h) $\left(\frac{9}{100}\right)^{\frac{3}{2}}$

4. Use the calculator button labelled $\boxed{x^y}$ or $\boxed{\wedge}$ to find these powers. Give each answer exactly if possible, or else correct to four significant figures.
(a) 13^4 (b) $3{\cdot}235^4$ (c) 2^{50} (d) $759\,375^{\frac{1}{5}}$ (e) $10^{\frac{1}{3}}$ (f) $17^{\frac{1}{4}}$ (g) 7^{-12} (h) $15^{-\frac{3}{5}}$

5. Use the index laws to simplify these expressions, leaving your answers in index form.

(a) $x^{\frac{1}{2}} \times x^{\frac{1}{2}}$ (d) $x^{4\frac{1}{2}} \div x^{3\frac{1}{2}}$ (g) $(x^{\frac{1}{3}})^6$

(b) $x^{2\frac{1}{2}} \times x^{3\frac{1}{2}}$ (e) $x \div x^{\frac{1}{2}}$ (h) $(x^{-\frac{2}{3}})^6$

(c) $x^4 \times x^{-\frac{1}{2}}$ (f) $x^{-7} \div x^{-2\frac{1}{2}}$ (i) $(x^9)^{\frac{2}{3}}$

6. Use the index laws to simplify these expressions, giving answers as integers or fractions.

(a) $2^{\frac{1}{2}} \times 2^{\frac{1}{2}}$ (d) $3^{4\frac{1}{2}} \div 3^{5\frac{1}{2}}$ (g) $3^3 \times 3^4 \div 3^{10}$

(b) $2^{\frac{1}{2}} \times 2^{-\frac{1}{2}}$ (e) $25 \div 25^{\frac{1}{2}}$ (h) $(3^{-\frac{1}{3}})^6$

(c) $2^{\frac{1}{2}} \times 2^{2\frac{1}{2}}$ (f) $7^{\frac{2}{3}} \div 7^{\frac{2}{3}}$ (i) $(9^8)^{\frac{1}{4}}$

7. Write down the solutions of these index equations.

(a) $9^x = 3$ (c) $81^x = 3$ (e) $(\frac{1}{25})^x = \frac{1}{5}$

(b) $121^x = 11$ (d) $64^x = 2$ (f) $(\frac{1}{8})^x = \frac{1}{2}$

8. Rewrite each expression, using surds instead of fractional indices.

(a) $x^{\frac{1}{2}}$ (c) $(7x)^{\frac{1}{2}}$ (e) $15x^{\frac{1}{4}}$ (g) $6x^{\frac{5}{2}}$

(b) $7x^{\frac{1}{2}}$ (d) $x^{\frac{1}{3}}$ (f) $x^{\frac{3}{2}}$ (h) $x^{\frac{4}{3}}$

9. Rewrite each expression, using fractional indices instead of surds.

(a) $\sqrt{x}$ (c) $\sqrt{3x}$ (e) $9\sqrt[6]{x}$ (g) $\sqrt{x^9}$

(b) $3\sqrt{x}$ (d) $12\sqrt[3]{x}$ (f) $\sqrt{x^3}$ (h) $25\sqrt[5]{x^6}$

DEVELOPMENT

10. Simplify these powers. First take the reciprocal as indicated by the negative index.

(a) $25^{-\frac{1}{2}}$ (c) $125^{-\frac{1}{3}}$ (e) $16^{-\frac{3}{4}}$ (g) $81^{-\frac{3}{4}}$

(b) $100^{-\frac{1}{2}}$ (d) $16^{-\frac{1}{4}}$ (f) $27^{-\frac{2}{3}}$ (h) $49^{-\frac{3}{2}}$

11. Simplify these powers.

(a) $(\frac{1}{16})^{-\frac{1}{4}}$ (c) $(\frac{1}{49})^{-\frac{1}{2}}$ (e) $(\frac{1}{16})^{-\frac{3}{4}}$ (g) $(\frac{4}{9})^{-\frac{3}{2}}$

(b) $(\frac{1}{125})^{-\frac{1}{3}}$ (d) $(\frac{1}{27})^{-\frac{1}{3}}$ (f) $(\frac{1}{81})^{-\frac{3}{4}}$ (h) $(\frac{125}{8})^{-\frac{2}{3}}$

12. Simplify each expression, giving the answer in index form.

(a) $3x^{\frac{1}{2}}y \times 3x^{\frac{1}{2}}y^2$ (d) $x^2y^3 \div x^{\frac{1}{2}}y^{\frac{1}{2}}$ (g) $(8x^3y^{-6})^{\frac{1}{3}}$

(b) $5a^{\frac{1}{3}}b^{\frac{2}{3}} \times 7a^{-\frac{1}{3}}b^{\frac{1}{3}}$ (e) $a^{\frac{1}{2}}b^{\frac{1}{2}} \div a^{-\frac{1}{2}}b^{\frac{1}{2}}$ (h) $(p^{\frac{1}{5}}q^{-\frac{3}{5}})^{10}$

(c) $\frac{1}{8}s^{2\frac{1}{2}} \times 24s^{-2}$ (f) $(a^{-2}b^4)^{\frac{1}{2}}$ (i) $(x^{\frac{3}{4}})^4 \times (x^{\frac{4}{3}})^3$

13. Rewrite these expressions, using fractional and negative indices.

(a) $\dfrac{1}{\sqrt{x}}$ (c) $-\dfrac{5}{\sqrt{x}}$ (e) $-\dfrac{4}{\sqrt[3]{x^2}}$ (g) $\dfrac{5}{x\sqrt{x}}$

(b) $\dfrac{12}{\sqrt{x}}$ (d) $\dfrac{15}{\sqrt[3]{x}}$ (f) $x\sqrt{x}$ (h) $8x^2\sqrt{x}$

14. Given that $x = 16$ and $y = 25$, evaluate:

(a) $x^{\frac{1}{2}} + y^{\frac{1}{2}}$ (b) $x^{\frac{1}{4}} - y^{\frac{1}{2}}$ (c) $x^{-\frac{1}{2}} - y^{-\frac{1}{2}}$ (d) $(y - x)^{\frac{1}{2}} \times (4y)^{-\frac{1}{2}}$

CHALLENGE

15. Expand and simplify, giving answers in index form:

(a) $(x^{\frac{1}{2}} + x^{-\frac{1}{2}})^2$ (b) $(x^{\frac{1}{2}} - x^{-\frac{1}{2}})^2$ (c) $(x^{\frac{5}{2}} - x^{-\frac{5}{2}})^2$

16. Write down the solutions of these index equations.

(a) $49^x = \frac{1}{7}$ (c) $8^x = 4$ (e) $4^x = 8$ (g) $81^x = 27$ (i) $(\frac{1}{25})^x = 5$

(b) $81^x = \frac{1}{3}$ (d) $8^x = \frac{1}{4}$ (f) $4^x = \frac{1}{8}$ (h) $27^x = \frac{1}{81}$ (j) $(\frac{8}{125})^x = \frac{25}{4}$

17. By taking 6th powers of both sides, show that $11^{\frac{1}{3}} < 5^{\frac{1}{2}}$. Using similar methods (followed perhaps by a check on the calculator), compare:

(a) $3^{\frac{1}{3}}$ and $2^{\frac{1}{2}}$ (b) $2^{\frac{1}{2}}$ and $5^{\frac{1}{5}}$ (c) $7^{\frac{3}{2}}$ and 20 (d) $5^{\frac{1}{5}}$ and $3^{\frac{1}{3}}$

7 C Logarithms

The introduction to this chapter mentioned that the mould growing on cheese was doubling every day. Thus in 2 days the amount of mould will increase by a factor of $2^2 = 4$, in 3 days the amount will increase by a factor of $2^3 = 8$, and so on.

Now suppose that someone asks a question backwards, asking how many days it takes for the mould to increase by a factor of 64. The answer is 6 days, because $64 = 2^6$. This index 6 is the *logarithm of 64 base 2*, and is written as $\log_2 64 = 6$.

Logarithms: Here is the general definition of a logarithm. The base a must always be positive and not equal to 1.

8 DEFINITION OF A LOGARITHM: The *logarithm base a* of a positive number x is the index, when the number x is expressed as a power of the base a:

$$y = \log_a x \quad \text{means that} \quad x = a^y.$$

The basic skill with logarithms is converting between statements about indices and statements about logarithms. The next box should be learnt off by heart.

9 A SENTENCE TO COMMIT TO MEMORY: This one sentence should fix most problems:

'$\log_2 8 = 3$ because $8 = 2^3$.'

EXAMINE THE SENTENCE AND NOTICE THAT:

- The base of the log is the base of the power (in this case 2).
- The log is the index (in this case 3).

WORKED EXERCISE:

Rewrite each index statement in logarithmic form.

(a) $10^3 = 1000$ (b) $3^4 = 81$

SOLUTION:

(a) $10^3 = 1000$
$\log_{10} 1000 = 3$

(b) $3^4 = 81$
$\log_3 81 = 4$

WORKED EXERCISE:

Rewrite each logarithmic statement in index form. Then state whether it is true or false.

(a) $\log_2 16 = 4$ (b) $\log_3 27 = 4$

SOLUTION:

(a) $\log_2 16 = 4$
$16 = 2^4$, which is true.

(b) $\log_3 27 = 4$
$27 = 3^4$, which is false.

Finding Logarithms: Change questions about logarithms to questions about indices.

WORKED EXERCISE:

(a) Find $\log_2 32$. (b) Find $\log_{10} 1\,000\,000$.

SOLUTION:

(a) Let $x = \log_2 32$.
Then $2^x = 32$.
Hence $x = 5$.

(b) Let $x = \log_{10} 1\,000\,000$.
Then $10^x = 1\,000\,000$.
Hence $x = 6$.

Negative and Fractional Indices: A logarithmic equation can involve a negative or a fractional index.

WORKED EXERCISE:

Solve each logarithmic equation by changing it to an index equation.

(a) $x = \log_7 \frac{1}{7}$ (b) $\frac{1}{3} = \log_8 x$ (c) $\log_x 9 = -1$

SOLUTION:

(a) $x = \log_7 \frac{1}{7}$
$\frac{1}{7} = 7^x$
$x = -1$

(b) $\frac{1}{3} = \log_8 x$
$x = 8^{\frac{1}{3}}$
$= 2$

(c) $\log_x 9 = -1$
$x^{-1} = 9$
$x = \frac{1}{9}$

Logarithms on the Calculator: Most scientific calculators only allow direct calculations of logarithms base 10, using the button labelled [log]. (They also allow calculations base e using the button marked [ln], as will be explained in Year 12.)

The function [10^x] is usually on the same key as [log], and is reached by pressing [shift] followed by [log]. These two functions [log] and [10^x] are inverses of each other — when used one after the other, the original number returns.

WORKED EXERCISE:

Write each statement below in logarithmic form, then use the function labelled [log] on your calculator to approximate x, correct to four significant figures. Check your calculation using the button labelled [10^x].

(a) $10^x = 750$ (b) $10^x = 0{\cdot}003$

SOLUTION:

(a) $10^x = 750$
$x = \log_{10} 750$
$\doteqdot 2{\cdot}875$, using [log].
Using [10^x], $10^{2{\cdot}875} \doteqdot 750$.

(b) $10^x = 0{\cdot}003$
$x = \log_{10} 0{\cdot}003$
$\doteqdot -2{\cdot}523$, using [log].
Using [10^x], $10^{-2{\cdot}523} \doteqdot 0{\cdot}003$.

Locating a Logarithm Between Two Integers: Consider again the cheese with a mould that doubles in amount every day. Suppose that someone asks how many days it takes for the amount of mould to increase by a factor of 10.

The answer is $\log_2 10$ days, but this is an awkward question, because the answer is not an integer. In three days the mould increases 8-fold, and in four days it increases 16-fold, thus we can at least say that the answer lies between 3 and 4.

WORKED EXERCISE:
Use a list of powers of 2 to explain why $\log_2 10$ is between 3 and 4.

SOLUTION:
We know that $2^3 < 10 < 2^4$, because $2^3 = 8$ and $2^4 = 16$.
Taking logarithms base 2, $3 < \log_2 10 < 4$.

Exercise 7C

NOTE: Do not use a calculator in this exercise unless the question asks for it. Make sure that you can refer easily to the the list of powers of 2, 3, 5, ... from Exercise 7A.

1. Copy and complete each sentence.
(a) '$\log_2 8 = 3$ because ...'
(b) '$\log_5 25 = 2$ because ...'
(c) '$\log_{10} 1000 = 3$ because ...'
(d) '$7^2 = 49$, and so $\log_7 49 = \ldots$'
(e) '$3^4 = 81$, and so ...'
(f) '$10^5 = 100\,000$, and so ...'

2. Copy and complete the following statements of the meaning of logarithms.
(a) '$y = \log_a x$ means that ...'
(b) '$y = a^x$ means that ...'

3. Rewrite each equation in index form, then solve it for x.
(a) $x = \log_{10} 10\,000$
(b) $x = \log_{10} 1000$
(c) $x = \log_{10} 100$
(d) $x = \log_{10} 10$
(e) $x = \log_{10} 1$
(f) $x = \log_{10} \frac{1}{10}$
(g) $x = \log_{10} \frac{1}{100}$
(h) $x = \log_{10} \frac{1}{1000}$

4. Rewrite each equation in index form, then solve it for x.
(a) $x = \log_3 9$
(b) $x = \log_5 125$
(c) $x = \log_7 49$
(d) $x = \log_2 64$
(e) $x = \log_4 64$
(f) $x = \log_8 64$
(g) $x = \log_8 8$
(h) $x = \log_8 1$
(i) $x = \log_7 \frac{1}{7}$
(j) $x = \log_{12} \frac{1}{12}$
(k) $x = \log_{11} \frac{1}{121}$
(l) $x = \log_6 \frac{1}{36}$
(m) $x = \log_4 \frac{1}{64}$
(n) $x = \log_8 \frac{1}{64}$
(o) $x = \log_2 \frac{1}{64}$
(p) $x = \log_5 \frac{1}{125}$

5. Rewrite each equation in index form, then solve it for x.
(a) $\log_7 x = 2$
(b) $\log_9 x = 2$
(c) $\log_5 x = 3$
(d) $\log_2 x = 5$
(e) $\log_4 x = 3$
(f) $\log_{100} x = 3$
(g) $\log_7 x = 1$
(h) $\log_{11} x = 0$
(i) $\log_{13} x = -1$
(j) $\log_7 x = -1$
(k) $\log_{10} x = -2$
(l) $\log_{12} x = -2$
(m) $\log_5 x = -3$
(n) $\log_7 x = -3$
(o) $\log_2 x = -5$
(p) $\log_3 x = -4$

6. Rewrite each equation in index form, then solve it for x.
(a) $\log_x 49 = 2$
(b) $\log_x 8 = 3$
(c) $\log_x 27 = 3$
(d) $\log_x 10\,000 = 4$
(e) $\log_x 10\,000 = 2$
(f) $\log_x 64 = 6$
(g) $\log_x 64 = 2$
(h) $\log_x 125 = 1$
(i) $\log_x 11 = 1$
(j) $\log_x \frac{1}{17} = -1$
(k) $\log_x \frac{1}{6} = -1$
(l) $\log_x \frac{1}{7} = -1$
(m) $\log_x \frac{1}{9} = -2$
(n) $\log_x \frac{1}{49} = -2$
(o) $\log_x \frac{1}{8} = -3$
(p) $\log_x \frac{1}{81} = -2$

DEVELOPMENT

7. Rewrite each equation in index form and then solve it for x (where a is a constant).
(a) $x = \log_a a$
(b) $\log_a x = 1$
(c) $\log_x a = 1$
(d) $x = \log_a \frac{1}{a}$
(e) $\log_a x = -1$
(f) $\log_x \frac{1}{a} = -1$
(g) $x = \log_a 1$
(h) $\log_a x = 0$
(i) $\log_x 1 = 0$

8. Given that a is a positive real number not equal to 1, evaluate:

(a) $\log_a a$
(b) $\log_a \dfrac{1}{a}$
(c) $\log_a a^3$
(d) $\log_a \dfrac{1}{a^2}$
(e) $\log_a \dfrac{1}{a^5}$
(f) $\log_a \sqrt{a}$
(g) $\log_a \dfrac{1}{\sqrt{a}}$
(h) $\log_a 1$

9. Use the list of powers of 2 prepared in Exercise 7A to find which two integers each expression lies between.

(a) $\log_2 3$
(b) $\log_2 7$
(c) $\log_2 1{\cdot}8$
(d) $\log_2 13$
(e) $\log_2 50$
(f) $\log_2 1000$
(g) $\log_2 \frac{4}{5}$
(h) $\log_2 \frac{1}{3}$

10. Use the tables of powers of 2, 3, 5, … prepared in Exercise 7A to find which two integers each expression lies between.

(a) $\log_{10} 35$
(b) $\log_{10} 6$
(c) $\log_{10} 2000$
(d) $\log_3 2$
(e) $\log_3 50$
(f) $\log_3 100$
(g) $\log_5 100$
(h) $\log_7 20$
(i) $\log_{10} 0{\cdot}4$
(j) $\log_{10} 0{\cdot}05$

11. Use the calculator buttons [log] and [10^x] to approximate each expression. Give answers correct to three significant figures, then check each answer using the other button.

(a) $\log_{10} 2$
(b) $\log_{10} 20$
(c) $10^{0{\cdot}301}$
(d) $10^{1{\cdot}301}$
(e) $10^{0{\cdot}5}$
(f) $10^{1{\cdot}5}$
(g) $\log_{10} 3{\cdot}16$
(h) $\log_{10} 31{\cdot}6$
(i) $\log_{10} 1000$
(j) $\log_{10} 1\,000\,000$
(k) 10^3
(l) 10^6
(m) $\log_{10} 0{\cdot}7$
(n) $\log_{10} 0{\cdot}007$
(o) $10^{-0{\cdot}155}$
(p) $10^{-2{\cdot}15}$

CHALLENGE

12. Solve each equation for x. These questions involve fractional indices.

(a) $x = \log_7 \sqrt{7}$
(b) $x = \log_{11} \sqrt{11}$
(c) $\log_9 x = \frac{1}{2}$
(d) $\log_{144} x = \frac{1}{2}$
(e) $\log_x 3 = \frac{1}{2}$
(f) $\log_x 13 = \frac{1}{2}$
(g) $x = \log_6 \sqrt[3]{6}$
(h) $x = \log_9 3$
(i) $\log_{64} x = \frac{1}{3}$
(j) $\log_{16} x = \frac{1}{4}$
(k) $\log_x 2 = \frac{1}{3}$
(l) $\log_x 2 = \frac{1}{6}$
(m) $x = \log_8 2$
(n) $x = \log_{125} 5$
(o) $\log_7 x = \frac{1}{2}$
(p) $\log_7 x = -\frac{1}{2}$
(q) $\log_x \frac{1}{7} = -\frac{1}{2}$
(r) $\log_x \frac{1}{20} = -\frac{1}{2}$
(s) $x = \log_4 \frac{1}{2}$
(t) $x = \log_{27} \frac{1}{3}$
(u) $\log_{121} x = -\frac{1}{2}$
(v) $\log_{81} x = -\frac{1}{4}$
(w) $\log_x \frac{1}{2} = -\frac{1}{4}$
(x) $\log_x 2 = -\frac{1}{4}$

7 D The Laws for Logarithms

Logarithms are indices, so the laws for manipulating indices can be rewritten as laws for manipulating logarithms.

THREE LAWS FOR LOGARITHMS:

- The log of a product is the sum of the logs:
$$\log_a xy = \log_a x + \log_a y$$
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- The log of a quotient is the difference of the logs:
$$\log_a \frac{x}{y} = \log_a x - \log_a y$$
- The log of a power is the multiple of the log:
$$\log_a x^n = n \log_a x$$

WORKED EXERCISE:
Suppose that it has been found that $\log_3 2 \doteqdot 0{\cdot}63$ and $\log_3 5 \doteqdot 1{\cdot}46$. Use the log laws to find approximations for:
(a) $\log_3 10$ (b) $\log_3 \frac{2}{5}$ (c) $\log_3 32$ (d) $\log_3 18$

SOLUTION:
(a) Because $10 = 2 \times 5$,
$\log_3 10 = \log_3 2 + \log_3 5$
$\doteqdot 0{\cdot}63 + 1{\cdot}46$
$\doteqdot 2{\cdot}09.$

(b) $\log_3 \frac{2}{5} = \log_3 2 - \log_3 5$
$\doteqdot 0{\cdot}63 - 1{\cdot}46$
$\doteqdot -0{\cdot}83.$

(c) Because $32 = 2^5$,
$\log_3 32 = 5 \log_3 2$
$\doteqdot 3{\cdot}15.$

(d) Because $18 = 2 \times 3^2$,
$\log_3 18 = \log_3 2 + 2 \log_3 3.$
$= \log_3 2 + 2$, since $\log_3 3 = 1$,
$\doteqdot 2{\cdot}63.$

Some Particular Values of Logarithmic Functions: Some particular values of logarithmic functions occur very often and are worth committing to memory.

11 SOME PARTICULAR VALUES AND IDENTITIES OF LOGARITHMIC FUNCTIONS:

$\log_a 1 = 0$, because $1 = a^0$.

$\log_a a = 1$, because $a = a^1$.

$\log_a \sqrt{a} = \frac{1}{2}$, because $\sqrt{a} = a^{\frac{1}{2}}$.

$\log_a \dfrac{1}{a} = -1$, because $\dfrac{1}{a} = a^{-1}$.

$\log_a \dfrac{1}{x} = -\log_a x$, because $\log_a \dfrac{1}{x} = \log_a x^{-1} = -\log_a x$.

WORKED EXERCISE:
Use the log laws to expand:
(a) $\log_3 7x^3$ (b) $\log_5 \dfrac{5}{x}$

SOLUTION:
(a) $\log_3 7x^3 = \log_3 7 + \log_3 x^3$ (The log of a product is the sum of the logs.)
$= \log_3 7 + 3 \log_3 x$ (The log of a power is the multiple of the log.)

(b) $\log_5 \dfrac{5}{x} = \log_5 5 - \log_5 x$ (The log of a quotient is the difference of the logs.)
$= 1 - \log_5 x$ ($\log_5 5 = 1$, because $5 = 5^1$.)

Combining Logarithmic and Exponential Functions: We have already seen that the functions [log] and [10^x] are inverses of each other. That is, when applied one after the other to any number, the original number returns. Check, for example, using the buttons [log] and [10^x] on your calculator, that

$$\log_{10} 10^3 = 3 \quad \text{and} \quad 10^{\log_{10} 3} = 3.$$

This relationship is true whatever the base of the logarithm:

12 THE FUNCTIONS $y = a^x$ AND $y = \log_a x$ ARE MUTUALLY INVERSE:

$$\log_a a^x = x \qquad \text{and} \qquad a^{\log_a x} = x$$

Two further examples:

$$\begin{aligned}\log_2 2^7 &= 7\log_2 2 \\ &= 7 \times 1 \\ &= 7\end{aligned} \qquad \begin{aligned}2^{\log_2 8} &= 2^3 \\ &= 8\end{aligned}$$

WORKED EXERCISE:

(a) Simplify $\log_7 7^{12}$ and $5^{\log_5 11}$.

(b) Write 3 as a power of 10, and write 7 as a power of 2.

SOLUTION:

(a) Using the identities above, $\log_7 7^{12} = 12$ and $5^{\log_5 11} = 11$.

(b) Using the second identity above, $3 = 10^{\log_{10} 3}$ and $7 = 2^{\log_2 7}$.

Exercise 7D

NOTE: Do not use a calculator in this exercise. Make sure that you can refer easily to the list of powers of 2, 3, 5, ... from Exercise 7A.

1. Use the log law $\log_a x + \log_a y = \log_a xy$ to simplify:
(a) $\log_6 2 + \log_6 3$ (b) $\log_{15} 3 + \log_{15} 5$ (c) $\log_{10} 4 + \log_{10} 25$ (d) $\log_{12} 72 + \log_{12} 2$ (e) $\log_{10} 50 + \log_{10} 20$ (f) $\log_6 18 + \log_6 2$

2. Use the log law $\log_a x - \log_a y = \log_a \dfrac{x}{y}$ to simplify:
(a) $\log_3 15 - \log_3 5$ (b) $\log_4 20 - \log_4 5$ (c) $\log_2 24 - \log_2 3$ (d) $\log_5 50 - \log_5 2$ (e) $\log_3 810 - \log_3 10$ (f) $\log_2 96 - \log_2 3$

3. Use the log laws to simplify:
(a) $\log_{30} 2 + \log_{30} 3 + \log_{30} 5$ (b) $\log_{12} 9 + \log_{12} 8 + \log_{12} 2$ (c) $\log_2 12 + \log_2 6 - \log_2 9$ (d) $\log_3 6 + \log_3 12 - \log_3 8$ (e) $\log_5 12 + \log_5 2 - \log_5 24$ (f) $\log_5 2 - \log_5 50$ (g) $\log_2 6 - \log_2 48$ (h) $\log_2 12 + \log_2 \frac{1}{3}$ (i) $\log_7 \frac{1}{9} + \log_7 9$

4. Use the log law $\log_a x^n = n\log_a x$ to write each expression in terms of $\log_a 2$.
(a) $\log_a 8$ (b) $\log_a 16$ (c) $\log_a 64$ (d) $\log_a \frac{1}{2}$ (e) $\log_a \frac{1}{8}$ (f) $\log_a \frac{1}{32}$ (g) $\log_a \sqrt{2}$ (h) $\log_a \dfrac{1}{\sqrt{2}}$

5. Express each logarithm in terms of $\log_2 3$ and $\log_2 5$. Remember that $\log_2 2 = 1$.
(a) $\log_2 9$ (b) $\log_2 25$ (c) $\log_2 6$ (d) $\log_2 10$ (e) $\log_2 18$ (f) $\log_2 20$ (g) $\log_2 \frac{2}{3}$ (h) $\log_2 2\frac{1}{2}$

6. Given that $\log_2 3 \doteqdot 1{\cdot}58$ and $\log_2 5 \doteqdot 2{\cdot}32$, use the log laws to find approximations for:
(a) $\log_2 15$ (b) $\log_2 9$ (c) $\log_2 25$ (d) $\log_2 10$ (e) $\log_2 50$ (f) $\log_2 24$ (g) $\log_2 \frac{3}{2}$ (h) $\log_2 \frac{3}{5}$ (i) $\log_2 \frac{9}{5}$ (j) $\log_2 \frac{2}{3}$ (k) $\log_2 54$ (l) $\log_2 75$

DEVELOPMENT

7. Use the log law $\log_a x^n = n \log_a x$ and the identity $\log_a a = 1$ to simplify:
(a) $\log_a a^2$ (b) $5 \log_a a^3$ (c) $\log_a \frac{1}{a}$ (d) $12 \log_a \sqrt{a}$

8. Use the log law $\log_a x^n = n \log_a x$ to write each expression in terms of $\log_a x$.
(a) $\log_a x^3$ (c) $\log_a \sqrt{x}$ (e) $\log_a x^3 - \log_a x^5$ (g) $2 \log_a a^4 - \log_a x^8$
(b) $\log_a \dfrac{1}{x}$ (d) $\log_a \dfrac{1}{x^2}$ (f) $\log_a x^4 + \log_a \dfrac{1}{x^2}$ (h) $\log_a \dfrac{1}{\sqrt{x}} + 3 \log_a \sqrt{x}$

9. Write these expressions in terms of $\log_a x$, $\log_a y$ and $\log_a z$.
(a) $\log_a yz$ (c) $\log_a y^4$ (e) $\log_a xy^3$ (g) $\log_a \sqrt{y}$
(b) $\log_a \dfrac{z}{y}$ (d) $\log_a \dfrac{1}{x^2}$ (f) $\log_a \dfrac{x^2 y}{z^3}$ (h) $\log_a \sqrt{xz}$

10. Given that $\log_{10} 2 \doteqdot 0{\cdot}30$ and $\log_{10} 3 \doteqdot 0{\cdot}48$, use the fact that $\log_{10} 5 = \log_{10} 10 - \log_{10} 2$ to find an approximation for $\log_{10} 5$. Then use the log laws to find approximations for:
(a) $\log_{10} 20$ (c) $\log_{10} 360$ (e) $\log_{10} \sqrt{8}$ (g) $\log_{10} \sqrt{12}$
(b) $\log_{10} 0{\cdot}2$ (d) $\log_{10} \sqrt{2}$ (f) $\log_{10} \dfrac{1}{\sqrt{10}}$ (h) $\log_{10} \dfrac{1}{\sqrt{5}}$

11. If $x = \log_a 2$, $y = \log_a 3$ and $z = \log_a 5$, simplify:
(a) $\log_a 64$ (c) $\log_a 27a^5$ (e) $\log_a 1{\cdot}5$ (g) $\log_a 0{\cdot}04$
(b) $\log_a \dfrac{1}{30}$ (d) $\log_a \dfrac{100}{a}$ (f) $\log_a \dfrac{18}{25a}$ (h) $\log_a \dfrac{8}{15a^2}$

CHALLENGE

12. Use the identities $\log_a a^x = x$ and $a^{\log_a x} = x$ to simplify:
(a) $\log_7 7^5$ (b) $3^{\log_3 7}$ (c) $\log_{12} 12^n$ (d) $6^{\log_6 y}$

13. Using the identity $x = a^{\log_a x}$, express:
(a) 10 as a power of 3, (b) 3 as a power of 10, (c) $0{\cdot}1$ as a power of 2.

7 E Equations Involving Logarithms and Indices

When the base is 10, solutions to index and logarithmic equations can be approximated using a calculator. This section deals with equations involving other bases.

The Change-of-Base Formula: Finding approximations of logarithms to other bases requires a formula that converts logarithms from one base to another.

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THE CHANGE-OF-BASE FORMULA: To write logs base b in terms of logs base a:

$$\log_b x = \frac{\log_a x}{\log_a b}$$

Remember this as 'the log of the number over the log of the base'.

USING A CALCULATOR TO FIND LOGARITHMS TO OTHER BASES:
The button log gives logs base 10. Hence to find logs to another base b:

$$\log_b x = \frac{\log_{10} x}{\log_{10} b}$$

PROOF: To prove this formula, let $y = \log_b x$.
Then by the definition of logs, $x = b^y$
and taking logs base a of both sides, $\log_a x = \log_a b^y$.
Using the log laws, $\log_a x = y \log_a b$
and rearranging, $y = \dfrac{\log_a x}{\log_a b}$, as required.

WORKED EXERCISE:
Find, correct to four significant figures: (a) $\log_2 5$, (b) $\log_3 0{\cdot}02$.
Then check your approximations using the button labelled $\boxed{x^y}$ or $\boxed{\wedge}$.

SOLUTION:

(a) $\log_2 5 = \dfrac{\log_{10} 5}{\log_{10} 2}$
$\doteqdot 2{\cdot}322$
Checking, $2^{2{\cdot}322} \doteqdot 5$.

(b) $\log_3 0{\cdot}02 = \dfrac{\log_{10} 0{\cdot}02}{\log_{10} 3}$
$\doteqdot -3{\cdot}561$
Checking, $3^{-3{\cdot}561} \doteqdot 0{\cdot}02$.

WORKED EXERCISE:
Solve, correct to four significant figures: (a) $2^x = 7$, (b) $3^x = 0{\cdot}05$.
Then check your approximations using the button labelled $\boxed{x^y}$ or $\boxed{\wedge}$.

SOLUTION:

(a) Since $2^x = 7$, $x = \log_2 7$
$= \dfrac{\log_{10} 7}{\log_{10} 2}$
$\doteqdot 2{\cdot}807$
Checking, $2^{2{\cdot}807} \doteqdot 7$.

(b) Since $3^x = 0{\cdot}05$, $x = \log_3 0{\cdot}05$
$= \dfrac{\log_{10} 0{\cdot}05}{\log_{10} 3}$
$\doteqdot -2{\cdot}727$
Checking, $3^{-2{\cdot}727} \doteqdot 0{\cdot}05$.

Exercise 7E

1. Use the change-of-base formula, $\log_b x = \dfrac{\log_{10} x}{\log_{10} b}$, to approximate these logarithms, correct to four significant figures. Check each answer using the button labelled $\boxed{x^y}$ or $\boxed{\wedge}$.

(a) $\log_2 7$ (d) $\log_3 4690$ (g) $\log_6 3$ (j) $\log_3 0{\cdot}0004$ (m) $\log_{0{\cdot}03} 0{\cdot}89$
(b) $\log_2 26$ (e) $\log_5 2$ (h) $\log_{12} 2$ (k) $\log_{11} 1000$ (n) $\log_{0{\cdot}99} 0{\cdot}003$
(c) $\log_2 0{\cdot}07$ (f) $\log_7 31$ (i) $\log_3 0{\cdot}1$ (l) $\log_{0{\cdot}8} 0{\cdot}2$ (o) $\log_{0{\cdot}99} 1000$

2. Rewrite each equation with x as the subject, using logarithms. Then use the change-of-base formula to solve it, giving your answers correct to four significant figures. Check each answer using the button labelled $\boxed{x^y}$ or $\boxed{\wedge}$.

(a) $2^x = 15$ (d) $2^x = 0{\cdot}1$ (g) $3^x = 0{\cdot}01$ (j) $8^x = \frac{7}{9}$ (m) $0{\cdot}7^x = 0{\cdot}1$
(b) $2^x = 5$ (e) $2^x = 0{\cdot}0007$ (h) $5^x = 10$ (k) $6^x = 1{\cdot}4$ (n) $0{\cdot}98^x = 0{\cdot}03$
(c) $2^x = 1{\cdot}45$ (f) $3^x = 10$ (i) $12^x = 150$ (l) $30^x = 2$ (o) $0{\cdot}99^x = 0{\cdot}01$

3. Give exact solutions to these inequations. Do not use a calculator.

(a) $2^x > 32$ (c) $2^x < 64$ (e) $5^x > 5$ (g) $2^x < \frac{1}{2}$
(b) $2^x \le 32$ (d) $3^x \ge 81$ (f) $4^x \le 1$ (h) $10^x \le 0{\cdot}001$

DEVELOPMENT

4. Give exact solutions to these inequations. Do not use a calculator.

(a) $\log_2 x < 3$ (c) $\log_{10} x > 3$ (e) $\log_5 x > 0$ (g) $\log_5 x \le 3$
(b) $\log_2 x \ge 3$ (d) $\log_{10} x \ge 1$ (f) $\log_6 x < 1$ (h) $\log_6 x > 2$

5. Rewrite each inequation in terms of logarithms, with x as the subject. Then use the change-of-base formula to solve it, giving your answer correct to three significant figures.

(a) $2^x > 12$ (c) $2^x < 0{\cdot}02$ (e) $5^x < 100$ (g) $1{\cdot}2^x > 10$
(b) $2^x < 100$ (d) $2^x > 0{\cdot}1$ (f) $3^x < 0{\cdot}007$ (h) $1{\cdot}001^x > 100$

CHALLENGE

6. Solve these equation and inequations, correct to three significant figures if necessary.

(a) $2^{x+1} = 16$ (c) $5^{x-1} < 1$ (e) $(\frac{1}{2})^{x-3} = 8$ (g) $2^{x-2} < 7$
(b) $3^{2x} = 81$ (d) $10^{\frac{1}{3}x} \le 1\,000$ (f) $(\frac{1}{10})^{5x} = \frac{1}{10}$ (h) $3^{x+5} > 10$

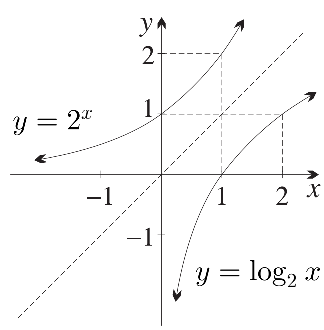
7. (a) Solve $2^x < 10^{10}$. How many positive integer powers of 2 are less than 10^{10}?
(b) Solve $3^x < 10^{50}$. How many positive integer powers of 3 are less than 10^{50}?

8. (a) Explain why $\log_{10} 300$ lies between 2 and 3.
(b) If x is a two-digit number, what two integers does $\log_{10} x$ lie between?
(c) If $\log_{10} x = 4{\cdot}7$, how many digits does x have to the left of the decimal point?
(d) Find $\log_{10} 25^{20}$, and hence find the number of digits in 25^{20}.
(e) Find $\log_{10} 2^{1000}$, and hence find the number of digits in 2^{1000}.

7 F Graphs of Exponential and Logarithmic Functions

The function $y = a^x$ is an *exponential function*, because the variable x is in the *exponent* or *index*. The function $y = \log_a x$ is a *logarithmic function*.

The Graphs of $y = 2^x$ and $y = \log_2 x$: These two graphs will demonstrate the characteristic features of all exponential and logarithmic graphs. Here are their tables of values:



$y = 2^x$

x	-2	-1	0	1	2
y	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4

$y = \log_2 x$

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
y	-2	-1	0	1	2

- The two graphs are reflections of each other in the diagonal line $y = x$. This is because they are inverse functions of each other, and so their tables of values are the same, except that the x-values and y-values have been swapped.
- For $y = 2^x$, the domain is all real x and the range is $y > 0$.
  For $y = \log_2 x$, the domain is $x > 0$ and the range is all real y.
- For $y = 2^x$, the x-axis is a horizontal asymptote.
  For $y = \log_2 x$, the y-axis is a vertical asymptote.
- The graph of $y = 2^x$ is concave up, but $y = \log_2 x$ is concave down.
- As x increases, $y = 2^x$ also increases, getting steeper all the time.
  As x increases, $y = \log_2 x$ also increases, but gets flatter all the time.

The Word 'Logarithm': The Scottish mathematician John Napier (1550–1617) constructed his new word 'log|arithm' from the two Greek words 'logos', meaning 'ratio' or 'calculation', and 'arithmos', meaning 'number'. Until the invention of calculators, the routine method for performing difficult calculations in arithmetic was to use tables of logarithms, invented by Napier, to convert products to sums and quotients to differences.

Exercise 7F

1. (a) Copy and complete these two tables of values.

(i)

x	-3	-2	-1	0	1	2	3
2^x							

(ii)

x	-3	-2	-1	0	1	2	3
2^{-x}							

(b) Hence sketch the graphs of $y = 2^x$ and $y = 2^{-x}$ on one set of axes.

(c) How are the two tables of values related to each other?

(d) What symmetry does the diagram of the two graphs display?

(e) Write down the domains and ranges of: (i) $y = 2^x$, (ii) $y = 2^{-x}$.

(f) Write down the equations of the asymptotes of: (i) $y = 2^x$, (ii) $y = 2^{-x}$.

(g) Copy and complete:
(i) 'As $x \to -\infty$, $2^x \to \ldots$' (ii) 'As $x \to \infty$, $2^x \to \ldots$'

(h) Copy and complete:
(i) 'As $x \to -\infty$, $2^{-x} \to \ldots$' (ii) 'As $x \to \infty$, $2^{-x} \to \ldots$'

2. (a) Copy and complete these two tables of values.

(i)

x	-2	-1	0	1	2
3^x					

(ii)

x	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9
$\log_3 x$					

(b) Hence sketch the graphs of $y = 3^x$ and $y = \log_3 x$ on one set of axes.

(c) How are the two tables of values related to each other?

(d) What symmetry does the diagram of the two graphs display?

(e) Write down the domains and ranges of: (i) $y = 3^x$, (ii) $y = \log_3 x$.

(f) Write down the equations of the asymptotes of: (i) $y = 3^x$, (ii) $y = \log_3 x$.

(g) Copy and complete:
(i) 'As $x \to -\infty$, $3^x \to \ldots$' (ii) 'As $x \to 0^+$, $\log_3 x \to \ldots$'

3. (a) Use the calculator button [log] to complete the following table of values, giving each entry correct to two significant figures where appropriate.

x	0·1	0·25	0·5	0·75	1	2	3	4	5	6	7	8	9	10
$\log_{10} x$														

(b) Hence sketch the graph of $y = \log_{10} x$.

DEVELOPMENT

4. (a) Sketch on one set of axes the graphs of $y = 3^x$ and $y = 3^{-x}$.
(b) Sketch on one set of axes the graphs of $y = 10^x$ and $y = 10^{-x}$.

5. Sketch the four graphs below on one set of axes.
(a) $y = 2^x$ (b) $y = -2^x$ (c) $y = 2^{-x}$ (d) $y = -2^{-x}$

6. Sketch the three graphs in each part below on one set of axes, clearly indicating the asymptote, the y-intercept, and the x-intercept if it exists. Use shifting of the graphs in the previous questions, but also use a table of values to confirm your diagram.

(a) $y = 2^x$
$y = 2^x + 3$
$y = 2^x - 1$

(b) $y = -2^x$
$y = 2 - 2^x$
$y = -2 - 2^x$

(c) $y = 2^{-x}$
$y = 2^{-x} + 1$
$y = 2^{-x} - 2$

7. Use reflections in the x-axis and y-axis to sketch the four graphs below on one set of axes.

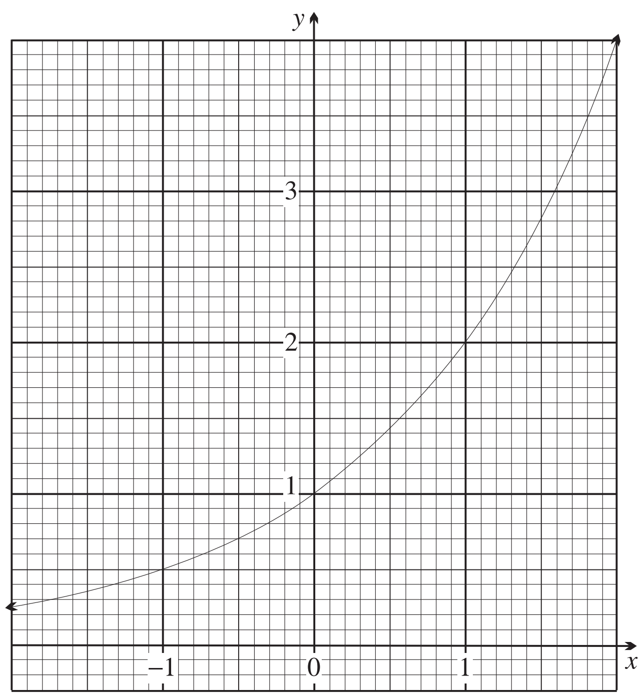
(a) $y = \log_2 x$
(b) $y = -\log_2 x$
(c) $y = \log_2(-x)$
(d) $y = -\log_2(-x)$

8. Use shifting to sketch the three graphs in each part below on one set of axes, clearly indicating the asymptote and the intercepts with the axes.

(a) $y = \log_2 x$
$y = \log_2 x + 1$
$y = \log_2 x - 1$

(b) $y = -\log_2 x$
$y = 2 - \log_2 x$
$y = -2 - \log_2 x$

9.



The diagram above show the graph of $y = 2^x$. Use the graph to answer the questions below, giving your answers correct to no more than two decimal places.

(a) Read off the graph the values of:
(i) 2^2 (ii) 2^{-2} (iii) $2^{1\cdot5}$ (iv) $2^{0\cdot4}$ (v) $2^{-0\cdot6}$

(b) Find x from the graph if:
(i) $2^x = 2$ (ii) $2^x = 3$ (iii) $2^x = 1\cdot2$ (iv) $2^x = 0\cdot4$

(c) Find the values of x for which:
(i) $1 \le 2^x \le 4$ (ii) $1 \le 2^x \le 2$ (iii) $1\cdot5 \le 2^x \le 3$ (iv) $0\cdot5 \le 2^x \le 2$

(d) Read the graph backwards to find:
(i) $\log_2 4$ (ii) $\log_2 3$ (iii) $\log_2 1\cdot4$ (iv) $\log_2 0\cdot8$

CHALLENGE

10. Sketch, on separate axes, the graphs of:
(a) $y = 2^{x-2}$ (c) $y = \log_2(x+1)$ (e) $y = \frac{1}{2}(2^x + 2^{-x})$
(b) $y = 2^{x+1}$ (d) $y = \log_2(x-1)$ (f) $y = \frac{1}{2}(2^x - 2^{-x})$

7G Chapter Review Exercise

1. Write each expression as an integer or fraction.
(a) 5^3 (f) 2^{-3} (k) $(\frac{5}{6})^{-2}$ (p) $(\frac{4}{49})^{\frac{1}{2}}$
(b) 2^8 (g) 3^{-4} (l) $36^{\frac{1}{2}}$ (q) $(\frac{14}{59})^0$
(c) 10^9 (h) 27^0 (m) $27^{\frac{1}{3}}$ (r) $(\frac{9}{25})^{-\frac{1}{2}}$
(d) 17^{-1} (i) $(\frac{2}{3})^3$ (n) $8^{\frac{2}{3}}$ (s) $(\frac{8}{27})^{\frac{2}{3}}$
(e) 9^{-2} (j) $(\frac{7}{12})^{-1}$ (o) $9^{\frac{5}{2}}$ (t) $(\frac{9}{100})^{-\frac{3}{2}}$

2. Write each expression in index form.
(a) $\dfrac{1}{x}$ (c) $-\dfrac{1}{2x}$ (e) $5\sqrt{36x}$ (g) $\dfrac{y}{x}$
(b) $\dfrac{7}{x^2}$ (d) $\sqrt{x}$ (f) $\dfrac{4}{\sqrt{x}}$ (h) $2y\sqrt{x}$

3. Simplify:
(a) $(x^4)^5$ (b) $\left(\dfrac{3}{a^3}\right)^4$ (c) $(25x^6)^{\frac{1}{2}}$ (d) $\left(\dfrac{8r^3}{t^6}\right)^{\frac{1}{3}}$

4. Simplify each expression, leaving the answer in index form.
(a) $x^2y \times y^2x$ (d) $4a^{-2}bc \times a^5b^2c^{-2}$ (g) $m^{\frac{1}{2}}n^{-\frac{1}{2}} \times m^{1\frac{1}{2}}n^{-\frac{1}{2}}$
(b) $15xyz \times 4y^2z^4$ (e) $x^3y \div xy^3$ (h) $(2st)^3 \times (3s^3)^2$
(c) $3x^{-2}y \times 6xy^{-3}$ (f) $14x^{-2}y^{-1} \div 7xy^{-2}$ (i) $(4x^2y^{-2})^3 \div (2xy^{-1})^3$

5. Solve each equation for x.
(a) $3^x = 81$ (c) $7^x = \frac{1}{7}$ (e) $(\frac{1}{3})^x = \frac{1}{9}$ (g) $25^x = 5$
(b) $5^x = 25$ (d) $2^x = \frac{1}{32}$ (f) $(\frac{2}{3})^x = \frac{8}{27}$ (h) $8^x = 2$

6. Rewrite each logarithmic equation as an index equation and solve it for x.
(a) $x = \log_2 8$ (e) $x = \log_7 \frac{1}{49}$ (i) $2 = \log_7 x$ (m) $2 = \log_x 36$
(b) $x = \log_3 9$ (f) $x = \log_{13} 1$ (j) $-1 = \log_{11} x$ (n) $3 = \log_x 1000$
(c) $x = \log_{10} 10\,000$ (g) $x = \log_9 3$ (k) $\frac{1}{2} = \log_{16} x$ (o) $-1 = \log_x \frac{1}{7}$
(d) $x = \log_5 \frac{1}{5}$ (h) $x = \log_2 \sqrt{2}$ (l) $\frac{1}{3} = \log_{27} x$ (p) $\frac{1}{2} = \log_x 4$

7. Use the log laws and identities to simplify:
(a) $\log_{22} 2 + \log_{22} 11$ (c) $\log_7 98 - \log_7 2$ (e) $\log_3 54 + \log_3 \frac{1}{6}$
(b) $\log_{10} 25 + \log_{10} 4$ (d) $\log_5 6 - \log_5 150$ (f) $\log_{12} \frac{2}{7} + \log_{12} \frac{7}{2}$

8. Write each expression in terms of $\log_a x$, $\log_a y$ and $\log_a z$.
(a) $\log_a xyz$ (c) $\log_a x^3$ (e) $\log_a x^2y^5$ (g) $\log_a \sqrt{x}$
(b) $\log_a \dfrac{x}{y}$ (d) $\log_a \dfrac{1}{z^2}$ (f) $\log_a \dfrac{y^2}{xz^2}$ (h) $\log_a \sqrt{xyz}$

9. Between what two consecutive integers do the following logarithms lie?

(a) $\log_{10} 34$ (c) $\log_3 90$ (e) $\log_{10} 0{\cdot}4$ (g) $\log_2 0{\cdot}1$

(b) $\log_7 100$ (d) $\log_2 35$ (f) $\log_{10} 0{\cdot}007$ (h) $\log_7 0{\cdot}1$

10. Use a calculator, and the change-of-base formula if necessary, to approximate these logarithms, correct to four significant figures.

(a) $\log_{10} 215$ (c) $\log_7 50$ (e) $\log_5 0{\cdot}215$ (g) $\log_{0{\cdot}5} 8$

(b) $\log_{10} 0{\cdot}0045$ (d) $\log_2 1000$ (f) $\log_{1{\cdot}01} 2$ (h) $\log_{0{\cdot}99} 0{\cdot}001$

11. Use logarithms to solve these index equations, correct to four significant figures.

(a) $2^x = 11$ (c) $7^x = 350$ (e) $1{\cdot}01^x = 5$ (g) $0{\cdot}8^x = \frac{1}{10}$

(b) $2^x = 0{\cdot}04$ (d) $3^x = 0{\cdot}67$ (f) $1{\cdot}01^x = 0{\cdot}2$ (h) $0{\cdot}99^x = \frac{1}{100}$

12. (a) Sketch on one set of axes $y = 3^x$, $y = 3^{-x}$, $y = -3^x$ and $y = -3^{-x}$.

(b) Sketch on one set of axes $y = 2^x$ and $y = \log_2 x$.

(c) Sketch on one set of axes $y = 3^x$, $y = 3^x - 1$ and $y = 3^x + 2$

Sequences and Series

Many situations in nature result in a sequence of numbers with a simple pattern. For example, when cells continually divide into two, the numbers in successive generations descending from a single cell form the sequence

$$1,\ 2,\ 4,\ 8,\ 16,\ 32,\ \ldots$$

Again, someone thinking about the half-life of a radioactive substance will need to ask what happens when we add up more and more terms of the series

$$\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32} + \tfrac{1}{64} + \cdots$$

Sequences and series are reviewed in the Year 12 volume, which also deals more fully with their applications, particularly in financial situations. Readers who want to study both terms and sums of arithmetic series before considering geometric series at all should work on Sections 8E and 8F before Sections 8C and 8D.

8 A Sequences and How to Specify Them

A typical *infinite sequence* is formed by arranging the positive odd integers in increasing order:

$$1, 3, 5, 7, 9, 11, 13, 15, 17, 19, \ldots$$

The three dots ... indicate that the sequence goes on forever, with no last term. The sequence starts with the first term 1, then has second term 3, third term 5, and so on. The symbol T_n will usually be used to stand for the nth term, thus

$$T_1 = 1, \quad T_2 = 3, \quad T_3 = 5, \quad T_4 = 7, \quad T_5 = 9, \quad \ldots$$

The two-digit odd numbers less than 100 form a *finite sequence*:

$$1,\ 3,\ 5,\ 7,\ \ldots,\ 99$$

where the dots ... stand for the 45 terms that have been omitted.

There are three different ways to specify a sequence, and it is important to be able to display a given sequence in each of these different ways.

Write Out the First Few Terms: The easiest way is to write out the first few terms until the pattern is clear. Continuing with our example of the positive odd integers, we could write the sequence as

$$1,\ 3,\ 5,\ 7,\ 9,\ \ldots$$

This sequence clearly continues $\ldots,\ 11,\ 13,\ 15,\ 17,\ 19,\ \ldots$, and with a few more calculations, it becomes clear that $T_{11} = 21$, $T_{14} = 27$, and $T_{16} = 31$.

Give a Formula for the nth Term: The formula for the nth term of this sequence is

$$T_n = 2n - 1,$$

because the nth term is always 1 less than $2n$. Giving the formula does not rely on the reader recognising a pattern, and any particular term of the sequence can now be calculated quickly:

$$T_{30} = 60 - 1 = 59 \qquad T_{100} = 200 - 1 = 199 \qquad T_{244} = 488 - 1 = 487$$

Say Where to Start and How to Proceed: The sequence of odd positive integers starts with 1, then each term is 2 more than the previous one. Thus the sequence is completely specified by writing down these two statements:

$T_1 = 1,$ (Start the sequence with 1.)

$T_n = T_{n-1} + 2,$ for $n \geq 2$. (Every term is 2 more than the previous term.)

Such a specification is called a *recursive* formula of a sequence. Most of the sequences studied in this chapter are based on this idea.

WORKED EXERCISE:

(a) Write down the first five terms of the sequence given by $T_n = 7n - 3$.

(b) Describe how each term T_n can be obtained from the previous term T_{n-1}.

SOLUTION:

(a) $T_1 = 7 - 3 = 4$ $\quad T_2 = 14 - 3 = 11$ $\quad T_3 = 21 - 3 = 18$ $\quad T_4 = 28 - 3 = 25$ $\quad T_5 = 35 - 3 = 32$

(b) Each term is 7 more than the previous term. That is, $T_n = T_{n-1} + 7$.

WORKED EXERCISE:

(a) Find the first five terms of the sequence given by $T_1 = 14$ and $T_n = T_{n-1} + 10$.

(b) Write down a formula for the nth term T_n.

SOLUTION:

(a) $T_1 = 14$ $\quad T_2 = T_1 + 10 = 24$ $\quad T_3 = T_2 + 10 = 34$ $\quad T_4 = T_3 + 10 = 44$ $\quad T_5 = T_4 + 10 = 54$

(b) From this pattern, the formula for the nth term is clearly $T_n = 10n + 4$.

Using the Formula for T_n to Solve Problems: Many problems about sequences can be solved by forming an equation using the formula for T_n.

WORKED EXERCISE:

Find whether 300 and 400 are members of the sequence $T_n = 7n + 20$.

SOLUTION:

Put $T_n = 300$.
Then $7n + 20 = 300$
$7n = 280$
$n = 40$.
Hence 300 is the 40th term.

Put $T_n = 400$.
Then $7n + 20 = 400$
$7n = 380$
$n = 54\frac{2}{7}$.
Hence 400 is not a term of the sequence.

WORKED EXERCISE:

(a) Find how many negative terms there are in the sequence $T_n = 12n - 100$.

(b) Find the first positive term. (State its number and its value.)

SOLUTION:

(a) Put $T_n < 0$.
Then $12n - 100 < 0$
$n < 8\frac{1}{3}$,
so there are eight negative terms.

(b) The last negative term is T_8,
so the first positive term is
$T_9 = 108 - 100$
$= 8$.

Exercise 8A

1. Write down the next four terms of each sequence.

(a) 5, 10, 15, ...
(b) 6, 16, 26, ...
(c) 2, 4, 8, ...
(d) 3, 6, 12, ...
(e) 38, 34, 30, ...
(f) 39, 30, 21, ...
(g) 24, 12, 6, ...
(h) 81, 27, 9, ...
(i) $-1, 1, -1, \ldots$
(j) 1, 4, 9, ...
(k) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$
(l) $16, -8, 4, \ldots$

2. Find the first four terms of each sequence. You will need to substitute $n = 1$, $n = 2$, $n = 3$ and $n = 4$ into the formula for the nth term T_n.

(a) $T_n = 6n$
(b) $T_n = 5n - 2$
(c) $T_n = 2^n$
(d) $T_n = 5^n$
(e) $T_n = 20 - n$
(f) $T_n = 6 - 2n$
(g) $T_n = 3 \times 2^n$
(h) $T_n = 7 \times 10^n$
(i) $T_n = n^3$
(j) $T_n = n(n + 1)$
(k) $T_n = (-1)^n$
(l) $T_n = (-3)^n$

3. Write down the first four terms of each sequence described below.

(a) The first term is 6, and every term after that is 2 more than the previous term.
(b) The first term is 11, and every term after that is 50 more than the previous term.
(c) The first term is 15, and every term after that is 3 less than the previous term.
(d) The first term is 12, and every term after that is 8 less than the previous term.
(e) The first term is 5, and every term after that is twice the previous term.
(f) The first term is $\frac{1}{3}$, and every term after that is three times the previous term.
(g) The first term is 18, and every term after that is half the previous term.
(h) The first term is -100, and every term after that is one fifth of the previous term.

4. Write out the first twelve terms of the sequence 7, 12, 17, 22, ...

(a) How many terms are less than 30?
(b) How many terms are less than 60?
(c) How many terms lie between 20 and 40?
(d) How many terms lie between 10 and 50?
(e) What is the 10th term?
(f) What number term is 37?
(g) Is 87 a term in the sequence?
(h) Is 201 a term in the sequence?
(i) Find the first term greater than 45.
(j) Find the last term less than 43.

5. Write out the first twelve terms of the sequence $\frac{3}{4}, 1\frac{1}{2}, 3, 6, \ldots$

(a) How many terms are less than 30?
(b) How many terms are less than 400?
(c) How many terms lie between 20 and 100?
(d) How many terms lie between 1 and 1000?
(e) What is the 10th term?
(f) What number term is 192?
(g) Is 96 a term in the sequence?
(h) Is 100 a term in the sequence?
(i) Find the first term greater than 200.
(j) Find the last term less than 50.

DEVELOPMENT

6. For each sequence, write out the first five terms. Then explain how each term is obtained from the previous term.

(a) $T_n = 12 + n$
(b) $T_n = 4 + 5n$
(c) $T_n = 15 - 5n$
(d) $T_n = 3 \times 2^n$
(e) $T_n = 7 \times (-1)^n$
(f) $T_n = 80 \times (\frac{1}{2})^n$

7. The nth term of a sequence is given by $T_n = 3n + 1$.

(a) Put $T_n = 40$, and hence show that 40 is the 13th member of the sequence.
(b) Put $T_n = 30$, and hence show that 30 is not a member of the sequence.
(c) Similarly, find whether 100, 200 and 1000 are members of the sequence.

8. Answer each of these questions by forming an equation and solving it.

(a) Find whether 16, 35 and 111 are members of the sequence $T_n = 2n - 5$.
(b) Find whether 44, 200 and 306 are members of the sequence $T_n = 10n - 6$.
(c) Find whether 40, 72 and 200 are members of the sequence $T_n = 2n^2$.
(d) Find whether 8, 96 and 128 are members of the sequence $T_n = 2^n$.

9. The nth term of a sequence is given by $T_n = 10n + 4$.

(a) Put $T_n < 100$, and hence show that the nine terms T_1 to T_9 are less than 100.
(b) Put $T_n > 56$, and hence show that the first term greater than 56 is $T_6 = 64$.
(c) Similarly, find how many terms are less than 500.
(d) Find the first term greater than 203, giving its number and its value.

10. Answer each of these questions by forming an inequation and solving it.

(a) How many terms of the sequence $T_n = 2n - 5$ are less than 100?
(b) How many terms of the sequence $T_n = 4n + 6$ are less than 300?
(c) What is the first term of the sequence $T_n = 3n + 5$ that is greater than 127?
(d) What is the first term of the sequence $T_n = 7n - 44$ that is greater than 100?

11. In each part, the two lines define a sequence T_n. The first line gives the first term T_1. The second line defines how each subsequent term T_n is obtained from the previous term T_{n-1}. Write down the first four terms of each sequence.

(a) $T_1 = 3$,
$T_n = T_{n-1} + 2$, for $n \ge 2$

(b) $T_1 = 5$,
$T_n = T_{n-1} + 12$, for $n \ge 2$

(c) $T_1 = 6$,
$T_n = T_{n-1} - 3$, for $n \ge 2$

(d) $T_1 = 12$,
$T_n = T_{n-1} - 10$, for $n \ge 2$

(e) $T_1 = 5$,
$T_n = 2T_{n-1}$, for $n \ge 2$

(f) $T_1 = 4$,
$T_n = 5T_{n-1}$, for $n \ge 2$

(g) $T_1 = 20$,
$T_n = \frac{1}{2}T_{n-1}$, for $n \ge 2$

(h) $T_1 = 1$,
$T_n = -T_{n-1}$, for $n \ge 2$

CHALLENGE

12. Write down the first four terms of each sequence, then state which terms are zero.

(a) $T_n = \sin 90n^\circ$
(b) $T_n = \cos 90n^\circ$
(c) $T_n = \cos 180n^\circ$
(d) $T_n = \sin 180n^\circ$

13. The *Fibonacci sequence* is defined by

$$F_1 = 1, \qquad F_2 = 1, \qquad F_n = F_{n-1} + F_{n-2}, \text{ for } n \ge 3.$$

Write out the first 12 terms of the sequence. Explain why every third term of the sequence is even and the rest are odd.

8 B Arithmetic Sequences

A very simple type of sequence is an *arithmetic sequence*. This is a sequence like

$$3,\ 13,\ 23,\ 33,\ 43,\ 53,\ 63,\ 73,\ 83,\ 93,\ \ldots,$$

in which the difference between successive terms is constant — in this example each term is 10 more than the previous term. Notice that all the terms can be generated from the *first term* 3 by repeated addition of this *common difference* 10.

Definition of an Arithmetic Sequence: Arithmetic sequences are called APs for short. The initials stand for 'arithmetic progression' — an old name for the same thing.

1 DEFINITION OF AN AP: A sequence T_n is called an *arithmetic sequence* if

$$T_n - T_{n-1} = d, \text{ for } n \geq 2,$$

where d is a constant, called the *common difference*.

WORKED EXERCISE:
Test whether each sequence is an AP. If the sequence is an AP, find its first term a and its common difference d.
(a) $46,\ 43,\ 40,\ 37,\ \ldots$ (b) $1,\ 4,\ 9,\ 16,\ \ldots$

SOLUTION:
(a) $T_2 - T_1 = 43 - 46 = -3$ $\quad T_3 - T_2 = 40 - 43 = -3$ $\quad T_4 - T_3 = 37 - 40 = -3$
Hence the sequence is an AP with $a = 46$ and $d = -3$.
(b) $T_2 - T_1 = 4 - 1 = 3$ $\quad T_3 - T_2 = 9 - 4 = 5$ $\quad T_4 - T_3 = 16 - 9 = 7$
Since the differences are not all the same, the sequence is not an AP.

A Formula for the nth Term of an AP: Let the first term of an AP be a and the common difference be d. Then the first few terms of the sequence are

$$T_1 = a, \qquad T_2 = a + d, \qquad T_3 = a + 2d, \qquad T_4 = a + 3d, \qquad \ldots$$

From this pattern, the general formula for the nth term is clear:

2 THE nTH TERM OF AN AP:

$$T_n = a + (n-1)d$$

where a is the first term and d is the common difference.

WORKED EXERCISE:
Write out the first five terms, and calculate the 20th term, of the AP with:
(a) $a = 2$ and $d = 5$, (b) $a = 20$ and $d = -3$.

SOLUTION:
(a) $2,\ 7,\ 12,\ 17,\ 22,\ \ldots$
$$\begin{aligned} T_{20} &= a + 19d \\ &= 2 + 19 \times 5 \\ &= 97 \end{aligned}$$
(b) $20,\ 17,\ 14,\ 11,\ 8,\ \ldots$
$$\begin{aligned} T_{20} &= a + 19d \\ &= 20 + 19 \times (-3) \\ &= -37 \end{aligned}$$

WORKED EXERCISE:

(a) Find a formula for the nth term of the sequence 26, 35, 44, 53, ...

(b) How many terms are there in the sequence 26, 35, 44, 53, ..., 917?

SOLUTION:

(a) The sequence is an AP with $a = 26$ and $d = 9$.

Hence $T_n = a + (n-1)d$
$= 26 + 9(n-1)$
$= 26 + 9n - 9$
$= 17 + 9n.$

(b) Put $T_n = 917.$

Then $17 + 9n = 917$
$9n = 900$
$n = 100,$

so there are 100 terms in the sequence.

Solving Problems Involving APs: Now that we have the formula for the nth term T_n, many problems can be solved by forming an equation and solving it.

WORKED EXERCISE:

(a) Show that the sequence 200, 193, 186, ... is an AP.

(b) Find a formula for the nth term.

(c) Find the first negative term.

SOLUTION:

(a) Since $T_2 - T_1 = -7$
and $T_3 - T_2 = -7$,
it is an AP with $a = 200$ and $d = -7$.

(b) Hence $T_n = 200 - 7(n-1)$
$= 200 - 7n + 7$
$= 207 - 7n.$

(c) Put $T_n < 0.$
Then $207 - 7n < 0$
$207 < 7n$
$n > 29\frac{4}{7},$
so the first negative term is $T_{30} = -3.$

WORKED EXERCISE:

The first term of an AP is 105 and the 10th term is 6. Find the common difference and write out the first five terms.

SOLUTION:

First, we know that $T_1 = 105$
that is, $a = 105.$ (1)
Secondly, we know that $T_{10} = 6$
so using the formula for the 10th term, $a + 9d = 6.$ (2)
Substituting (1) into (2), $105 + 9d = 6$
$9d = -99$
$d = -11.$

So the common difference is $d = -11$ and the sequence is 105, 94, 83, 72, 61, ...

Exercise 8B

1. Write out the next three terms of these sequences. They are all APs.
(a) $2,\ 6,\ 10,\ \ldots$
(b) $3,\ 8,\ 13,\ \ldots$
(c) $35,\ 25,\ 15,\ \ldots$
(d) $11,\ 5,\ -1,\ \ldots$
(e) $4\frac{1}{2},\ 6,\ 7\frac{1}{2},\ \ldots$
(f) $8,\ 7\frac{1}{2},\ 7,\ \ldots$

2. Find the differences $T_2 - T_1$ and $T_3 - T_2$ for each sequence to test whether it is an AP. If the sequence is an AP, state the values of the first term a and the common difference d.
(a) $3,\ 7,\ 11,\ \ldots$
(b) $11,\ 7,\ 3,\ \ldots$
(c) $10,\ 17,\ 24,\ \ldots$
(d) $10,\ 20,\ 40,\ \ldots$
(e) $50,\ 35,\ 20,\ \ldots$
(f) $23,\ 34,\ 45,\ \ldots$
(g) $-12,\ -7,\ -2,\ \ldots$
(h) $-40,\ 20,\ -10,\ \ldots$
(i) $1,\ 11,\ 111,\ \ldots$
(j) $8,\ -2,\ -12,\ \ldots$
(k) $-17,\ 0,\ 17,\ \ldots$
(l) $10,\ 7\frac{1}{2},\ 5,\ \ldots$

3. Write out the first four terms of the APs whose first terms and common differences are:
(a) $a = 3$ and $d = 2$
(b) $a = 7$ and $d = 2$
(c) $a = 7$ and $d = -4$
(d) $a = 17$ and $d = 11$
(e) $a = 30$ and $d = -11$
(f) $a = -9$ and $d = 4$
(g) $a = 4\frac{1}{2}$ and $d = -\frac{1}{2}$
(h) $a = 3\frac{1}{2}$ and $d = -2$
(i) $a = 0{\cdot}9$ and $d = 0{\cdot}7$

4. Use the formula $T_n = a + (n-1)d$ to find the 11th term T_{11} of the APs in which:
(a) $a = 7$ and $d = 6$
(b) $a = 15$ and $d = -7$
(c) $a = 10\frac{1}{2}$ and $d = 4$

5. Use the formula $T_n = a + (n-1)d$ to find the eighth term T_8 of the APs in which:
(a) $a = 1$ and $d = 4$
(b) $a = 100$ and $d = -7$
(c) $a = -13$ and $d = 6$

6. (a) Find the first term a and the common difference d of the AP $6,\ 16,\ 26,\ \ldots$
(b) Find the ninth term T_9, the 21st term T_{21} and the 100th term T_{100}.
(c) Use the formula $T_n = a + (n-1)d$ to find a formula for the nth term T_n.

7. (a) Find the first term a and the common difference d of the AP $-20,\ -9,\ 2,\ \ldots$
(b) Find the eighth term T_8, the 31st term T_{31} and the 200th term T_{200}.
(c) Use the formula $T_n = a + (n-1)d$ to find a formula for the nth term T_n.

8. (a) Find the first term a and the common difference d of the AP $300,\ 260,\ 220,\ \ldots$
(b) Find the seventh term T_7, the 51st term T_{51} and the 1000th term T_{1000}.
(c) Use the formula $T_n = a + (n-1)d$ to find a formula for the nth term T_n.

DEVELOPMENT

9. Find $T_3 - T_2$ and $T_2 - T_1$ to test whether each sequence is an AP. If the sequence is an AP, use the formula $T_n = a + (n-1)d$ to find a formula for the nth term T_n.
(a) $8,\ 11,\ 14,\ \ldots$
(b) $21,\ 15,\ 9,\ \ldots$
(c) $8,\ 4,\ 2,\ \ldots$
(d) $-3,\ 1,\ 5,\ \ldots$
(e) $1\frac{3}{4},\ 3,\ 4\frac{1}{4},\ \ldots$
(f) $12,\ -5,\ -22,\ \ldots$
(g) $\sqrt{2},\ 2\sqrt{2},\ 3\sqrt{2},\ \ldots$
(h) $1,\ 4,\ 9,\ 16,\ \ldots$
(i) $-2\frac{1}{2},\ 1,\ 4\frac{1}{2},\ \ldots$

10. (a) Use the formula $T_n = a + (n-1)d$ to find the nth term T_n of $165,\ 160,\ 155,\ \ldots$
(b) Solve $T_n = 40$ to find the number of terms in the finite sequence $165,\ 160,\ 155,\ \ldots,\ 40$.
(c) Solve $T_n < 0$ to find the first negative term of the sequence $165,\ 160,\ 155,\ \ldots$

11. Use the formula $T_n = a + (n-1)d$ to find the number of terms in each finite sequence.
(a) $10,\ 12,\ 14,\ \ldots,\ 30$
(b) $1,\ 4,\ 7,\ \ldots,\ 100$
(c) $105,\ 100,\ 95,\ \ldots,\ 30$
(d) $100,\ 92,\ 84,\ \ldots,\ 4$
(e) $-12,\ -10\frac{1}{2},\ -9,\ \ldots,\ 0$
(f) $2,\ 5,\ 8,\ \ldots,\ 2000$

12. Find T_n for each AP. Then solve $T_n < 0$ to find the first negative term.
(a) $20,\ 17,\ 14,\ \dots$
(b) $50,\ 45,\ 40,\ \dots$
(c) $67,\ 60,\ 53,\ \dots$
(d) $82,\ 79,\ 76,\ \dots$
(e) $345,\ 337,\ 329,\ \dots$
(f) $24\frac{1}{2},\ 24,\ 23\frac{1}{2},\ \dots$

13. The nth term of an arithmetic sequence is $T_n = 7 + 4n$.
(a) Write out the first four terms, and hence find the values of a and d.
(b) Find the sum and the difference of the 50th and the 25th terms.
(c) Prove that $5T_1 + 4T_2 = T_{27}$.
(d) Which term of the sequence is 815?
(e) Find the last term less than 1000 and the first term greater than 1000.
(f) Find which terms are between 200 and 300, and how many of them there are.

14. (a) Let T_n be the sequence $8,\ 16,\ 24,\ \dots$ of positive multiples of 8.
(i) Show that the sequence is an AP, and find a formula for T_n.
(ii) Find the first term of the sequence greater than 500 and the last term less than 850.
(iii) Hence find the number of multiples of 8 between 500 and 850.
(b) Use the same steps to find the number of multiples of 11 between 1000 and 2000.
(c) Use the same steps to find the number of multiples of 7 between 800 and 2000.

15. (a) The first term of an AP is $a = 7$ and the fourth term is $T_4 = 16$. Use the formula $T_n = a+(n-1)d$ to find the common difference d. Then write down the first four terms.
(b) The first term of an AP is $a = 100$ and the sixth term is $T_6 = 10$. Find the common difference d using the formula $T_n = a+(n-1)d$. Then write down the next four terms.
(c) Find the 20th term of an AP with first term 28 and 11th term 108.
(d) Find the 100th term of an AP with first term 32 and 20th term -6.

16. Ionian Windows charges \$500 for the first window, then \$300 for each additional window.
(a) Write down the cost of 1 window, 2 windows, 3 windows, 4 windows, ...
(b) Show that this is an AP, and write down the first term a and common difference d.
(c) Use the formula $T_n = a + (n-1)d$ to find the cost of 15 windows.
(d) Use the formula $T_n = a + (n-1)d$ to find a formula for the cost of n windows.
(e) Form an inequation and solve it to find the maximum number of windows whose total cost is less than \$10 000.

17. Many years ago, 160 km of a railway line from Nevermore to Gindarinda was built. On 1st January 2001, work was resumed, with 20 km of new track completed each month.
(a) Write down the lengths of track 1 month later, 2 months later, 3 months later, ...
(b) Show that this is an AP, and write down the first term a and common difference d.
(c) Use the formula $T_n = a + (n-1)d$ to find how much track there was after 12 months.
(d) Use the formula $T_n = a + (n-1)d$ to find a formula for the length after n months.
(e) The distance from Nevermore to Gindarinda is 540 km. Form an equation and solve it to find how many months it took to complete the track.

CHALLENGE

18. Find the common difference of each AP. Then find x if $T_{11} = 36$.
(a) $5x - 9,\ 5x - 5,\ 5x - 1,\ \dots$
(b) $16,\ 16 + 6x,\ 16 + 12x,\ \dots$

19. Find the common difference of each AP. Then find a formula for the nth term T_n.
(a) $\log_3 2,\ \log_3 4,\ \log_3 8,\ \dots$
(b) $\log_a 54,\ \log_a 18,\ \log_a 6,\ \dots$
(c) $x - 3y,\ 2x + y,\ 3x + 5y,\ \dots$
(d) $5 - 6\sqrt{5},\ 1 + \sqrt{5},\ -3 + 8\sqrt{5},\ \dots$
(e) $1{\cdot}36,\ -0{\cdot}52,\ -2{\cdot}4,\ \dots$
(f) $\log_a 3x^2,\ \log_a 3x,\ \log_a 3,\ \dots$

8 C Geometric Sequences

A *geometric sequence* is a sequence like this:

$$2,\ 6,\ 18,\ 54,\ 162,\ 486,\ 1458,\ \ldots$$

in which the *ratio* of successive terms is constant — in this example, each term is 3 times the previous term. Because the ratio is constant, all the terms can be generated from the *first term* 2 by repeated multiplication by this *common ratio* 3.

Definition of a Geometric Sequence: The old name was 'geometric progession' and so geometric sequences are called GPs for short.

3 DEFINITION OF A GP: A sequence T_n is called a *geometric sequence* if

$$\frac{T_n}{T_{n-1}} = r, \quad \text{for } n \ge 2,$$

where r is a non-zero constant, called the *common ratio*.

Thus arithmetic sequences have a common difference and geometric sequences have a common ratio, so the methods of dealing with them are quite similar.

WORKED EXERCISE:
Test whether each sequence is a GP. If the sequence is a GP, find its first term a and its ratio r.

(a) $40,\ 20,\ 10,\ 5,\ \ldots$ (b) $5,\ 10,\ 100,\ 200,\ \ldots$

SOLUTION:

(a) Here $\dfrac{T_2}{T_1} = \dfrac{20}{40} = \frac{1}{2}$ and $\dfrac{T_3}{T_2} = \dfrac{10}{20} = \frac{1}{2}$ and $\dfrac{T_4}{T_3} = \dfrac{5}{10} = \frac{1}{2}$,

so the sequence is a GP with $a = 40$ and $r = \frac{1}{2}$.

(b) Here $\dfrac{T_2}{T_1} = \dfrac{10}{5} = 2$ and $\dfrac{T_3}{T_2} = \dfrac{100}{10} = 10$ and $\dfrac{T_4}{T_3} = \dfrac{200}{100} = 2$.

Since the ratios are not all the same, the sequence is not a GP.

A Formula for the nth Term of a GP: Let the first term of a GP be a and the common ratio be r. Then the first few terms of the sequence are

$$T_1 = a, \quad T_2 = ar, \quad T_3 = ar^2, \quad T_4 = ar^3, \quad T_5 = ar^4, \quad \ldots$$

From this pattern, the general formula for the nth term is clear:

4 THE nTH TERM OF A GP:

$$T_n = ar^{n-1}$$

where a is the first term and r is the common ratio.

WORKED EXERCISE:
Write out the first five terms, and calculate the 10th term, of the GP with:
(a) $a = 3$ and $r = 2$, (b) $a = 7$ and $r = 10$.

SOLUTION:
(a) 3, 6, 12, 24, 48, ...
$$T_{10} = ar^9 = 3 \times 2^9 = 1536$$
(b) 7, 70, 700, 7000, 70 000, ...
$$T_{10} = a \times r^9 = 7 \times 10^9 = 7\,000\,000\,000$$

Negative Ratios and Alternating Signs: The sequence

$$2,\ -6,\ 18,\ -54,\ \ldots$$

is an important type of GP. Its ratio is $r = -3$, which is negative, and so the terms are alternately positive and negative.

WORKED EXERCISE:
(a) Show that 2, −6, 18, −54, ... is a GP and find its first term a and ratio r.
(b) Find a formula for the nth term, and hence find T_6 and T_{15}.

SOLUTION:
(a) Here $\dfrac{T_2}{T_1} = \dfrac{-6}{2} = -3$ and $\dfrac{T_3}{T_2} = \dfrac{18}{-6} = -3$ and $\dfrac{T_4}{T_3} = \dfrac{-54}{18} = -3$

so the sequence is a GP with $a = 2$ and $r = -3$.

(b) Using the formula for the nth term,
$$T_n = ar^{n-1} = 2 \times (-3)^{n-1}.$$
Hence $T_6 = 2 \times (-3)^5 = -486$ and $T_{15} = 2 \times (-3)^{14} = 2 \times 3^{14}$, since 14 is even.

Solving Problems Involving GPs: As with APs, the formula for the nth term allows many problems to be solved by forming an equation and solving it.

WORKED EXERCISE:
(a) Find a formula for the nth term of the GP 5, 10, 20, ...
(b) Hence find whether 320 and 720 are terms of this sequence.

SOLUTION:
(a) The sequence is a GP with $a = 5$ and $r = 2$.
Hence $T_n = ar^{n-1} = 5 \times 2^{n-1}$.

(b) Put $T_n = 320$.
Then $5 \times 2^{n-1} = 320$
$2^{n-1} = 64$
$n - 1 = 6$
$n = 7$,
so 320 is the seventh term T_7.

(c) Put $T_n = 720$.
Then $5 \times 2^{n-1} = 720$
$2^{n-1} = 144$.
But 144 is not a power of 2,
so 720 is not a term of the sequence.

WORKED EXERCISE:
The first term of a GP is 448 and the seventh term is 7. Find the common ratio and write out the first seven terms.

SOLUTION:

First, we know that $T_1 = 448$

that is, $a = 448.$ (1)

Secondly, we know that $T_7 = 7$

so using the formula for the 7th term, $ar^6 = 7.$ (2)

Substituting (1) into (2), $448\,r^6 = 7$

$$r^6 = \tfrac{1}{64}$$

$$r = \tfrac{1}{2} \text{ or } -\tfrac{1}{2}.$$

Thus either the ratio is $r = \frac{1}{2}$, and the sequence is

$$448,\ 224,\ 112,\ 56,\ 28,\ 14,\ 7,\ \ldots$$

or else the ratio is $r = -\frac{1}{2}$, and the sequence is

$$448,\ -224,\ 112,\ -56,\ 28,\ -14,\ 7,\ \ldots$$

Exercise 8C

1. Write out the next three terms of each sequence. They are all GPs.

(a) $1,\ 2,\ 4,\ \ldots$
(b) $81,\ 27,\ 9,\ \ldots$
(c) $-7,\ -14,\ -28,\ \ldots$
(d) $-2500,\ -500,\ -100, \ldots$
(e) $3,\ -6,\ 12,\ \ldots$
(f) $-25,\ 50,\ -100,\ \ldots$
(g) $5,\ -5,\ 5,\ \ldots$
(h) $-1000,\ 100,\ -10,\ \ldots$
(i) $0{\cdot}04,\ 0{\cdot}4,\ 4,\ \ldots$

2. Find the ratios $\dfrac{T_3}{T_2}$ and $\dfrac{T_2}{T_1}$ for each sequence to test whether it is a GP. If the sequence is a GP, write down the first term a and the common ratio r.

(a) $4,\ 8,\ 16,\ \ldots$
(b) $16,\ 8,\ 4,\ \ldots$
(c) $7,\ 21,\ 63,\ \ldots$
(d) $-4,\ -20,\ -100,\ \ldots$
(e) $2,\ 4,\ 6,\ \ldots$
(f) $-1000,\ -100,\ -10,\ \ldots$
(g) $-80,\ 40,\ -20,\ \ldots$
(h) $29,\ 29,\ 29,\ \ldots$
(i) $1,\ 4,\ 9,\ \ldots$
(j) $-14,\ 14,\ -14,\ \ldots$
(k) $6,\ 1,\ \frac{1}{6},\ \ldots$
(l) $-\frac{1}{3},\ 1,\ -3,\ \ldots$

3. Write out the first four terms of the GPs whose first terms and common ratios are:

(a) $a = 1$ and $r = 3$
(b) $a = 12$ and $r = 2$
(c) $a = 5$ and $r = -2$
(d) $a = 18$ and $r = \frac{1}{3}$
(e) $a = 18$ and $r = -\frac{1}{3}$
(f) $a = 50$ and $r = \frac{1}{5}$
(g) $a = 6$ and $r = -\frac{1}{2}$
(h) $a = -13$ and $r = 2$
(i) $a = -7$ and $r = -1$

4. Use the formula $T_n = ar^{n-1}$ to find the fourth term of the GP with:

(a) $a = 5$ and $r = 2$
(b) $a = 300$ and $r = \frac{1}{10}$
(c) $a = -7$ and $r = 2$
(d) $a = -64$ and $r = \frac{1}{2}$
(e) $a = 11$ and $r = -2$
(f) $a = -15$ and $r = -2$

5. Use the formula $T_n = ar^{n-1}$ to find an expression for the 70th term of the GP with:

(a) $a = 1$ and $r = 3$
(b) $a = 5$ and $r = 7$
(c) $a = 8$ and $r = -3$

6. (a) Find the first term a and the common ratio r of the GP $7,\ 14,\ 28,\ \ldots$
(b) Find the sixth term T_6 and an expression for the 50th term T_{50}.
(c) Find a formula for the nth term T_n.

7. (a) Find the first term a and the common ratio r of the GP $10,\ -30,\ 90,\ \dots$
(b) Find the sixth term T_6 and an expression for the 25th term T_{25}.
(c) Find a formula for the nth term T_n.

8. (a) Find the first term a and the common ratio r of the GP $-80,\ -40,\ -20,\ \dots$
(b) Find the 10th term T_{10} and an expression for the 100th term T_{100}.
(c) Find a formula for the nth term T_n.

DEVELOPMENT

9. Find $\dfrac{T_3}{T_2}$ and $\dfrac{T_2}{T_1}$ to test whether each sequence is a GP. If the sequence is a GP, use the formula $T_n = ar^{n-1}$ to find a formula for the nth term, then find T_6.
(a) $10,\ 20,\ 40,\ \dots$
(b) $180,\ 60,\ 20,\ \dots$
(c) $64,\ 81,\ 100,\ \dots$
(d) $35,\ 50,\ 65,\ \dots$
(e) $\frac{3}{4},\ 3,\ 12,\ \dots$
(f) $-48,\ -24,\ -12,\ \dots$

10. Find the common ratio of each GP, find a formula for T_n, and find T_6.
(a) $1,\ -1,\ 1,\ \dots$
(b) $-2,\ 4,\ -8,\ \dots$
(c) $-8,\ 24,\ -72,\ \dots$
(d) $60,\ -30,\ 15,\ \dots$
(e) $-1024,\ 512,\ -256,\ \dots$
(f) $\frac{1}{16},\ -\frac{3}{8},\ \frac{9}{4},\ \dots$

11. Use the formula $T_n = ar^{n-1}$ to find how many terms there are in each finite sequence.
(a) $1,\ 2,\ 4,\ \dots,\ 64$
(b) $-1,\ -3,\ -9,\ \dots,\ -81$
(c) $8,\ 40,\ 200,\ \dots,\ 125\,000$
(d) $7,\ 14,\ 28,\ \dots,\ 224$
(e) $2,\ 14,\ 98,\ \dots,\ 4802$
(f) $\frac{1}{25},\ \frac{1}{5},\ 1,\ \dots,\ 625$

12. (a) The first term of a GP is $a = 25$ and the fourth term is $T_4 = 200$. Use the formula $T_n = ar^{n-1}$ to find the common ratio r, then write down the first five terms.
(b) The first term of a GP is $a = 3$ and the sixth term is $T_6 = 96$. Use the formula $T_n = ar^{n-1}$ to find the common ratio r, then write down the first six terms.
(c) The first term of a GP is $a = 1$ and the fifth term is $T_5 = 81$. Use the formula $T_n = ar^{n-1}$ to find the common ratio r, then write down the first five terms.

13. Use the formula $T_n = ar^{n-1}$ to find the common ratio r of a GP for which:
(a) $a = 486$ and $T_5 = \frac{2}{27}$
(b) $a = 1000$ and $T_7 = 0{\cdot}001$
(c) $a = 32$ and $T_6 = -243$
(d) $a = 5$ and $T_7 = 40$

14. The nth term of a geometric sequence is $T_n = 25 \times 2^n$.
(a) Write out the first six terms and hence find the values of a and r.
(b) Which term of the sequence is 6400?
(c) Find in factored form $T_{50} \times T_{25}$ and $T_{50} \div T_{25}$.
(d) Prove that $T_9 \times T_{11} = 25 \times T_{20}$.
(e) Write out the terms between 1000 and 100 000. How how many of them are there?
(f) Verify by calculations that $T_{11} = 51\,200$ is the last term less than 100 000 and that $T_{12} = 102\,400$ is the first term greater than 100 000.

15. A piece of paper 0·1 mm thick is folded successively 100 times. How thick is it now?

CHALLENGE

16. Find the nth term of each GP.
(a) $\sqrt{6},\ 2\sqrt{3},\ 2\sqrt{6},\ \dots$
(b) $ax,\ a^2x^3,\ a^3x^5,\ \dots$
(c) $-\dfrac{x}{y},\ -1,\ -\dfrac{y}{x},\ \dots$

17. (a) Find a formula for T_n in $2x,\ 2x^2,\ 2x^3,\ \dots$ Then find x if $T_6 = 2$.
(b) Find a formula for T_n in $x^4,\ x^2,\ 1,\ \dots$ Then find x if $T_6 = 3^6$.
(c) Find a formula for T_n in $2^{-16}x,\ 2^{-12}x,\ 2^{-8}x,\ \dots$ Then find x if $T_6 = 96$.

8 D Solving Problems about APs and GPs

This section deals with APs and GPs together and presents some further approaches to problems about the terms of APs and GPs.

A Condition for Three Numbers to be in AP or GP: The three numbers 10, 25, 40 form an AP because the differences $25 - 10 = 15$ and $40 - 25 = 15$ are equal.

Similarly, 10, 20, 40 form a GP because the ratios $\frac{20}{10} = 2$ and $\frac{40}{20} = 2$ are equal.

These situations occur quite often and a formal statement is worthwhile:

5

THREE NUMBERS IN AP: Three numbers a, b and c form an AP if

$$b - a = c - b$$

THREE NUMBERS IN GP: Three numbers a, b and c form a GP if

$$\frac{b}{a} = \frac{c}{b}$$

WORKED EXERCISE:

(a) Find the value of x if $3,\ x,\ 12$ forms an AP.

(b) Find the value of x if $3,\ x,\ 12$ forms a GP.

SOLUTION:

(a) Since $3,\ x,\ 12$ forms an AP,

$$\begin{aligned} x - 3 &= 12 - x \\ 2x &= 15 \\ x &= 7\tfrac{1}{2}. \end{aligned}$$

(b) Since $3,\ x,\ 12$ forms a GP,

$$\begin{aligned} \frac{x}{3} &= \frac{12}{x} \\ x^2 &= 36 \\ x &= 6 \text{ or } -6. \end{aligned}$$

Solving Problems Leading to Simultaneous Equations: Many problems about APs and GPs lead to simultaneous equations. These are best solved by elimination.

6

PROBLEMS ON APS AND GPS LEADING TO SIMULTANEOUS EQUATIONS:

- With APs, eliminate a by subtracting one equation from the other.
- With GPs, eliminate a by dividing one equation by the other.

WORKED EXERCISE:

The third term of an AP is 16 and the 12th term is 79. Find the 41st term.

SOLUTION:

Let the first term be a and the common difference be d.

Since $T_3 = 16$, $\quad a + 2d = 16,$ (1)

and since $T_{12} = 79$, $\quad a + 11d = 79.$ (2)

Subtracting (1) from (2), $\quad 9d = 63$ (This is the key step; it eliminates a.)

$d = 7.$

Substituting into (1), $\quad a + 14 = 16$

$a = 2.$

Hence $\quad T_{41} = a + 40d$

$= 282.$

WORKED EXERCISE:
Find the first term a and the common ratio r of a GP in which the fourth term is 6 and the seventh term is 162.

SOLUTION:

Since $T_4 = 6$, $ar^3 = 6$, (1)
and since $T_7 = 162$, $ar^6 = 162$. (2)
Dividing (2) by (1), $r^3 = 27$ (This is the key step; it eliminates a.)
$r = 3$.
Substituting into (1), $a \times 27 = 6$
$a = \frac{2}{9}$.

Solving GP Problems Involving Trial-and-Error or Logarithms: Equations and inequations involving the terms of a GP are index equations, and so logarithms are needed for a systematic approach.

Trial-and-error, however, is quite satisfactory for simpler problems, and the reader may prefer to leave the application of logarithms until the Year 12 volume.

WORKED EXERCISE:

(a) Find a formula for the nth term of the GP 2, 6, 18, ...
(b) Use trial-and-error to find the first term greater than 1 000 000.
(c) Use logarithms to find the first term greater than 1 000 000.

SOLUTION:

(a) This is a GP with $a = 2$ and $r = 3$,
so $T_n = ar^{n-1}$
$= 2 \times 3^{n-1}$.

(b) Put $T_n > 1\,000\,000$.
Using the calculator, $T_{12} = 354\,294$
and $T_{13} = 1\,062\,882$.
Hence the first term over 1 000 000 is $T_{13} = 1\,062\,882$.

(c) Put $T_n > 1\,000\,000$.
Then $2 \times 3^{n-1} > 1\,000\,000$
$3^{n-1} > 500\,000$
$n - 1 > \log_3 500\,000$ (Remember that $2^3 = 8$ means $3 = \log_2 8$.)
$n - 1 > \dfrac{\log_{10} 500\,000}{\log_{10} 3}$ (This is the change-of-base formula.)
$n - 1 > 11{\cdot}94\ldots$
$n > 12{\cdot}94\ldots$
Hence the first term over 1 000 000 is $T_{13} = 1\,062\,882$.

Exercise 8D

1. Find the value of x if each set of numbers below forms an arithmetic sequence.
[HINT: Form an equation using the identity $T_2 - T_1 = T_3 - T_2$, then solve it to find x.]

(a) $5,\ x,\ 17$
(b) $32,\ x,\ 14$
(c) $-12,\ x,\ -50$
(d) $-23,\ x,\ 7$
(e) $x,\ 22,\ 32$
(f) $-20,\ -5,\ x$

2. Find the value of x if each of the following sets of numbers forms a geometric sequence. [HINT: Form an equation using the identity $\dfrac{T_2}{T_1} = \dfrac{T_3}{T_2}$, then solve it to find x.]

(a) $2,\ x,\ 18$
(b) $48,\ x,\ 3$
(c) $-10,\ x,\ -90$
(d) $-98,\ x,\ -2$
(e) $x,\ 20,\ 80$
(f) $-1,\ 4,\ x$

3. Find x if each set of three numbers forms: (i) an AP, (ii) a GP.

(a) $4,\ x,\ 16$
(b) $1,\ x,\ 49$
(c) $16,\ x,\ 25$
(d) $-5,\ x,\ -20$
(e) $x,\ 10,\ 50$
(f) $x,\ 12,\ 24$
(g) $x,\ -1,\ 1$
(h) $x,\ 6,\ -12$
(i) $20,\ 30,\ x$
(j) $-36,\ 24,\ x$
(k) $-\frac{1}{4},\ -3,\ x$
(l) $7,\ -7,\ x$

DEVELOPMENT

4. In these questions, substitute the last term into $T_n = a + (n-1)d$ or $T_n = ar^{n-1}$.

(a) Find the first six terms of the AP with first term $a = 7$ and sixth term $T_6 = 42$.
(b) Find the first four terms of the GP with first term $a = 27$ and fourth term $T_4 = 8$.
(c) Find the first eleven terms of the AP with $a = 40$ and $T_{11} = 5$.
(d) Find the first seven terms of the GP with $a = 1$ and $T_7 = 1\,000\,000$.
(e) Find the first five terms of the AP with $a = 3$ and $T_5 = 48$.
(f) Find the first five terms of the GP with $a = 3$ and $T_5 = 48$.

5. Use simultaneous equations and the formula $T_n = a + (n-1)d$ to solve these problems.

(a) Find the first term and common difference of the AP with $T_{10} = 18$ and $T_{20} = 48$.
(b) Find the first term and common difference of the AP with $T_2 = 3$ and $T_{10} = 35$.
(c) Find the first term and common difference of the AP with $T_5 = 24$ and $T_9 = -12$.
(d) Find the first term and common difference of the AP with $T_4 = 6$ and $T_{12} = 34$.

6. Use simultaneous equations and the formula $T_n = ar^{n-1}$ to solve these problems.

(a) Find the first term and common ratio of the GP with $T_3 = 16$ and $T_6 = 128$.
(b) Find the first term and common ratio of the GP with $T_3 = 1$ and $T_6 = 64$.
(c) Find the first term and common ratio of the GP with $T_2 = \frac{1}{3}$ and $T_6 = 27$.
(d) Find the first term and common ratio of the GP with $T_5 = 6$ and $T_9 = 24$.

7. (a) The third term of an AP is 7 and the seventh term is 31. Find the eighth term.
(b) The common difference of an AP is -7 and the 10th term is 3. Find the second term.
(c) The common ratio of a GP is 2 and the sixth term is 6. Find the second term.

8. Use either trial-and-error or logarithms to solve these problems.

(a) Find the smallest value of n such that $3^n > 1\,000\,000$.
(b) Find the largest value of n such that $5^n < 1\,000\,000$.
(c) Find the smallest value of n such that $7^n > 1\,000\,000\,000$.
(d) Find the largest value of n such that $12^n < 1\,000\,000\,000$.

9. Let T_n be the sequence $2,\ 4,\ 8,\ \dots$ of powers of 2.

(a) Show that the sequence is a GP, and show that the nth term is $T_n = 2^n$.
(b) Find how many terms are less than $1\,000\,000$. [HINT: You will need to solve the inequation $T_n < 1\,000\,000$ using trial-and-error or logarithms.]
(c) Use the same method to find how many terms are less than $1\,000\,000\,000$.
(d) Use the same method to find how many terms are less than 10^{20}.

(e) How many terms are between 1 000 000 and 1 000 000 000?

(f) How many terms are between 1 000 000 000 and 10^{20}?

10. Find a formula for T_n for these GPs. Then find how many terms exceed 10^{-6}. [HINT: You will need to solve the inequation $T_n > 10^{-6}$ using trial-and-error or logarithms.]

(a) 98, 14, 2, ... (b) 25, 5, 1, ... (c) 1, 0·9, 0·81, ...

11. When light passes through one sheet of very thin glass, its intensity is reduced by 3%. What is the minimum number of sheets that will reduce the intensity of the light below 1%? [HINT: 97% of the light gets though each sheet.]

12. (a) Find a and d for the AP in which $T_6 + T_8 = 44$ and $T_{10} + T_{13} = 35$.

(b) Find a and r for the GP in which $T_2 + T_3 = 4$ and $T_4 + T_5 = 36$.

(c) The fourth, sixth and eighth terms of an AP add to -6. Find the sixth term.

13. Each set of three numbers forms an AP. Find x and write out the numbers.

(a) $x - 1,\ 17,\ x + 15$ (c) $x - 3,\ 5,\ 2x + 7$

(b) $2x + 2,\ x - 4,\ 5x$ (d) $3x - 2,\ x,\ x + 10$

14. Each set of three numbers forms a GP. Find x and write out the numbers.

(a) $x,\ x + 1,\ x$ (b) $2 - x,\ 2,\ 5 - x$

15. Find x and write out the three numbers if they form: (i) an AP, (ii) a GP.

(a) $x,\ 24,\ 96$ (d) $x - 4,\ x + 1,\ x + 11$ (g) $\sqrt{2},\ x,\ \sqrt{8}$

(b) $0{\cdot}2,\ x,\ 0{\cdot}000\,02$. (e) $x - 2,\ x + 2,\ 5x - 2$ (h) $2^4,\ x,\ 2^6$

(c) $x,\ 0{\cdot}2,\ 0{\cdot}002$. (f) $\sqrt{5} + 1,\ x,\ \sqrt{5} - 1$ (i) $7,\ x,\ -7$

CHALLENGE

16. (a) Find a and b if $a,\ b,\ 1$ forms a GP, and $b,\ a,\ 10$ forms an AP.

(b) Find a and b if $a,\ 1,\ a + b$ forms a GP, and $b,\ \frac{1}{2},\ a - b$ forms an AP.

(c) The first, second and fourth terms of an AP $T_n = a + (n - 1)d$ form a geometric sequence. Show that either $d = 0$ or $d = a$.

(d) The first, second and fifth terms of an AP $T_n = a + (n - 1)d$ form a geometric sequence. Show that either $d = 0$ or $d = 2a$.

17. (a) Show that $2^5,\ 2^2,\ 2^{-1},\ 2^{-4},\ \ldots$ is a GP. Then find its nth term.

(b) Show that $\log_2 96,\ \log_2 24,\ \log_2 6,\ \ldots$ is an AP. Then show that $T_n = 7 - 2n + \log_2 3$.

18. [Extension — geometric sequences in musical intruments] The pipe lengths in a rank of organ pipes decrease from left to right. The lengths form a GP, and the 13th pipe along is exactly half the length of the first pipe (making an interval called an *octave*).

(a) Show that the ratio of the GP is $r = (\frac{1}{2})^{\frac{1}{12}}$.

(b) Show that the eighth pipe along is just over two-thirds the length of the first pipe (this interval is called a *perfect fifth*).

(c) Show that the fifth pipe along is just under four-fifths the length of the first pipe (a *major third*).

(d) Find which pipes are about three-quarters (a *perfect fourth*) and five-sixths (a *minor third*) the length of the first pipe.

(e) What simple fractions are closest to the relative lengths of the third pipe (a *major second*) and the second pipe (a *minor second*)?

8 E Adding Up the Terms of a Sequence

It is often important to add the terms of a sequence. For example, a boulder falling from the top of a high cliff falls 5 metres in the first second, 15 metres in the second second, 25 metres in the third second, and so on. The distance that it falls in the first 10 seconds is the sum of the 10 numbers

$$5 + 15 + 25 + 35 + \cdots + 95\,.$$

A Notation for the Sums of Terms of a Sequence: For any sequence $T_1,\ T_2,\ T_3\ \ldots,$ define S_n to be the sum of the first n terms of the sequence.

7 The sum of the first n terms of a sequence:

Given a sequence $T_1,\ T_2,\ T_3,\ \ldots,$ define

$$S_n = T_1 + T_2 + T_3 + \cdots + T_n\,.$$

The sum S_n is called the *sum of the first n terms* of the sequence. Alternatively, S_n is called the *the sum to n terms* of the sequence.

For example, the sum of the first 10 terms of the sequence $5,\ 15,\ 25,\ 35,\ \ldots$ is

$$\begin{aligned} S_{10} &= 5 + 15 + 25 + 35 + 45 + 55 + 65 + 75 + 85 + 95 \\ &= 500 \end{aligned}$$

The Sequence $S_1,\ S_2,\ S_3,\ S_4,\ \ldots$ of Sums: The sums $S_1,\ S_2,\ S_3,\ S_4,\ \ldots$ form another sequence. For example, with the sequence $5,\ 15,\ 25,\ 35,\ \ldots,$

$$\begin{aligned} S_1 &= 5 \end{aligned} \qquad \begin{aligned} S_2 &= 5 + 15 \\ &= 20 \end{aligned} \qquad \begin{aligned} S_3 &= 5 + 15 + 25 \\ &= 45 \end{aligned} \qquad \begin{aligned} S_4 &= 5 + 15 + 25 + 35 \\ &= 80 \end{aligned}$$

Worked Exercise:

Copy and complete the following table for the successive sums of a sequence.

T_n	5	15	25	35	45	55	65	75	85	95
S_n										

Solution:

Each entry for S_n is the sum of all the terms T_n up to that point.

T_n	5	15	25	35	45	55	65	75	85	95
S_n	5	20	45	80	125	180	245	320	405	500

Worked Exercise:

By taking successive differences, list the terms of the original sequence.

T_n										
S_n	1	5	12	22	35	51	70	92	117	145

Solution:

Each entry for T_n is the difference between two successive sums S_n.

T_n	1	4	7	10	13	16	19	22	25	28
S_n	1	5	12	22	35	51	70	92	117	145

Sigma Notation: This is a concise notation for the sums of a sequence. For example:

$$\sum_{n=1}^{5} n^2 = 1 + 4 + 9 + 16 + 25 \qquad \sum_{n=6}^{10} n^2 = 36 + 49 + 64 + 81 + 100$$
$$= 55 \qquad\qquad\qquad = 330$$

The first sum says 'evaluate the function n^2 for all the integers from $n = 1$ to $n = 5$, then add up the resulting values'. The final answer is 55.

8 SIGMA NOTATION: Suppose that $T_1, T_2, T_3, \ldots$ is a sequence. Then

$$\sum_{n=5}^{20} T_n = T_5 + T_6 + T_7 + T_8 + \cdots + T_{20}$$

(Any two integers can obviously be substituted for the numbers 5 and 20.)

The symbol $\sum$ used here is a large version of the Greek capital letter Σ called 'sigma' and pronounced 's'. It stands for the word 'sum'.

WORKED EXERCISE:

Evaluate the following sums.

(a) $\displaystyle\sum^{7}(5n+1)$ (b) $\displaystyle\sum^{5}(-2)^n$

SOLUTION:

(a) $\displaystyle\sum_{n=4}^{7}(5n+1) = 21 + 26 + 31 + 36$
$= 114$

(b) $\displaystyle\sum_{n=1}^{5}(-2)^n = -2 + 4 - 8 + 16 - 32$
$= -22$

Series: The word *series* is a rather imprecise term, but it always refers to the activity of adding up terms of a sequence. For example, the notation

'the series $1 + 4 + 9 + \cdots$'

means that one is considering the successive sums $S_1, S_2, S_3, \ldots$ of the sequence of positive squares. The words 'series' and 'sequence' tend to be used interchangeably.

Exercise 8E

1. Find the sum S_4 of the first four terms of each sequence.
(a) $3, 5, 7, 9, 11, 13, \ldots$
(b) $2, 6, 18, 54, 162, 486, \ldots$
(c) $6, 2, -2, -6, -10, -14, \ldots$
(d) $2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

2. Find the sum S_3 of the first three terms of each series.
(a) $200 + 150 + 100 + 50 + 0 + \cdots$
(b) $32 - 16 + 8 - 4 + 2 - 1 + \cdots$
(c) $-24 - 18 - 12 - 6 + 0 + 6 + \cdots$
(d) $5{\cdot}1 + 5{\cdot}2 + 5{\cdot}3 + 5{\cdot}4 + 5{\cdot}5 + 5{\cdot}6 + \cdots$

3. Find the sums S_1, S_2, S_3, S_4 and S_5 for each sequence.
(a) $10, 20\ 30, 40, 50, 60, \ldots$
(b) $1, -3, 9, -27, 81, -243, \ldots$
(c) $1, 4, 9, 16, 25, 36, \ldots$
(d) $3, 4\frac{1}{2}, 6, 7\frac{1}{2}, 9, 10\frac{1}{2}, 12, \ldots$

4. Find the sums S_4, S_5 and S_6 for each series. (You will need to continue each series first.)

(a) $1 - 2 + 3 - 4 + \cdots$

(b) $81 + 27 + 9 + 3 + \cdots$

(c) $30 + 20 + 10 + \cdots$

(d) $0{\cdot}1 + 0{\cdot}01 + 0{\cdot}001 + 0{\cdot}0001 + \cdots$

5. Copy and complete these tables of a sequence and its sums.

(a)

T_n	2	5	8	11	14	17	20
S_n							

(b)

T_n	40	38	36	34	32	30	28
S_n							

(c)

T_n	2	-4	6	-8	10	-12	14
S_n							

(d)

T_n	7	-7	7	-7	7	-7	7
S_n							

DEVELOPMENT

6. Each of the following tables gives the sums S_1, S_2, S_3, ... of a sequence. By taking successive differences, write out the terms of the original sequence.

(a)

T_n							
S_n	1	4	9	16	25	36	49

(b)

T_n							
S_n	2	6	14	30	62	126	254

(c)

T_n							
S_n	-3	-8	-15	-24	-35	-48	-63

(d)

T_n							
S_n	8	0	8	0	8	0	8

7. [The Fibonacci and Lucas sequences] Each of the following tables gives the sums S_n of a sequence. By taking successive differences, write out the terms of the original sequence.

(a)

T_n								
S_n	1	2	3	5	8	13	21	34

(b)

T_n								
S_n	3	4	7	11	18	29	47	76

8. Rewrite each sum without sigma notation, then evaluate it.

(a) $\displaystyle\sum_{n=1}^{6} 2n$

(b) $\displaystyle\sum_{n=1}^{6} (3n+2)$

(c) $\displaystyle\sum_{k=3}^{7} (18-3n)$

(d) $\displaystyle\sum_{n=5}^{8} n^2$

(e) $\displaystyle\sum_{n=1}^{4} n^3$

(f) $\displaystyle\sum_{n=0}^{5} 2^n$

(g) $\displaystyle\sum_{n=2}^{4} 3^n$

(h) $\displaystyle\sum_{\ell=1}^{31} (-1)^\ell$

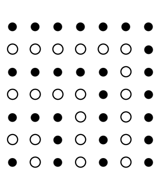
(i) $\displaystyle\sum_{\ell=1}^{40} (-1)^{\ell-1}$

(j) $\displaystyle\sum_{n=5}^{105} 4$

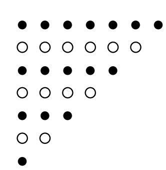
(k) $\displaystyle\sum_{n=0}^{4} (-1)^n (n+5)$

(l) $\displaystyle\sum_{n=0}^{4} (-1)^{n+1} (n+5)$

9. (a) Use the dot diagram on the right to explain why the sum of the first n odd positive integers is n^2.



(b) Use the dot diagram on the right to explain why the sum of the first n positive integers is $\frac{1}{2}n(n+1)$.



CHALLENGE

10. Rewrite each sum in sigma notation, starting each sum at $n = 1$. Do not evaluate it.

(a) $1^3 + 2^3 + 3^3 + \cdots + 40^3$

(b) $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{40}$

(c) $3 + 4 + 5 + \cdots + 22$

(d) $2 + 2^2 + 2^3 + \cdots + 2^{12}$

(e) $-1 + 2 - 3 + \cdots + 10$

(f) $1 - 2 + 3 - \cdots - 10$

8 F Summing an Arithmetic Series

There are two formulae for adding up the first n terms of an AP.

Adding the Terms of an AP: Consider adding the first six terms of the AP

$$5 + 15 + 25 + 35 + 45 + 55 + \dots$$

Writing out the sum, $S_6 = 5 + 15 + 25 + 35 + 45 + 55.$
Reversing the sum, $S_6 = 55 + 45 + 35 + 25 + 15 + 5,$
and adding the two, $2S_6 = 60 + 60 + 60 + 60 + 60 + 60$
$= 6 \times 60.$
Dividing by 2, $S_6 = \frac{1}{2} \times 6 \times 60$
$= 180.$

Notice that 60 is the sum of the first term $T_1 = 5$ and the last term $T_6 = 55$.

In general, let $\ell = T_n$ be the last term of an AP with first term a and difference d.
Then $S_n = a + (a+d) + (a+2d) + \cdots + (\ell - 2d) + (\ell - d) + \ell.$
Reversing the sum, $S_n = \ell + (\ell - d) + (\ell - 2d) + \cdots + (a+2d) + (a+d) + a,$
and adding, $2S_n = (a+\ell) + (a+\ell) + \cdots + (a+\ell) + (a+\ell) + (a+\ell)$
$= n \times (a + \ell).$ (There are n terms in the series.)
Dividing by 2, $S_n = \frac{1}{2}n(a+\ell).$

9 THE SUM OF THE FIRST n TERMS OF AN AP:

$$S_n = \tfrac{1}{2}n(a+\ell),$$

where a is the first term and $\ell = T_n$ is the last term.

WORKED EXERCISE:
Add up all the integers from 100 to 200 inclusive.

SOLUTION:
The sum $100 + 101 + \cdots + 200$ is an AP with 101 terms.
The first term is $a = 100$ and the last term is $\ell = 200$.

Using $S_n = \frac{1}{2}n(a+\ell),$

$S_{101} = \frac{1}{2} \times 101 \times (100 + 200)$
$= \frac{1}{2} \times 101 \times 300$
$= 15\,150.$

An Alternative Formula for Summing an AP: This alternative form is equally important.

The previous formula is $S_n = \frac{1}{2}n(a+\ell)$, where $\ell = T_n = a + (n-1)d.$

Substituting $\ell = a + (n-1)d$, $S_n = \frac{1}{2}n\Big(a + a + (n-1)d\Big)$

so $S_n = \frac{1}{2}n\Big(2a + (n-1)d\Big)$

10 THE TWO FORMULAE FOR SUMMING AN AP:

- When the last term $\ell = T_n$ is known, use $S_n = \frac{1}{2}n(a+\ell).$
- When the difference d is known, use $S_n = \frac{1}{2}n\Big(2a + (n-1)d\Big).$

WORKED EXERCISE:

Consider the AP $100 + 94 + 88 + 82 + \cdots$

(a) Find S_{10}.

(b) Find S_{41}.

SOLUTION:

The series is an AP with $a = 100$ and $d = -6$.

(a) Using $S_n = \frac{1}{2}n\big(2a + (n-1)d\big)$,

$$\begin{aligned} S_{10} &= \tfrac{1}{2} \times 10 \times (2a + 9d) \\ &= 5 \times (200 - 54) \\ &= 730. \end{aligned}$$

(b) Similarly,

$$\begin{aligned} S_{41} &= \tfrac{1}{2} \times 41 \times \big(2a + 40d\big) \\ &= \tfrac{1}{2} \times 41 \times \big(200 - 240)\big) \\ &= \tfrac{1}{2} \times 41 \times (-40) \\ &= -820. \end{aligned}$$

WORKED EXERCISE:

(a) Find how many terms are in the sum $41 + 45 + 49 + \cdots + 401$.

(b) Hence evaluate the sum $41 + 45 + 49 + \cdots + 401$.

SOLUTION:

(a) The series is an AP with first term $a = 41$ and difference $d = 4$.

To find the numbers of terms, put $T_n = 401$

$$\begin{aligned} a + (n-1)d &= 401 \\ 41 + 4(n-1) &= 401 \\ 4(n-1) &= 360 \\ n - 1 &= 90 \\ n &= 91. \end{aligned}$$

Thus there are 91 terms in the series.

(b) Since we now know both the difference d and the last term $\ell = T_{91}$, either formula can be used. It's always easier to use $S_n = \frac{1}{2}n(a + \ell)$ if you can.

Using $S_n = \frac{1}{2}n(a + \ell)$,

$$\begin{aligned} S_{91} &= \tfrac{1}{2} \times 91 \times (41 + 401) \\ &= \tfrac{1}{2} \times 91 \times 442 \\ &= 20\,111. \end{aligned}$$

OR

Using $S_n = \frac{1}{2}n\big(2a + (n-1)d\big)$,

$$\begin{aligned} S_{91} &= \tfrac{1}{2} \times 91 \times (2a + 90d) \\ &= \tfrac{1}{2} \times 91 \times (82 + 360) \\ &= 20\,111. \end{aligned}$$

Solving Problems Involving the Sums of APs: Problems involving sums of APs are solved using the formulae developed for the nth term T_n and the sum S_n of the first n terms.

WORKED EXERCISE:

(a) Find an expression for the sum S_n of n terms of the series $40 + 37 + 34 + \cdots$

(b) Hence find the least value of n for which the sum S_n is negative.

SOLUTION:

The sequence is an AP with $a = 40$ and $d = -3$.

(a)

$$\begin{aligned} S_n &= \tfrac{1}{2}n\big(2a + (n-1)d\big) \\ &= \tfrac{1}{2} \times n \times \big(80 - 3(n-1)\big) \\ &= \tfrac{1}{2} \times n \times (80 - 3n + 3) \\ &= \frac{n(83 - 3n)}{2} \end{aligned}$$

(b) Put $S_n < 0.$

Then $\dfrac{n(83-3n)}{2} < 0$

$\boxed{\times 2}$ $n(83-3n) < 0$

$\boxed{\div n}$ $83 - 3n < 0,$ since n is positive,

$83 < 3n$

$n > 27\frac{2}{3}.$

Hence S_{28} is the first sum that is negative.

WORKED EXERCISE:

The sum of the first 10 terms of an AP is zero, and the sum of the first and second terms is 24. Find the first three terms.

SOLUTION:

The first piece of information given is $S_{10} = 0$

$5(2a + 9d) = 0$

$\boxed{\div 5}$ $2a + 9d = 0.$ (1)

The second piece of information given is $T_1 + T_2 = 24$

$a + (a + d) = 24$

$2a + d = 24.$ (2)

Subtracting (2) from (1), $8d = -24$

$d = -3,$

and substituting this into (2), $2a - 3 = 24$

$a = 13\frac{1}{2}.$

Hence the AP is $13\frac{1}{2} + 10\frac{1}{2} + 7\frac{1}{2} + \cdots$

Exercise 8F

1. State how many terms each sum has, then find the sum using $S_n = \frac{1}{2}n(a + \ell)$.

(a) $1 + 2 + 3 + 4 + \cdots + 100$

(b) $1 + 3 + 5 + 7 + \cdots + 99$

(c) $2 + 4 + 6 + 8 + \cdots + 100$

(d) $3 + 6 + 9 + 12 + \cdots + 300$

(e) $101 + 103 + 105 + \cdots + 199$

(f) $1001 + 1002 + 1003 + \cdots + 10\,000$

2. Use $S_n = \frac{1}{2}n\big(2a + (n-1)d\big)$ to find the sum S_6 of the first 6 terms of the series with:

(a) $a = 5$ and $d = 10$

(b) $a = 8$ and $d = 2$

(c) $a = -3$ and $d = -9$

(d) $a = -7$ and $d = -12$

3. State the first term a and the difference d for each series below. Then use the formula $S_n = \frac{1}{2}n\big(2a + (n-1)d\big)$ to find the sum S_{21} of the first 21 terms of each series.

(a) $2 + 6 + 10 + \cdots$

(b) $3 + 10 + 17 + \cdots$

(c) $-6 - 1 + 4 + \cdots$

(d) $10 + 5 + 0 - \cdots$

(e) $-7 - 10 - 13 - \cdots$

(f) $1\frac{1}{2} + 3\frac{1}{2} + 5\frac{1}{2} + \cdots$

4. Use the formula $S_n = \frac{1}{2}n\big(2a + (n-1)d\big)$ to find the sum of the stated number of terms.

(a) $2 + 5 + 8 + \cdots$ (12 terms)

(b) $40 + 33 + 26 + \cdots$ (21 terms)

(c) $-6 - 2 + 2 + \cdots$ (200 terms)

(d) $33 + 30 + 27 + \cdots$ (23 terms)

(e) $-10 - 7\frac{1}{2} - 5 + \cdots$ (13 terms)

(f) $10\frac{1}{2} + 10 + 9\frac{1}{2} + \cdots$ (40 terms)

5. First use the formula $T_n = a + (n-1)d$ to find how many terms there are in each sum. Then use the formula $S_n = \frac{1}{2}n(a+\ell)$ to find the sum, where ℓ is the last term T_n.

(a) $50 + 51 + 52 + \cdots + 150$
(b) $8 + 15 + 22 + \cdots + 92$
(c) $-10 - 3 + 4 + \cdots + 60$
(d) $4 + 7 + 10 + \cdots + 301$
(e) $6\frac{1}{2} + 11 + 15\frac{1}{2} + \cdots + 51\frac{1}{2}$
(f) $-1\frac{1}{3} + \frac{1}{3} + 2 + \cdots + 13\frac{2}{3}$

6. Find these sums by any appropriate method.

(a) $2 + 4 + 6 + \cdots + 1000$
(b) $1000 + 1001 + \cdots + 3000$
(c) $1 + 5 + 9 + \cdots$ (40 terms)
(d) $10 + 30 + 50 + \cdots$ (12 terms)

DEVELOPMENT

7. Use $S_n = \frac{1}{2}n\big(2a + (n-1)d\big)$ to find and simplify the sum of the first n terms of each series.

(a) $5 + 10 + 15 + \cdots$
(b) $10 + 13 + 16 + \cdots$
(c) $3 + 7 + 11 + \cdots$
(d) $-9 - 4 + 1 + \cdots$
(e) $5 + 4\frac{1}{2} + 4 + \cdots$
(f) $(1 - \sqrt{2}\,) + 1 + (1 + \sqrt{2}) + \cdots$

8. Use either standard formula for S_n to find a formula for the sum of the first n:

(a) positive integers,
(b) odd positive integers,
(c) positive integers divisible by 3,
(d) odd positive multiples of 100.

9. (a) How many legs are there on 15 fish, 15 ducks, 15 dogs, 15 beetles, 15 spiders, and 15 ten-legged grubs? How many of these creatures have the mean number of legs?

(b) Matthew Flinders High School has 1200 pupils, with equal numbers of each age from 6 to 17 years inclusive. It also has 100 teachers and ancilliary staff, all aged 30 years, and one Principal aged 60 years. What is the total of the ages of everyone in the school?

(c) An advertising graduate earns \$28 000 per annum in her first year, then each successive year her salary rises by \$1600. What are her total earnings over 10 years?

10. By substituting appropriate values of k, find the first term a and last term ℓ of each sum. Then evaluate the sum using $S_n = \frac{1}{2}n(a+\ell)$. (Note that all four series are APs.)

(a) $\displaystyle\sum_{k=1}^{200}(600 - 2k)$
(b) $\displaystyle\sum_{k=1}^{61}(93 - 3k)$
(c) $\displaystyle\sum_{k=1}^{40}(3k - 50)$
(d) $\displaystyle\sum_{k=10}^{30}(5k + 3)$

11. Solve these questions using the formula $S_n = \frac{1}{2}n(a+\ell)$ whenever possible — otherwise use the formula $S_n = \frac{1}{2}n\big(2a + (n-1)d\big)$.

(a) Find the last term if a series with 10 terms and first term -23 has sum -5.
(b) Find the first term if a series with 40 terms and last term $8\frac{1}{2}$ has sum 28.
(c) Find the common difference if a series with 8 terms and first term 5 has sum 348.
(d) Find the first term if a series with 15 terms and difference $\frac{2}{7}$ has sum -15.

12. Beware! These questions require quadratic equations to find solutions for n.

(a) Show that the sum to n terms of the AP $60 + 52 + 44 + 36 + \cdots$ is $S_n = 4n(16 - n)$.
(b) Hence find how many terms must be taken to make the sum: (i) zero, (ii) negative.
(c) Find the two values of n for which the sum S_n is 220.
(d) Show that $S_n = -144$ has two integer solutions, but that only one has meaning.
(e) For what values of n does the sum S_n exceed 156?
(f) Prove that no sum S_n can exceed 256.
(g) Write out the first 16 terms and sums, and check your results.

13. First use the formula $S_n = \frac{1}{2}n\big(2a+(n-1)d\big)$ to find the sum S_n for each arithmetic series. Then use quadratic equations to find the number of terms if S_n has the given value.

(a) $42 + 40 + 38 + \cdots$, $S_n = 0$

(b) $60 + 57 + 54 + \cdots$, $S_n = 0$

(c) $45 + 51 + 57 + \cdots$, $S_n = 153$

(d) $2\frac{1}{2} + 3 + 3\frac{1}{2} + \cdots$, $S_n = 22\frac{1}{2}$

CHALLENGE

14. (a) Logs of wood are stacked with 10 on the top row, 11 on the next, and so on. If there are 390 logs, find the number of rows, and the number of logs on the bottom row.

(b) A stone dropped from the top of a 245-metre cliff falls 5 metres in the first second, 15 metres in the second second, and so on in arithmetic sequence. Find a formula for the distance after n seconds, and find how long the stone takes to fall to the ground.

(c) A truck spends several days depositing truckloads of gravel from a quarry at equally spaced intervals along a straight road. The first load is deposited 20 km from the quarry, the last is 10 km further along the road. If the truck travels 550 km during the day, how many trips does it make, and how far apart are the deposits?

15. (a) The sum of the first and fourth terms of an AP is 16, and the sum of the third and eighth terms is 4. Find the sum of the first 10 terms.

(b) The sum of the first 10 terms of an AP is zero, and the 10th term is -9. Find the first and second terms.

(c) The sum to 16 terms of an AP is 96, and the sum of the second and fourth terms is 45. Find the fourth term, and show that the sum to four terms is also 96.

16. Find the sums of these APs, whose terms are logarithms.

(a) $\log_a 2 + \log_a 4 + \log_a 8 + \cdots + \log_a 1024$

(b) $\log_5 243 + \log_5 81 + \log_5 27 + \cdots + \log_5 \frac{1}{243}$

(c) $\log_b 36 + \log_b 18 + \log_b 9 + \cdots + \log_b \frac{9}{8}$

(d) $\log_x \frac{27}{8} + \log_x \frac{9}{4} + \log_x \frac{3}{2} + \cdots$ (10 terms)

8 G Summing a Geometric Series

There is also a simple formula for finding the sum of the first n terms of a GP. The approach, however, is quite different from the approach used for APs.

Adding up the Terms of a GP: This method is easier to understand with a general GP. Let us find the sum S_n of the first n terms of the GP $a + ar + ar^2 + \cdots$

Writing out the sum, $\quad S_n = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1}$. (1)

Multiplying both sides by r, $\quad rS_n = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n$. (2)

Subtracting (1) from (2), $\quad (r-1)S_n = ar^n - a$.

Then provided that $r \neq 1$, $\quad S_n = \dfrac{a(r^n - 1)}{r - 1}$.

This formula will always work, but if $r < 1$, there is a more convenient form. Taking opposites of numerator and denominator,

$$S_n = \frac{a(1 - r^n)}{1 - r}.$$

Method for Summing a GP: Thus again there are two forms to remember:

11 THE TWO FORMULAE FOR SUMMING A GP:

When $r > 1$, use the formula $S_n = \dfrac{a(r^n - 1)}{r - 1}$.

When $r < 1$, use the formula $S_n = \dfrac{a(1 - r^n)}{1 - r}$.

WORKED EXERCISE:

(a) Find the sum of all the powers of 5 from 5^0 to 5^7.

(b) Find the sum of the first six terms of the GP $2 - 6 + 18 - \cdots$

SOLUTION:

(a) The sum $5^0 + 5^1 + \cdots + 5^7$ is a GP with $a = 1$ and $r = 5$.

$$\text{Using } S_n = \frac{a(r^n - 1)}{r - 1}, \qquad \text{(In this case } r > 1.)$$

$$S_8 = \frac{a(r^8 - 1)}{r - 1} \qquad \text{(There are 8 terms.)}$$

$$= \frac{1 \times (5^8 - 1)}{5 - 1}$$

$$= 97\,656.$$

(b) The series $2 - 6 + 18 - \cdots$ is a GP with $a = 2$ and $r = -3$.

$$\text{Using } S_n = \frac{a(1 - r^n)}{1 - r}, \qquad \text{(In this case } r < 1.)$$

$$S_6 = \frac{a(1 - r^6)}{1 - r}$$

$$= \frac{2 \times \big(1 - (-3)^6\big)}{1 + 3}$$

$$= -364.$$

Solving Problems about the Sums of GPs: As always, read the question very carefully and write down all the information in symbolic form.

WORKED EXERCISE:

The sum of the first four terms of a GP with ratio 3 is 200. Find the four terms.

SOLUTION:

It is known that $S_4 = 200$.

Using the formula, $\dfrac{a(3^4 - 1)}{3 - 1} = 200$

$$\frac{80a}{2} = 200$$

$$40a = 200$$

$$a = 5.$$

So the series is $5 + 15 + 45 + 135 + \cdots$

Solving Problems Involving Trial-and-Error or Logarithms: As remarked already in Section 8D, logarithms are needed for solving GP problems systematically, but trial-and-error is quite satisfactory for simpler problems.

WORKED EXERCISE:

(a) Find a formula for the sum of the first n terms of the GP $2 + 6 + 18 + \cdots$

(b) How many terms of this GP must be taken for the sum to exceed one billion?

SOLUTION:

(a) The sequence is a GP with $a = 2$ and $r = 3$,

$$\text{so } S_n = \frac{a(r^n - 1)}{r - 1} = \frac{2(3^n - 1)}{3 - 1} = 3^n - 1.$$

(b) Put $S_n > 1\,000\,000\,000$.
Then $3^n - 1 > 1\,000\,000\,000$
$3^n > 1\,000\,000\,001$.
Using trial-and-error on the calculator,
$3^{18} = 387\,420\,489$
and $3^{19} = 1\,162\,261\,467$,
so S_{19} is the first sum over one billion.

OR

Put $S_n > 1\,000\,000\,000$.
Then $3^n - 1 > 1\,000\,000\,000$
$3^n > 1\,000\,000\,001$

$$n > \frac{\log_{10} 1\,000\,000\,001}{\log_{10} 3}$$

$n > 18{\cdot}86\ldots$,
so S_{19} is the first sum over one billion.

Two Exceptional Cases: If the ratio of a GP is 1, then the formula for S_n doesn't work, because the denominator $r - 1$ would be zero. All the terms, however, are equal to the first term a, and so the formula for the sum S_n is just $S_n = an$.

Secondly, the ratio of a GP cannot ever be zero, for then the second and third terms would both be zero and the quotient $T_3 \div T_2$ would be undefined.

Exercise 8G

1. 'As I was going to St Ives, I met a man with seven wives. Each wife had seven sacks, each sack had seven cats, each cat had seven kits. Kits, cats, sacks and wives, how many were going to St Ives?'
   Only the speaker was going *to* St Ives, but how many people were going the other way?

2. (a) Use the formula $S_7 = \dfrac{a(r^7 - 1)}{r - 1}$ to find $1 + 3 + 3^2 + 3^3 + 3^4 + 3^5 + 3^6$.
   (b) Use the formula $S_7 = \dfrac{a(1 - r^7)}{1 - r}$ to find $1 - 3 + 3^2 - 3^3 + 3^4 - 3^5 + 3^6$.

3. Find these sums using $S_n = \dfrac{a(r^n - 1)}{r - 1}$ when $r > 1$, or $S_n = \dfrac{a(1 - r^n)}{1 - r}$ when $r < 1$. Then find a formula for the sum S_n of the first n terms of each series.
   (a) $1 + 2 + 4 + 8 + \cdots$ (10 terms)
   (b) $2 + 6 + 18 + \cdots$ (5 terms)
   (c) $-1 - 10 - 100 - \cdots$ (5 terms)
   (d) $-1 - 5 - 25 - \cdots$ (5 terms)
   (e) $1 - 2 + 4 - 8 + \cdots$ (10 terms)
   (f) $2 - 6 + 18 - \cdots$ (5 terms)
   (g) $-1 + 10 - 100 + \cdots$ (5 terms)
   (h) $-1 + 5 - 25 + \cdots$ (5 terms)

4. Find these sums. Then find a formula for the sum S_n of the first n terms of each series.
NOTE: Be careful when dividing by $1 - r$, because $1 - r$ is a fraction in each case.

(a) $8 + 4 + 2 + \cdots$ (10 terms)
(b) $9 + 3 + 1 + \cdots$ (6 terms)
(c) $45 + 15 + 5 + \cdots$ (5 terms)
(d) $\frac{2}{3} + 1 + \frac{3}{2} + \frac{9}{4} + \frac{27}{8}$
(e) $8 - 4 + 2 - \cdots$ (10 terms)
(f) $9 - 3 + 1 - \cdots$ (6 terms)
(g) $-45 + 15 - 5 + \cdots$ (5 terms)
(h) $\frac{2}{3} - 1 + \frac{3}{2} - \frac{9}{4} + \frac{27}{8}$

DEVELOPMENT

5. Find an expression for S_n. Hence approximate S_{10} correct to four significant figures.

(a) $1 + 1{\cdot}2 + (1{\cdot}2)^2 + \cdots$
(b) $1 + 0{\cdot}95 + (0{\cdot}95)^2 + \cdots$
(c) $1 + 1{\cdot}01 + (1{\cdot}01)^2 + \cdots$
(d) $1 + 0{\cdot}99 + (0{\cdot}99)^2 + \cdots$

6. The King takes a chessboard of 64 squares, and places 1 grain of wheat on the first square, 2 grains on the next square, 4 grains on the next square, and so on.

(a) How many grains are on: (i) the last square (ii) the whole chessboard?
(b) Given that 1 litre of wheat contains about 30 000 grains, how many cubic kilometres of wheat are there on the chessboard?

7. Find S_n and S_{10} for each series, rationalising the denominators in your answers.

(a) $1 + \sqrt{2} + 2 + \cdots$
(b) $2 - 2\sqrt{5} + 10 - \cdots$

8. Find these sums. First write out some terms and identify a and r.

(a) $\displaystyle\sum_{n=1}^{7} 3 \times 2^n$
(b) $\displaystyle\sum_{n=3}^{8} 3^{n-1}$
(c) $\displaystyle\sum_{n=1}^{8} 3 \times 2^{3-n}$

9. (a) The first term of a GP is $\frac{1}{8}$ and the fifth term is 162. Find the first five terms of the GP, then find their sum.
(b) The first term of a GP is $-\frac{3}{4}$ and the fourth term is 6. Find the sum of the first six terms.
(c) The second term of GP is $0{\cdot}08$ and the third term is $0{\cdot}4$. Find the sum to eight terms.
(d) The ratio of a GP is $r = 2$ and the sum to eight terms is 1785. Find the first term.
(e) A GP has ratio $r = -\frac{1}{2}$ and the sum to eight terms is 425. Find the first term.

10. (a) Each year when the sunflower paddock is weeded, only half the previous weight of weed is dug out. In the first year, 6 tonnes of weed is dug out.
(i) How much is dug out in the 10th year?
(ii) What is the total dug out over 10 years (correct to four significant figures)?
(b) Every two hours, half of a particular medical isotope decays. If there was originally 20 grams, how much remains after a day (correct to two significant figures)?
(c) The price of Victoria shoes is increasing with inflation over a 10-year period by 10% per annum, so that the price in each of those 10 years is P, $1{\cdot}1 \times P$, $(1{\cdot}1)^2 \times P$, ... I buy one pair of these shoes each year.
(i) Find an expression for the total paid over 10 years (correct to the nearest cent).
(ii) Hence find the initial price P if the total paid is \$900.

11. Find a formula for S_n, and hence find n for the given value of S_n.

(a) $5 + 10 + 20 + \cdots$, $S_n = 315$
(b) $5 - 10 + 20 - \cdots$, $S_n = -425$
(c) $18 + 6 + 2 + \cdots$, $S_n = 26\frac{8}{9}$
(d) $48 - 24 + 12 - \cdots$, $S_n = 32\frac{1}{4}$

CHALLENGE

12. (a) Show that the sum S_n of the first n terms of $7 + 14 + 28 + \cdots$ is $S_n = 7(2^n - 1)$.
(b) For what value of n is S_n equal to 1785?
(c) Show that $T_n = 7 \times 2^{n-1}$, and find how many terms are less than 70 000.
(d) Use trial-and-error to find the first sum S_n that is greater than 70 000.
(e) Prove that the sum S_n of the first n terms is always 7 less than the $(n+1)$th term.

13. The powers of 3 that are greater than 1 form a GP 3, 9, 27, ...
(a) Find how many powers of 3 there are between 2 and 10^{20}.
(b) Show that $S_n = \frac{3}{2}(3^n - 1)$, and find the smallest value of n for which $S_n > 10^{20}$.

8 H The Limiting Sum of a Geometric Series

There is a sad story of a perishing frog, dying of thirst only 8 metres from the edge of a waterhole. He first jumps 4 metres towards it, his second jump is 2 metres, then each successive jump is half the previous jump. Does the frog perish?

The jumps form a GP, whose terms T_n and sums S_n are as follows:

T_n	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\cdots$
S_n	4	6	7	$7\frac{1}{2}$	$7\frac{3}{4}$	$7\frac{7}{8}$	$7\frac{15}{16}$	$\cdots$

The successive jumps $4,\ 2,\ 1,\ \frac{1}{2},\ \frac{1}{4},\ \ldots$ have limit zero, because they are halving each time. It seems too that the successive sums S_n have limit 8, meaning that the frog's total distance gets 'as close as we like' to 8 metres. So provided the frog can stick his tongue out even the merest fraction of a millimetre, eventually he will get some water to drink and be saved.

The Limiting Sum of a GP: We can describe all this more precisely by looking at the sum S_n of the first n terms and examining what happens as $n \to \infty$.

The series $4 + 2 + 1 + \frac{1}{2} + \cdots$ is a GP with $a = 4$ and $r = \frac{1}{2}$.
Using the formula for the sum to n terms of the series,

$$S_n = \frac{a(1 - r^n)}{1 - r} \qquad \text{(This formula is used because } r < 1.\text{)}$$

$$= \frac{4\left(1 - (\frac{1}{2})^n\right)}{1 - \frac{1}{2}}$$

$$= 4 \times \left(1 - (\tfrac{1}{2})^n\right) \div \tfrac{1}{2}$$

$$= 8\left(1 - (\tfrac{1}{2})^n\right).$$

As n increases, the term $(\frac{1}{2})^n$ gets progressively closer to zero:

$$(\tfrac{1}{2})^2 = \tfrac{1}{4}, \quad (\tfrac{1}{2})^3 = \tfrac{1}{8}, \quad (\tfrac{1}{2})^4 = \tfrac{1}{16}, \quad (\tfrac{1}{2})^5 = \tfrac{1}{32}, \quad (\tfrac{1}{2})^6 = \tfrac{1}{64}, \quad \ldots$$

so that $(\frac{1}{2})^n$ has limit zero as $n \to \infty$.

Hence S_n does indeed have limit $8(1 - 0) = 8$, as the table of values suggested.

There are several different common notations and words for this situation:

12 NOTATIONS FOR THE LIMITING SUM: Consider the series $4 + 2 + 1 + \frac{1}{2} + \cdots$

- $S_n \to 8$ as $n \to \infty$. (‘S_n has limit 8 as n increases without bound.’)
- $\lim\limits_{n\to\infty} S_n = 8$ (‘The limit of S_n, as n increases without bound, is 8.’)
- The series $4 + 2 + 1 + \frac{1}{2} + \cdots$ has *limiting sum* $S_\infty = 8$.
- The series $4 + 2 + 1 + \frac{1}{2} + \cdots$ *converges to the limit* $S_\infty = 8$.
- $4 + 2 + 1 + \frac{1}{2} + \cdots = 8$.

The General Case: Suppose now that T_n is a GP with first term a and ratio r, so that

$$T_n = ar^{n-1} \qquad \text{and} \qquad S_n = \frac{a(1-r^n)}{1-r}\,.$$

Suppose also that the ratio r lies in the interval $-1 < r < 1$.
Then as $n \to \infty$, the successive powers $r^1, r^2, r^3, r^4, \ldots$ get smaller and smaller:

$$\text{As } n \to \infty, \quad r^n \to 0.$$

Hence the difference $1 - r^n$ must get closer and closer to 1:

$$\text{As } n \to \infty, \quad 1 - r^n \to 1.$$

Thus both the nth term T_n and the sum S_n *converge* to a limit:

$$\lim_{n\to\infty} T_n = \lim_{n\to\infty} ar^{n-1} = 0, \qquad \text{and} \qquad \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r}\,.$$

All this can be summarised in two statements:

13 LIMITING SUMS OF GEOMETRIC SERIES:

- The sums S_n of a GP converge to a limit if and only if $-1 < r < 1$.
- The value of this limit is $S_\infty = \dfrac{a}{1-r}$.
- When $r \ge 1$ or $r \le -1$, the sums S_n do not converge.

The last statement will be demonstrated in the exercises.

WORKED EXERCISE:
Explain why these series have limiting sums, and find them.
(a) $18 + 6 + 2 + \cdots$ (b) $18 - 6 + 2 - \cdots$

SOLUTION:
(a) Here $a = 18$ and $r = \frac{1}{3}$.
Since $-1 < r < 1$,
the series must converge.
$$S_\infty = \frac{18}{1 - \frac{1}{3}} = 18 \times \tfrac{3}{2} = 27$$

(b) Here $a = 18$ and $r = -\frac{1}{3}$.
Since $-1 < r < 1$,
the series must converge.
$$S_\infty = \frac{18}{1 + \frac{1}{3}} = 18 \times \tfrac{3}{4} = 13\tfrac{1}{2}$$

WORKED EXERCISE:

(a) For what values of x does the series $1 + (x-2) + (x-2)^2 + \cdots$ converge?
(b) When the series does converge, what is its limiting sum?

SOLUTION:
The sequence is a GP with first term $a = 1$ and ratio $r = x - 2$.

(a) The GP converges when
$$-1 < r < 1$$
$$-1 < x - 2 < 1$$
$\boxed{+2}$
$$1 < x < 3.$$

(b) The limiting sum is then
$$S_\infty = \frac{1}{1-(x-2)} = \frac{1}{3-x}.$$

Solving Problems Involving Limiting Sums: As always, the first step is to write down in symbolic form everything that is given in the question.

WORKED EXERCISE:
Find the ratio of a GP whose first term is 10 and whose limiting sum is 40.

SOLUTION:
We know that $S_\infty = 40.$
Using the formula, $\dfrac{a}{1-r} = 40$
and substituting $a = 10$ gives $\dfrac{10}{1-r} = 40$
$$10 = 40(1-r)$$
$$1 = 4 - 4r$$
$$4r = 3$$
$$r = \tfrac{3}{4}.$$

Sigma Notation for Infinite Sums: When $-1 < r < 1$ and the GP converges, the limiting sum S_∞ can also be written as an infinite sum, either using sigma notation or using dots, so that
$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad \text{or} \quad a + ar + ar^2 + \cdots = \frac{a}{1-r},$$
and we say that 'the series $\displaystyle\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$ converges to $\dfrac{a}{1-r}$'.

Exercise 8H

1. (a) Copy and complete the table of values opposite for the GP with $a = 18$ and $r = \frac{1}{3}$.

(b) Find the limiting sum using $S_\infty = \dfrac{a}{1-r}$.

(c) Find the difference $S_\infty - S_6$.

n	1	2	3	4	5	6
T_n	18	6	2	$\frac{2}{3}$	$\frac{2}{9}$	$\frac{2}{27}$
S_n						

2. (a) Copy and complete the table of values opposite for the GP with $a = 24$ and $r = -\frac{1}{2}$.

n	1	2	3	4	5	6
T_n	24	-12	6	-3	$1\frac{1}{2}$	$-\frac{3}{4}$
S_n						

(b) Find the limiting sum using $S_\infty = \dfrac{a}{1-r}$.

(c) Find the difference $S_\infty - S_6$.

3. Each GP below has ratio $r = \frac{1}{2}$. Identify the first term a and hence find S_∞.
(a) $1 + \frac{1}{2} + \frac{1}{4} + \cdots$ (b) $8 + 4 + 2 + \cdots$ (c) $-4 - 2 - 1 - \cdots$

4. Each GP below has ratio $r = -\frac{1}{3}$. Identify the first term a and hence find S_∞.
(a) $1 - \frac{1}{3} + \frac{1}{9} - \cdots$ (b) $36 - 12 + 4 - \cdots$ (c) $-60 + 20 - 6\frac{2}{3} + \cdots$

5. Each GP below has first term $a = 60$. Identify the ratio r and hence find S_∞.
(a) $60 + 15 + 3\frac{3}{4} + \cdots$ (b) $60 - 30 + 15 - \cdots$ (c) $60 - 12 + 2\frac{2}{5} - \cdots$

6. Find each ratio r to test whether there is a limiting sum. Find the limiting sum if it exists.
(a) $1 - \frac{1}{2} + \frac{1}{4} - \cdots$
(b) $1 + \frac{1}{3} + \frac{1}{9} + \cdots$
(c) $1 - \frac{2}{3} + \frac{4}{9} - \cdots$
(d) $1 + \frac{3}{5} + \frac{9}{25} + \cdots$
(e) $4 - 6 + 9 - \cdots$
(f) $12 + 4 + \frac{4}{3} + \cdots$
(g) $1000 + 100 + 10 + \cdots$
(h) $1000 - 100 + 10 - \cdots$
(i) $1 - 1 + 1 - \cdots$
(j) $100 + 90 + 81 + \cdots$
(k) $-2 + \frac{2}{5} - \frac{2}{25} + \cdots$
(l) $-\frac{2}{3} - \frac{2}{15} - \frac{2}{75} - \cdots$

7. Bevin dropped the Nelson Bros Bouncy Ball from a height of 8 metres. It bounced continually, each successive height being half of the previous height.
(a) Show that the first distance travelled down-and-up is 12 metres, and explain why the successive down-and-up distances form a GP with $r = \frac{1}{2}$.
(b) Through what distance did the ball 'eventually' travel?

DEVELOPMENT

8. These examples will show that a GP does not have a limiting sum when $r \ge 1$ or $r \le -1$. Copy and complete the tables for these GPs, then describe the behaviour of S_n as $n \to \infty$.

(a) $r = 1$ and $a = 10$

n	1	2	3	4	5	6
T_n						
S_n						

(b) $r = -1$ and $a = 10$

n	1	2	3	4	5	6
T_n						
S_n						

(c) $r = 2$ and $a = 10$

n	1	2	3	4	5	6
T_n						
S_n						

(d) $r = -2$ and $a = 10$

n	1	2	3	4	5	6
T_n						
S_n						

9. For each series, find S_∞ and S_4, then find the difference $S_\infty - S_4$.
(a) $80 + 40 + 20 + \cdots$ (b) $100 + 10 + 1 + \cdots$ (c) $100 - 80 + 64 - \cdots$

10. When Brownleigh Council began offering free reflective house numbers to its 10 000 home owners, 20% installed them in the first month. The number installing them in the second month was only 20% of those in the first month, and so on.
(a) Show that the numbers installing them each month form a GP.
(b) How many home owners will 'eventually' install them? ('Eventually' means take S_∞.)
(c) How many eventual installations were not done in the first four months?

11. The Wellington Widget Factory has been advertising its unbreakable widgets every month. The first advertisement brought in 1000 sales, but every successive advertisement is only bringing in 90% of the previous month's sales.

(a) How many widget sales will the advertisements 'eventually' bring in?

(b) About how many eventual sales were not brought in by the first 10 advertisements?

12. Find, in terms of x, an expression for the limiting sum of each series on the left. Then solve the given equation to find x.

(a) $5 + 5x + 5x^2 + \cdots = 10$

(b) $5 + 5x + 5x^2 + \cdots = 3$

(c) $5 - 5x + 5x^2 - \cdots = 15$

(d) $x + \dfrac{x}{3} + \dfrac{x}{9} + \cdots = 2$

(e) $x - \dfrac{x}{3} + \dfrac{x}{9} - \cdots = 2$

(f) $x + \dfrac{2x}{3} + \dfrac{4x}{9} + \cdots = 2$

13. Find the condition for each GP to have a limiting sum, then find that limiting sum.

(a) $7 + 7x + 7x^2 + \cdots$

(b) $2x + 6x^2 + 18x^3 + \cdots$

(c) $1 + (x - 1) + (x - 1)^2 + \cdots$

(d) $1 + (1 + x) + (1 + x)^2 + \cdots$

14. Find the limiting sum of each series, if it exists.

(a) $1 + (1{\cdot}01) + (1{\cdot}01)^2 + \cdots$

(b) $1 - 0{\cdot}99 + (0{\cdot}99)^2 - \cdots$

(c) $1 + (1{\cdot}01)^{-1} + (1{\cdot}01)^{-2} + \cdots$

(d) $0{\cdot}72 - 0{\cdot}12 + 0{\cdot}02 - \cdots$

15. Find the limiting sums if they exist, rationalising denominators if necessary.

(a) $16\sqrt{5} + 4\sqrt{5} + \sqrt{5} + \cdots$

(b) $108\sqrt{7} - 36\sqrt{7} + 12\sqrt{7} - \cdots$

(c) $7 + \sqrt{7} + 1 + \cdots$

(d) $4 - 2\sqrt{2} + 2 - \cdots$

(e) $5 - 2\sqrt{5} + 4 - \cdots$

(f) $9 + 3\sqrt{10} + 10 + \cdots$

(g) $1 + (1 - \sqrt{3}) + (1 - \sqrt{3})^2 + \cdots$

(h) $1 + (2 - \sqrt{3}) + (2 - \sqrt{3})^2 + \cdots$

16. Expand each series for a few terms. Then write down a and r, and find the limiting sum.

(a) $\displaystyle\sum_{n=1}^{\infty} (\tfrac{1}{3})^n$

(b) $\displaystyle\sum_{n=1}^{\infty} 7 \times (\tfrac{1}{2})^n$

(c) $\displaystyle\sum_{n=1}^{\infty} 40 \times (-\tfrac{3}{5})^n$

CHALLENGE

17. Suppose that $T_n = ar^{n-1}$ is a GP with a limiting sum.

(a) Find the ratio r if the limiting sum equals 5 times the first term.

(b) Find the first three terms if the second term is 6 and the limiting sum is 27.

(c) Find the ratio if the sum of all terms except the first equals 5 times the first term.

(d) Show that the sum S of all terms from the third on is $\dfrac{ar^2}{1-r}$.

(i) Hence find r if S equals the first term.

(ii) Find r if S equals the second term.

(iii) Find r if S equals the sum of the first and second terms.

18. Find the condition for each GP to have a limiting sum, then find that limiting sum.

(a) $1 + (x^2 - 1) + (x^2 - 1)^2 + \cdots$

(b) $1 + \dfrac{1}{1 + x^2} + \dfrac{1}{(1 + x^2)^2} + \cdots$

8 I Recurring Decimals and Geometric Series

It is now possible to give a precise explanation of a recurring decimal. It is an infinite GP, and its value is the limiting sum of that GP.

WORKED EXERCISE:
Express these recurring decimals as infinite GPs. Then use the formula for the limiting sum to find their values as fractions reduced to lowest terms.

(a) $0{\cdot}\dot{2}\dot{7}$ (b) $2{\cdot}6\dot{4}\dot{5}$

SOLUTION:

(a) Expanding the decimal, $0{\cdot}\dot{2}\dot{7} = 0{\cdot}272727\ldots$

$$= 0{\cdot}27 + 0{\cdot}0027 + 0{\cdot}000027 + \cdots$$

This is an infinite GP with first term $a = 0{\cdot}27$ and ratio $r = 0{\cdot}01$.

Hence $0{\cdot}\dot{2}\dot{7} = \dfrac{a}{1-r}$

$= \frac{0{\cdot}27}{0{\cdot}99}$ $(1 - 0{\cdot}01 = 0{\cdot}99)$

$= \frac{27}{99}$ (Multiply top and bottom by 100.)

$= \frac{3}{11}$. (Divide top and bottom by 9.)

(b) This example is a little more complicated, beause the first part is not recurring.
Expanding the decimal, $2{\cdot}6\dot{4}\dot{5} = 2{\cdot}645454545\ldots$

$$= 2{\cdot}6 + (0{\cdot}045 + 0{\cdot}00045 + \cdots)$$

This is $2{\cdot}6$ plus an infinite GP with first term $a = 0{\cdot}045$ and ratio $r = 0{\cdot}01$.

Hence $2{\cdot}6\dot{4}\dot{5} = 2{\cdot}6 + \frac{0{\cdot}045}{0{\cdot}99}$ $(1 - 0{\cdot}01 = 0{\cdot}99)$

$= \frac{26}{10} + \frac{45}{990}$

$= \frac{286}{110} + \frac{5}{110}$ (Use a common denominator.)

$= \frac{291}{110}$.

Exercise 8I

NOTE: The following prime factorisations will be useful in this exercise:

$9 = 3^2$ $999 = 3^3 \times 37$ $99\,999 = 3^2 \times 41 \times 271$

$99 = 3^2 \times 11$ $9999 = 3^2 \times 11 \times 101$ $999\,999 = 3^3 \times 7 \times 11 \times 13 \times 37$

1. Write each of these recurring decimals as an infinite GP. Then use the formula for the limiting sum of a GP to express it as a rational number in lowest terms.

(a) $0{\cdot}\dot{3}$ (b) $0{\cdot}\dot{1}$ (c) $0{\cdot}\dot{7}$ (d) $0{\cdot}\dot{6}$

2. Write each of these recurring decimals as an infinite GP. Then use the formula for the limiting sum of a GP to express it as a rational number in lowest terms.

(a) $0{\cdot}\dot{2}\dot{7}$ (c) $0{\cdot}\dot{0}\dot{9}$ (e) $0{\cdot}\dot{7}\dot{8}$ (g) $0{\cdot}\dot{1}3\dot{5}$

(b) $0{\cdot}\dot{8}\dot{1}$ (d) $0{\cdot}\dot{1}\dot{2}$ (f) $0{\cdot}\dot{0}2\dot{7}$ (h) $0{\cdot}\dot{1}8\dot{5}$

3. Apply the earlier method in Section 2A — multiplying by 10^n where n is the cycle length, then subtracting — to every second recurring decimal in the previous questions.

DEVELOPMENT

4. Write each recurring decimal as the sum of an integer or terminating decimal and an infinite GP. Then express it as a fraction in lowest terms.

(a) $12{\cdot}\dot{4}$ (b) $7{\cdot}\dot{8}\dot{1}$ (c) $8{\cdot}4\dot{6}$ (d) $0{\cdot}2\dot{3}\dot{6}$

5. (a) Express the repeating decimal $0{\cdot}\dot{9}$ as an infinite GP, and hence show that it equals 1.

(b) Express $2{\cdot}7\dot{9}$ as $2{\cdot}7$ plus an infinite GP, and hence show that it equals $2{\cdot}8$.

CHALLENGE

6. Use GPs to express these as fractions in lowest terms.

(a) $0{\cdot}\dot{0}95\dot{7}$ (c) $0{\cdot}\dot{2}30\,76\dot{9}$ (e) $0{\cdot}25\dot{5}\dot{7}$ (g) $0{\cdot}0\dot{0}0\,27\dot{1}$

(b) $0{\cdot}\dot{2}47\dot{5}$ (d) $0{\cdot}\dot{4}28\,57\dot{1}$ (f) $1{\cdot}1\dot{0}3\dot{7}$ (h) $7{\cdot}7\dot{7}1\,428\,\dot{5}$

8J Chapter Review Exercise

1. Write out the first 12 terms of the sequence 50, 41, 32, 23, ...

(a) How many positive terms are there?

(b) How many terms lie between 0 and 40?

(c) What is the 10th term?

(d) What number term is -13?

(e) Is -100 a term in the sequence?

(f) What is the first term less than -35?

2. The nth term of a sequence is given by $T_n = 58 - 6n$.

(a) Find the first, 20th, 100th and the 1 000 000th terms.

(b) Find whether 20, 10, -56 and -100 are terms of the sequence.

(c) Find the first term less than -200, giving its number and its value.

(d) Find the last term greater than -600, giving its number and its value.

3. (a) Write out the first eight terms of the sequence $T_n = 5 \times (-1)^n$.

(b) Find the sum of the first seven terms and the sum of the first eight terms.

(c) How is each term obtained from the previous term?

(d) What are the 20th, 75th and 111th terms?

4. Test each sequence to see whether it is an AP, a GP or neither. State the common difference of any AP and the common ratio of any GP.

(a) 76, 83, 90, ...

(b) 100, -21, -142, ...

(c) 1, 4, 9, ...

(d) 6, 18, 54, ...

(e) 6, 10, 15, ...

(f) 48, -24, 12, ...

5. (a) State the first term and common difference of the AP 23, 35, 47, ...

(b) Use the formula $T_n = a + (n-1)d$ to find the 20th term and the 600th term.

(c) Show that the formula for the nth term is $T_n = 11 + 12n$.

(d) Hence find whether 143 and 173 are terms of the sequence.

(e) Hence find the first term greater than 1000 and the last term less than 2000.

(f) Hence find how many terms there are between 1000 and 2000.

6. A shop charges \$20 for one case of soft drink and \$16 for every subsequent case.

(a) Show that the costs of 1 case, 2 cases, 3 cases, ... form an AP and state its first term and common difference.

(b) Hence find a formula for the cost of n cases.

(c) What is the largest number of cases that I can buy with \$200, and what is my change?

(d) My neighbour paid \$292 for some cases. How many did he buy?

7. (a) Find the first term and common ratio of the GP $50,\ 100,\ 200,\ \ldots$

(b) Use the formula $T_n = ar^{n-1}$ to find a formula for the nth term.

(c) Hence find the eighth term and the 12th term.

(d) Find whether 1600 and 4800 are terms of the sequence.

(e) Find the product of the fourth and fifth terms.

(f) Use logarithms, or trial-and-error on the calculator, to find how many terms are less than 10 000 000.

8. On the first day that Barry exhibited his paintings, there were 486 visitors. On each subsequent day, there were only a third as many visitors as on the previous day.

(a) Show that the number of visitors on successive days forms a GP and state the first term and common ratio.

(b) Write out the terms of the GP until the numbers become absurd.

(c) For how many days were there at least 10 visitors?

(d) What was the total number of visitors while the formula was still valid?

(e) Use the formula $S_\infty = \dfrac{a}{1-r}$ to find the 'eventual' number of visitors if the absurdity of fractional numbers of people were ignored.

9. Find the second term x of the sequence $15,\ x,\ 135$:

(a) if the sequence is an AP, (b) if the sequence is a GP.

10. Use the formula $S_n = \frac{1}{2}n\big(2a + (n-1)d\big)$ to find the sum of the first 41 terms of each AP.

(a) $51 + 62 + 73 + \cdots$ (b) $100 + 75 + 50 + \cdots$ (c) $-35 - 32 - 29 - \cdots$

11. Use the formula $T_n = a + (n-1)d$ to find the number of terms in each AP, then use the formula $S_n = \frac{1}{2}n(a + \ell)$ to find the sum of the series.

(a) $23 + 27 + 31 + \cdots + 199$ (b) $200+197+194+\cdots-100$ (c) $12 + 12\frac{1}{2} + 13 + \cdots + 50$

12. Use $S_n = \dfrac{a(r^n - 1)}{r - 1}$ or $S_n = \dfrac{a(1 - r^n)}{1 - r}$ to find the sum of the first 6 terms of each GP.

(a) $3 + 6 + 12 + \cdots$ (b) $6 - 18 + 54 - \cdots$ (c) $-80 - 40 - 20 - \cdots$

13. Find the limiting sum of each GP, if it exists.

(a) $240 + 48 + 9\frac{3}{5} + \cdots$ (b) $-6 + 9 - 13\frac{1}{2} + \cdots$ (c) $-405 + 135 - 45 + \cdots$

14. (a) For what values of x does the GP $(2 + x) + (2 + x)^2 + (2 + x)^3 + \cdots$ have a limiting sum?

(b) Find a formula for the value of this limiting sum when it does exist.

15. Use the formula for the limiting sum of a GP to express as a fraction:

(a) $0{\cdot}\dot{3}\dot{9}$ (b) $0{\cdot}4\dot{6}\dot{8}$ (c) $12{\cdot}30\dot{4}\dot{5}$

16. (a) The second term of an AP is 21 and the ninth term is 56. Find the 100th term.

(b) Find the sum of the first 20 terms of an AP with third term 10 and 12th term -89.

(c) The third term of a GP is 3 and the eighth term is -96. Find the sixth term.

(d) Find the difference of the AP with first term 1 if the sum of the first 10 terms is -215.

(e) Find how many terms there are in an AP with first term $4\frac{1}{2}$ and difference -1 if the sum of all the terms is 8.

(f) Find the common ratio of a GP with first term 60 and limiting sum 45.

(g) The sum of the first 10 terms of a GP with ratio -2 is 682. Find the fourth term.

CHAPTER **NINE**

The Derivative

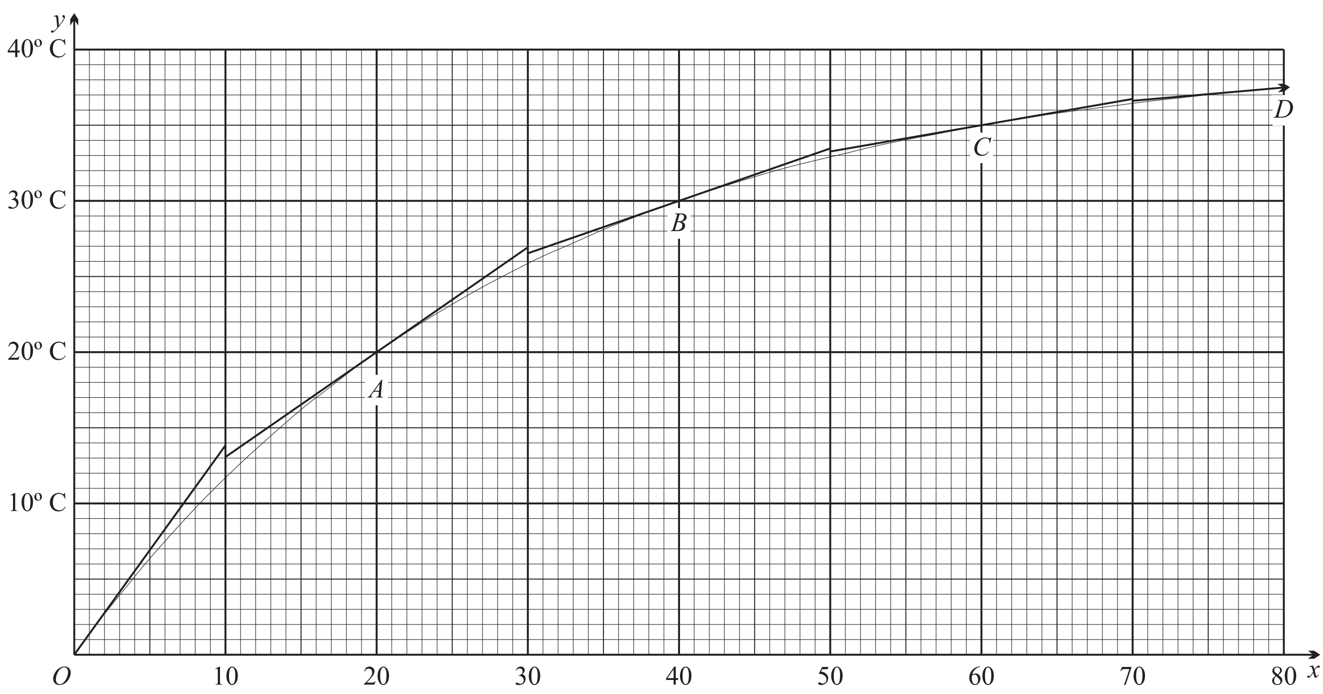
The next step in studying functions and their graphs is called *calculus*. Calculus begins with two processes called *differentiation* and *integration*.

- *Differentiation* looks at the changing steepness of a curve.
- *Integration* looks at the areas of regions bounded by curves.

Both processes involve taking limits. They were well known to the Greeks, but it was not until the late 17th century that Sir Isaac Newton in England and Gottfried Leibniz in Germany independently gave systematic accounts of them.

This chapter deals with differentiation, and introduces the derivative as the gradient of the tangent to a curve. Some readers may want to study Section 9I on limits before tackling the first-principles approach to the derivative in Section 9B.

9 A The Derivative — Geometric Definition



A bottle of water was taken out of a fridge on a hot day when the air temperature was 40°C. The graph $y = f(x)$ above shows how the temperature increased over the next 80 minutes. The horizontal axis gives the time x in minutes, and the vertical axis gives the temperature y°C.

The water temperature was originally 0°C and 20 minutes later it was 20°C. Thus during the first 20 minutes, the temperature was rising at an *average rate* of 1° per minute.

Measuring the *instantaneous rate of temperature increase*, however, requires tangents to be drawn at various points. The gradient of the tangent is the instantaneous rate of increase at any time x — these gradients are easy to measure on the graph paper by counting little divisions and using the formula gradient $= \dfrac{\text{rise}}{\text{run}}$.

The gradient of the curve $y = f(x)$ at any point is called the *derivative*, which is written as $f'(x)$. Measuring the gradients at the marked points O, A, B and C gives a table of values of the derivative:

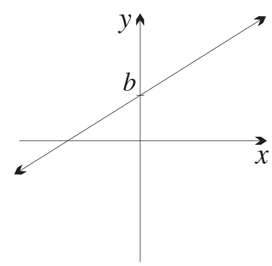
x	0	20	40	60
$f'(x)$	1·39	0·69	0·35	0·17

Geometric Definition of the Derivative: Here is the essential definition of the derivative.

1 THE DERIVATIVE $f'(x)$ DEFINED GEOMETRICALLY:

$f'(x)$ is the gradient of the tangent to $y = f(x)$ at each point on the curve.

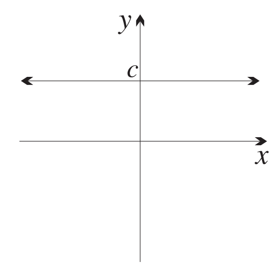
Linear Functions: When a graph is a straight line, the tangent at every point is just the line itself. Thus if $f(x) = mx + b$ is a line with gradient m, the derivative at every point is m. Hence the derivative is the constant function $f'(x) = m$.



2 THE DERIVATIVE OF A LINEAR FUNCTION:

The derivative of a linear function $f(x) = mx+b$ is the constant function $f'(x) = m$.

Constant Functions: In particular, a horizontal straight line has gradient zero. Hence the tangent to the graph of a constant function $f(x) = c$ at any point P is horizontal, and the derivative is the zero function $f'(x) = 0$.



3 THE DERIVATIVE OF A CONSTANT FUNCTION:

The derivative of a constant function $f(x) = c$ is the zero function $f'(x) = 0$.

WORKED EXERCISE:

Write down the derivative $f'(x)$ of each linear function.

(a) $f(x) = 3x + 2$ (b) $f(x) = 5 - 2x$ (c) $f(x) = -4$

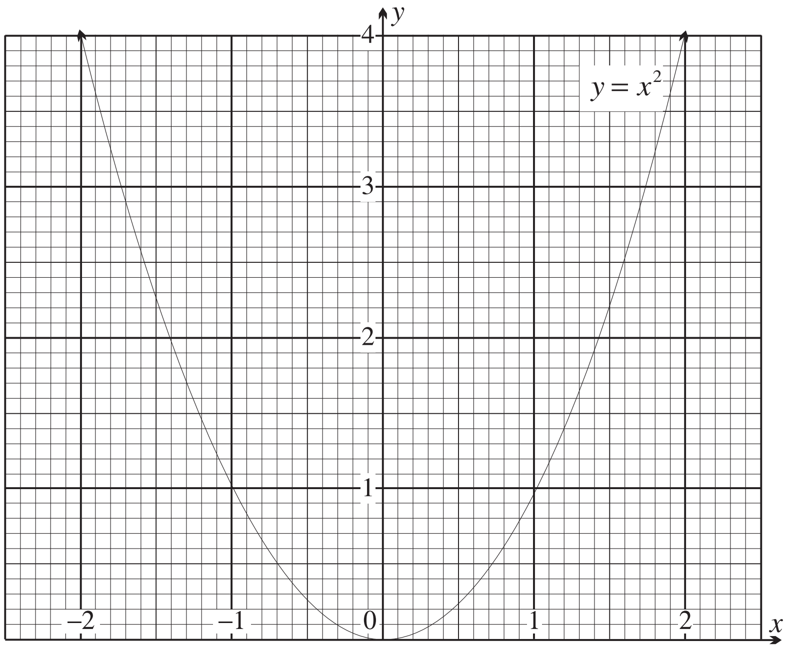
SOLUTION:

In each case the derivative is just the gradient of the line:

(a) $f'(x) = 3$ (b) $f'(x) = -2$ (c) $f'(x) = 0$

Exercise 9A

1.



(a) Photocopy the sketch above of $f(x) = x^2$.

(b) At the point $P(1, 1)$, construct the tangent. Place your pencil point on P, bring your ruler to the pencil, then rotate the ruler about P until it seems reasonably like a tangent.

(c) Use the definition gradient $= \dfrac{\text{rise}}{\text{run}}$ to measure the gradient of the tangent to at most two decimal places. Choose the run to be 10 little divisions, and count how many vertical divisions the tangent rises as it runs across the 10 horizontal divisions.

(d) Copy and complete the following table of values of the derivative $f'(x)$ by constructing a tangent at each of the nine points on the curve and measuring its gradient.

x	-2	$-1\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2
$f'(x)$									

(e) On a separate set of axes, use your table of values to sketch the curve $y = f'(x)$.

(f) Make a reasonable guess as to what the equation of the derivative $f'(x)$ is.

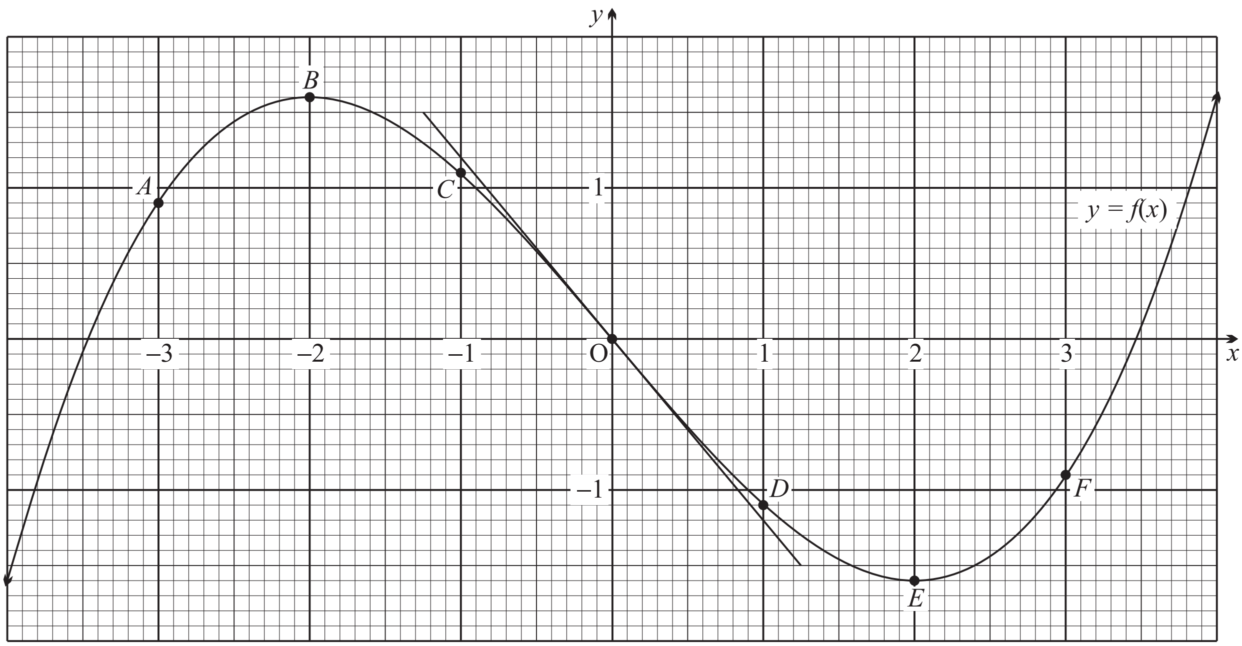
2. (a) Photocopy the graph printed at the top of the next page.

(b) Construct the tangent at the origin $O(0, 0)$ — it will actually cross the curve. Then use the definition gradient $= \dfrac{\text{rise}}{\text{run}}$ to measure the gradient of the tangent.

(c) Copy and complete the following table of values of the derivative $f'(x)$ by constructing a tangent at each of the seven points on the curve and measuring its gradient.

x	-3	-2	-1	0	1	2	3
$f'(x)$							

(d) On a separate set of axes, use your table of values to sketch the curve $y = f'(x)$.



3. Use the fact that the derivative of $f(x) = mx + b$ is $f'(x) = m$ to write down $f'(x)$.

(a) $f(x) = 2x + 3$ (c) $f(x) = -5x + 10$ (e) $f(x) = \frac{1}{2}x - 7$

(b) $f(x) = 7x - 4$ (d) $f(x) = 5 - 3x$ (f) $f(x) = -4$

DEVELOPMENT

4. Write each function in the form $f(x) = mx + b$. Then write down $f'(x)$.

(a) $f(x) = 5(2x + 4)$ (b) $f(x) = \frac{2}{3}(x + 4)$ (c) $f(x) = \frac{1}{4}(3 - 4x)$

CHALLENGE

5. Sketch a graph of each function, draw tangents at the points where $x = -2, -1, 0, 1, 2$, estimate their gradients, and hence draw a reasonable sketch of the derivative.

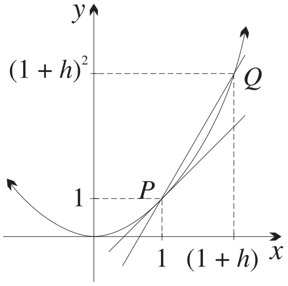
(a) $f(x) = 4 - x^2$ (b) $f(x) = \dfrac{1}{x}$ (c) $f(x) = 2^x$

9 B The Derivative as a Limit

Finding the gradient of a tangent at a point P on a curve is a limiting process. We need to look at secants through P that cross the curve again at another point Q near P, and then take the limit as Q moves towards P.

The Tangent as the Limit of Secants: The diagram below shows the graph of $f(x) = x^2$ and the tangent at the point $P(1, 1)$ on the curve.

Let Q be another point on the curve, and join the secant PQ.
Let the x-coordinate of Q be $1 + h$, where $h \neq 0$.
Then the y-coordinate of Q is $(1 + h)^2$.



$$\text{Hence gradient } PQ = \frac{(1+h)^2 - 1}{(1+h) - 1} \quad \text{(This is rise over run.)}$$
$$= \frac{2h + h^2}{h}$$
$$= 2 + h, \quad \text{since } h \neq 0.$$

As Q moves along the curve to the right or the left of P, the secant PQ changes. The closer Q is to the point P, the closer the secant PQ is to the tangent at P. The gradient of the secant PQ becomes 'as close as we like' to the gradient of the tangent as Q moves sufficiently close to P, that is, in the limit as $Q \to P$:

$$\begin{aligned}\text{gradient (tangent at } P) &= \lim_{Q \to P} (\text{gradient } PQ)\\ &= \lim_{h \to 0} (2 + h), \qquad \text{because } h \to 0 \text{ as } Q \to P\\ &= 2, \qquad \text{because } 2 + h \to 2 \text{ as } h \to 0.\end{aligned}$$

Thus the tangent at P has gradient 2, which means that $f'(1) = 2$.

Notice that Q cannot actually coincide with P, that is, h cannot be zero. Otherwise both rise and run would be zero, and the calculation would be invalid.

The Derivative as a Limit: This same process can be applied to any function $f(x)$.

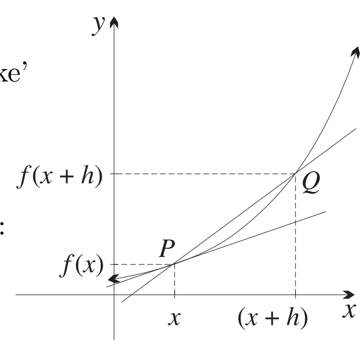
Let $P\big(x, f(x)\big)$ be any point on the curve.
Let Q be any other point on the curve, to the left or right of P,
and let Q have x-coordinate $x + h$, where $h \neq 0$, and y-coordinate $f(x + h)$.

Then $\quad \text{gradient of secant } PQ = \dfrac{f(x+h) - f(x)}{h}$ (rise over run).

As $h \to 0$, the point Q moves 'as close as we like' to P, and the gradient of the secant PQ becomes 'as close as we like' to the gradient of the tangent at P.

Hence gradient of tangent at $P = \displaystyle\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

This last expression is the *limiting formula for the derivative*:



4 THE DERIVATIVE $f'(x)$ AS A LIMIT:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Using the Definition of the Derivative — First-Principles Differentiation: Finding the derivative using the limiting formula above is called *first-principles differentiation*.

WORKED EXERCISE:

If $f(x) = x^2$, use first-principles differentiation to show that $f'(5) = 10$.

SOLUTION:

For all $h \neq 0$,

$$\begin{aligned}\frac{f(5+h) - f(5)}{h} &= \frac{(5+h)^2 - 5^2}{h}\\ &= \frac{25 + 10h + h^2 - 25}{h}\\ &= \frac{10h + h^2}{h}\\ &= 10 + h, \qquad \text{since } h \neq 0.\end{aligned}$$

Taking the limit as $h \to 0$, $\quad f'(5) = 10$.

WORKED EXERCISE:

(a) Use first-principles differentiation to find the derivative $f'(x)$ of $f(x) = x^2$.
(b) Substitute $x = 5$ to confirm that $f'(5) = 10$, as in the previous exercise.

SOLUTION:

(a) For all $h \neq 0$,

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{(x+h)^2 - x^2}{h} \\ &= \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \frac{2xh + h^2}{h} \\ &= 2x + h, \qquad \text{since } h \neq 0.\end{aligned}$$

Taking the limit as $h \to 0$, $f'(x) = 2x$.

(b) Substituting $x = 5$, $f'(5) = 10$, as established previously.

WORKED EXERCISE:

(a) Find the derivative of $f(x) = x^2 + 4x$ by first-principles differentiation.
(b) Find the gradient of the tangent at the point $(1, 5)$ on the curve $y = x^2 + 4x$.

SOLUTION:

(a) For all $h \neq 0$,

$$\begin{aligned}\lim_{h \to 0} \frac{f(x+h)-f(x)}{h} &= \frac{(x+h)^2 + 4(x+h) - (x^2 + 4x)}{h} \\ &= \frac{x^2 + 2xh + h^2 + 4x + 4h - x^2 - 4x}{h} \\ &= \frac{2xh + h^2 + 4h}{h} \\ &= 2x + h + 4, \qquad \text{since } h \neq 0.\end{aligned}$$

Taking the limit as $h \to 0$, $f'(x) = 2x + 4$.

(b) At the point $(1, 5)$, gradient of tangent $= f'(1)$ (This is what $f'(1)$ means.)
$= 6$.

Exercise 9B

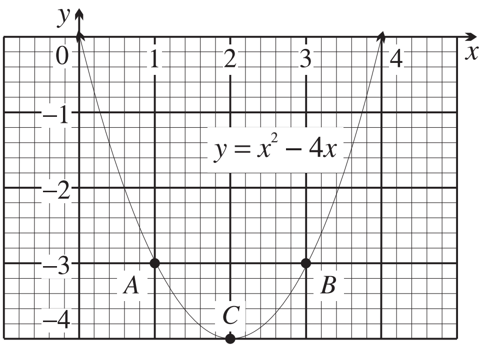
NOTE: All questions in this exercise use the formula for the deriative as a limit:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

1. Consider the function $f(x) = 5x^2$.
   (a) Show that $f(1) = 5$ and $f(1+h) = 5 + 10h + 5h^2$.
   (b) Hence find $f(1+h) - f(1)$.
   (c) Show that $\dfrac{f(1+h) - f(1)}{h} = 10 + 5h$.
   (d) Take the limit as $h \to 0$ to show that $f'(1) = 10$.

2. Consider again the function $f(x) = 5x^2$.
(a) Show that $f(x+h) = 5x^2 + 10xh + 5h^2$.
(b) Hence find $f(x+h) - f(x)$
(c) Show that $\dfrac{f(x+h) - f(x)}{h} = 10x + 5h$.
(d) Take the limit as $h \to 0$ to show that $f'(x) = 10x$.
(e) Substitute $x = 1$ into $f'(x)$ to confirm that $f(1) = 10$, as found in question 1.

3. Consider the function $f(x) = x^2 - 4x$.
(a) Show that $f(x+h) = x^2 + 2xh + h^2 - 4x - 4h$.
(b) Show that $\dfrac{f(x+h) - f(x)}{h} = 2x + h - 4$.
(c) Show that $f'(x) = 2x - 4$ by taking the limit as $h \to 0$.
(d) Evaluate $f'(1)$ to find the gradient of the tangent at $A(1, -3)$.
(e) Similarly, find the gradients of the tangents at $B(3, -3)$ and $C(2, -4)$.



(f) The function $f(x) = x^2 - 4x$ is graphed above. Place your ruler on the curve at A, B and C to check the reasonableness of the results obtained above.

4. Consider the function $f(x) = x^2 + 2$.
(a) Find $f(x+h) - f(x)$ and hence show that $\dfrac{f(x+h) - f(x)}{h} = 2x + h$.
(b) Take the limit as $h \to 0$ to find $f'(x)$.
(c) Evaluate $f'(0)$ to find the gradient of the tangent at the point where $x = 0$.
(d) Evaluate $f'(3)$ to find the gradient of the tangent at the point where $x = 3$.

5. Consider the function $f(x) = x^2 + 4x$.
(a) Simplify $\dfrac{f(x+h) - f(x)}{h}$, then take the limit as $h \to 0$ to find $f'(x)$.
(b) Find the gradients of the tangents to $y = x^2 + 4x$ at the points where $x = 0$ and $x = -2$.

6. (a) Find $\dfrac{f(x+h) - f(x)}{h}$ for the function $f(x) = x^2 - 2x$ and hence find its derivative.
(b) Hence find the gradients of the tangents at the points where $x = 0$ and $x = 2$.

7. (a) Find $\dfrac{f(x+h) - f(x)}{h}$ for the function $f(x) = x^2 + 6x + 8$ and hence find its derivative.
(b) Hence find the gradients of the tangents at the points where $x = 0$ and $x = -3$.

8. (a) Find $\dfrac{f(2+h) - f(2)}{h}$ for the function $f(x) = 1 + x^2$ and hence find $f'(2)$.
(b) Find $\dfrac{f(h) - f(0)}{h}$ for the function $f(x) = 2x^2 + 3x$ and hence find $f'(0)$.
(c) Find $\dfrac{f(-1+h) - f(-1)}{h}$ for the function $f(x) = x^2 - 4x$ and hence find $f'(-1)$.

DEVELOPMENT

9. (a) Find $\dfrac{f(x+h)-f(x)}{h}$ for $f(x) = x^2 - 6x$ and hence show that $f'(x) = 2x - 6$.

(b) Use the factoring $f(x) = x(x-6)$ to find the x-intercepts of $y = f(x)$, then sketch it.

(c) Find the gradients of the tangents at these two x-intercepts.

(d) Show that the tangent is horizontal at the point where $x = 3$.

10. (a) Find $\dfrac{f(x+h)-f(x)}{h}$ for $f(x) = 8x - x^2$ and hence show that $f'(x) = 8 - 2x$.

(b) Use the factoring $f(x) = x(8-x)$ to find the x-intercepts of $y = f(x)$, then sketch it.

(c) Find the gradients of the tangents at these two x-intercepts.

(d) Show that the tangent is horizontal at the point where $x = 4$.

11. (a) If $f(x) = x^3$, show that $\dfrac{f(x+h)-f(x)}{h} = 3x^2 + 3xh + h^2$ and hence find $f'(x)$.
You will need to use the expansion $(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$.

(b) If $f(x) = x^4$, show that $\dfrac{f(x+h)-f(x)}{h} = 4x^3 + 6x^2h + 4xh^2 + h^3$, then find $f'(x)$.
You will need to use the expansion $(x+h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$.

CHALLENGE

12. (a) Use first-principles differentiation to show that if $f(x) = 5x + 7$, then $f'(x) = 5$.

(b) Show similarly that if $f(x) = mx + b$, where m and b are constants, then $f'(x) = m$.

(c) Show similarly that if $f(x) = c$, where c is a constant, then $f'(x) = 0$.

(d) Show similarly that if $f(x) = ax^2 + bx + c$, then $f'(x) = 2ax + b$.

(e) Show similarly that if $f(x) = x^3 + x^2 + x + 1$, then $f'(x) = 3x^2 + 2x + 1$.

(f) Show similarly that if $f(x) = ax^3 + bx^2 + cx + d$, then $f'(x) = 3ax^2 + 2bx + c$.

13. The point $P(1, 5)$ lies on the graph of $f(x) = 5x^2$. Let Q be any other point on the curve, and let Q have x-coordinate $1 + h$, where $h \neq 0$, and y-coordinate $5(1+h)^2$.

(a) Show that the the secant PQ has gradient $10 + 5h$.

(b) By taking limits as $h \to 0$, show that the tangent at P has gradient 10.

(c) For what values of h is the difference between the gradients of PQ and the tangent:
(i) less than 1, (ii) less than 0·1, (iii) less than 0·001, (iv) less than 10^{-10}.

9 C A Rule for Differentiating Powers of x

The long calculations of $f'(x)$ in the previous exercise had quite simple answers, as the reader will probably have noticed. Here is the pattern:

$f(x)$	1	x	x^2	x^3	x^4
$f'(x)$	0	1	$2x$	$3x^2$	$4x^3$

These results are examples of a simple rule for differentiating any power of x:

5 THE DERIVATIVE OF ANY POWER OF x:

Let $f(x) = x^n$, where n is any real number. Then the derivative is

$$f'(x) = nx^{n-1}.$$

'Take the index as a factor, and reduce the index by 1.'

The proof is complicated, and is given in the appendix to this chapter only for positive integers n. The first questions in the Challenge sections of Exercises 9E and 9F develop the further proofs for $n = -1$ and $n = \frac{1}{2}$ respectively.

This course, however, will assume that the result is true for all real numbers n. Here are some examples of the formula with positive integers.

WORKED EXERCISE:
Differentiate each function.
(a) $f(x) = x^5$ (b) $f(x) = x^8$ (c) $f(x) = x^{37}$

SOLUTION:
(a) $f(x) = x^5$
$f'(x) = 5x^4$

(b) $f(x) = x^8$
$f'(x) = 8x^7$

(c) $f(x) = x^{37}$
$f'(x) = 37x^{36}$

Linear Combinations of Functions: Functions formed by taking sums and multiples of simpler functions can be differentiated in the obvious way, one term at a time. The proofs are not difficult and are given in the appendix.

6
DERIVATIVE OF A SUM: If $f(x) = g(x) + h(x)$, then $f'(x) = g'(x) + h'(x)$.
DERIVATIVE OF A MULTIPLE: If $f(x) = ag(x)$, then $f'(x) = ag'(x)$.

WORKED EXERCISE:
Differentiate each function.
(a) $f(x) = x^2 + 3x + 5$
(b) $f(x) = 4x^2 - 3x + 2$
(c) $f(x) = 3x^{10} + 4x^9$
(d) $f(x) = \frac{1}{2}x^6 - \frac{1}{6}x^3$

SOLUTION:
(a) $f(x) = x^2 + 3x + 5$
$f'(x) = 2x + 3 + 0$
$= 2x + 3$

(b) $f(x) = 4x^2 - 3x + 2$
$f'(x) = 4 \times 2x - 3 + 0$
$= 8x - 3$

(c) $f(x) = 3x^{10} + 4x^9$
$f'(x) = 3 \times 10x^9 + 4 \times 9x^8$
$= 30x^9 + 36x^8$

(d) $f(x) = \frac{1}{2}x^6 - \frac{1}{6}x^3$
$f'(x) = \frac{1}{2} \times 6x^5 - \frac{1}{6} \times 3x^2$
$= 3x^5 - \frac{1}{2}x^2$

Expanding Products: Sometimes a product needs to be expanded before the function can be differentiated.

WORKED EXERCISE: Differentiate each function after first expanding the brackets.
(a) $f(x) = x^3(x - 10)$ (b) $f(x) = (x + 2)(2x + 3)$

SOLUTION:
(a) $f(x) = x^3(x - 10)$
$= x^4 - 10x^3$
$f'(x) = 4x^3 - 30x^2$

(b) $f(x) = (x + 2)(2x + 3)$
$= 2x^2 + 7x + 6$
$f'(x) = 4x + 7$

The Angle of Inclination of a Tangent: The steepness of a tangent can be expressed either by giving its gradient or by giving its angle of inclination.

WORKED EXERCISE:

(a) Differentiate $f(x) = x^2 + 2x$.
(b) Find the gradient and angle of inclination of the curve at the origin.
(c) Find the gradient and angle of inclination of the curve at the point $A(-2, 0)$.
(d) Sketch the curve and the tangents, marking their angles of inclination.

SOLUTION:

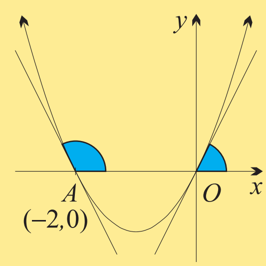
(a) Differentiating $f(x) = x^2 + 2x$ gives $f'(x) = 2x + 2$.

(b) At the origin $O(0, 0)$, gradient of tangent $= f'(0)$
$= 2.$

Let α be the angle of inclination of the tangent.
Then $\tan\alpha = 2$, where $0° \le \alpha < 180°$,
and $\alpha \doteqdot 63°26'$.

(c) At $A(-2, 0)$, gradient of tangent $= f'(-2)$
$= -4 + 2$
$= -2.$

Let β be the angle of inclination of the tangent.
Then $\tan\beta = -2$, where $0° \le \beta < 180°$,
and $\beta \doteqdot 116°34'$.



Finding Points on a Curve with a Given Gradient: The derivative can be used to find the points on a curve where the tangent has a particular gradient.

7 FINDING POINTS ON A CURVE WITH A GIVEN GRADIENT:

- To find the points where the tangent has a given gradient, solve the equation

$$f'(x) = m.$$

- To find the the y-coordinate of the points, substitute back into $f(x)$.

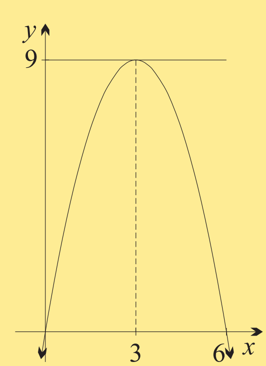
The points on the curve where the tangent is horizontal are particularly important.

WORKED EXERCISE:

Find the point on $f(x) = 6x - x^2$ where the tangent is horizontal, then sketch the curve.

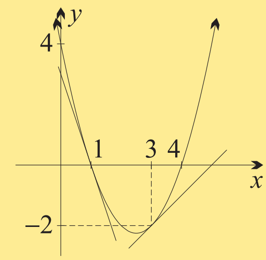
SOLUTION:

Here $f(x) = 6x - x^2$,
so $f'(x) = 6 - 2x$.
Put $f'(x) = 0$.
Then $6 - 2x = 0$
$x = 3$.
Substituting, $f(3) = 18 - 9$
$= 9$.
So the tangent is horizontal at $(3, 9)$.
[This is, of course, the vertex of the parabola.]



WORKED EXERCISE:

Find the points on the graph of $f(x) = x^2 - 5x + 4$ where:
(a) the tangent has gradient -3,
(b) the tangent has angle of inclination $45°$.



SOLUTION:

Here $f(x) = x^2 - 5x + 4$,
so $f'(x) = 2x - 5$.

(a) Put $f'(x) = -3$.
Then $2x - 5 = -3$
$2x = 2$
$x = 1$.
Substituting $x = 1$ into the function,
$f(1) = 1 - 5 + 4$
$= 0$.
So the tangent has gradient -3 at the point $(1, 0)$.

(b) First, $\tan 45° = 1$,
so put $f'(x) = 1$.
Then $2x - 5 = 1$
$2x = 6$
$x = 3$.
Substituting $x = 3$ into the function,
$f(3) = 9 - 15 + 4$
$= -2$.
So the tangent has angle of inclination $45°$ at $(3, -2)$.

Exercise 9C

1. Use the rule that the derivative of x^n is nx^{n-1} to write down the derivative $f'(x)$ of:
(a) $f(x) = x^7$
(b) $f(x) = x^5$
(c) $f(x) = 9x^5$
(d) $f(x) = 3x^2$
(e) $f(x) = \frac{1}{3}x^6$
(f) $f(x) = \frac{1}{2}x^8$
(g) $f(x) = 3$
(h) $f(x) = -2$
(i) $f(x) = 0$

2. For each function, use the rule to find the derivative $f'(x)$, then find $f'(0)$ and $f'(1)$.
(a) $f(x) = 5x + 7$
(b) $f(x) = 8 - x$
(c) $f(x) = x^2 + 5x + 7$
(d) $f(x) = 3x^2 - 5x$
(e) $f(x) = x^4 - 5x^2 + 5$
(f) $f(x) = 2 - 3x - 5x^3$
(g) $f(x) = x^4 + x^3 + x^2 + x + 1$
(h) $f(x) = \frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2$
(i) $f(x) = \frac{1}{3}x^6 - \frac{1}{2}x^4 + x^2 - 2$

3. Expand each product, then find the derivative.
(a) $f(x) = x(4 - x)$
(b) $f(x) = x(x^2 + 1)$
(c) $f(x) = x^2(3 - 4x^2)$
(d) $f(x) = (x + 4)(x - 2)$
(e) $f(x) = (2x + 1)(2x - 1)$
(f) $f(x) = (x^2 + 3)^2$
(g) $f(x) = (7 - x)^2$
(h) $f(x) = (x^2 + 3)(x - 5)$
(i) $f(x) = (3x - 5)^2$

4. (a) Find the derivative of $f(x) = x^2 + x + 1$.
(b) Evaluate $f'(0)$ to show that the tangent at the point $(0, 1)$ has gradient 1.
(c) Solve $\tan\alpha = 1$ to find the angle of inclination of the tangent at $(0, 1)$.

5. (a) Find the derivative of $f(x) = 9 - x + x^2$.
(b) Evaluate $f'(2)$ to show that the tangent at the point $(2, 11)$ has gradient 3.
(c) Find the angle of inclination of the tangent at $(2, 11)$, correct to the nearest minute.

6. (a) Find the derivative of $f(x) = x^2 + 8x + 7$.
(b) Solve $f'(x) = 0$. Hence find the point on $y = f(x)$ where the tangent is horizontal.
(c) Solve $f'(x) = 12$. Hence find the point on $y = f(x)$ where the tangent has gradient 12.

7. (a) Find the derivative of $f(x) = 3 - 2x^2$.
(b) Solve $f'(x) = 0$. Hence find the point on $y = f(x)$ where the tangent is horizontal.
(c) Solve $f'(x) = -20$. Hence find the point where the tangent has gradient -20.

DEVELOPMENT

8. Differentiate $f(x) = x^2 - 3x - 6$. Hence find the gradient and the angle of inclination (correct to the nearest minute, where appropriate) of the tangent at the point where:
(a) $x = 3$ (b) $x = 2$ (c) $x = 1\frac{1}{2}$ (d) $x = 1$ (e) $x = 0$

9. Differentiate, then find the points on each curve where the tangent is horizontal.
(a) $f(x) = x^2 - 2x + 7$ (b) $f(x) = x^2 + 4x - 10$ (c) $f(x) = x^2 - 10x + 15$

10. Differentiate $f(x) = x^2 - 5x + 1$ and hence find the points on the curve where:
(a) the gradient is 3,
(b) the gradient is -5,
(c) the angle of inclination is 45°,
(d) the angle of inclination is 135°.

11. (a) Differentiate $f(x) = x^3 - 3x + 2$. Hence find the two points on $y = x^3 - 3x + 2$ where the tangent is horizontal.
(b) Differentiate $f(x) = x^4 - 18x^2$. Hence find the three points on $y = x^4 - 18x^2$ where the tangent is horizontal.
(c) Differentiate $f(x) = x^3 + 6$. Hence find the two points on $y = x^3 + 6$ where the tangent has gradient 75.

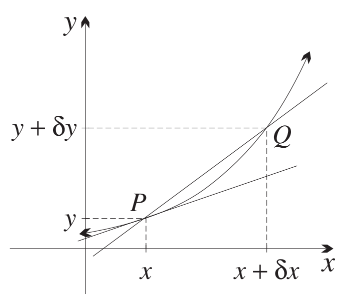
CHALLENGE

12. Sketch the graph of $f(x) = x^2 - 6x$ and find the gradients of the tangent and normal at the point $A(a, a^2 - 6a)$ on the curve. Hence find the value of a if:
(a) the tangent has gradient (i) 0, (ii) 2, (iii) $\frac{1}{2}$,
(b) the tangent has angle of inclination 135°,
(c) the tangent is (i) parallel, (ii) perpendicular, to $2x - 3y + 4 = 0$.

9 D Tangents and Normals — The Notation $\frac{dy}{dx}$

Leibniz's original notation for the derivative remains the most widely used and best-known notation. It is even said that Dee Why Beach was named after the derivative $\dfrac{dy}{dx}$. The notation is extremely flexible, as will soon become evident, and clearly expresses the fact that the derivative is very like a fraction.

Small Changes in x and in y:



Let $P(x, y)$ be any point on the graph of a function.
Let x change by a small amount δx to $x + \delta x$,
and let y change by a corresponding amount δy to $y + \delta y$.
Let the new point be $Q(x + \delta x, y + \delta y)$. Then

$$\text{gradient } PQ = \frac{\delta y}{\delta x} \quad \text{(rise over run)}.$$

When δx is small, the secant PQ is almost the same as the tangent at P, and, as before, the derivative is the limit of $\dfrac{\delta y}{\delta x}$ as $\delta x \to 0$.

This is the basis for Leibniz's notation:

8 **AN ALTERNATIVE NOTATION FOR THE DERIVATIVE:**
Let δy be the small change in y resulting from a small change δx in x. Then

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}\,.$$

The object dx is intuitively understood as an 'infinitesimal change' in x, dy as the corresponding 'infinitesimal change' in y, and the derivative $\frac{dy}{dx}$ as the ratio of these infinitesimal changes. Infinitesimal changes, however, are for the intuition only — the logic of the situation is:

9 **FRACTION NOTATION AND THE DERIVATIVE $\frac{dy}{dx}$:**
The derivative $\frac{dy}{dx}$ is not a fraction, but is the limit of the fraction $\frac{\delta y}{\delta x}$.

Nevertheless, the notation is very clever because the derivative is a gradient, the gradient is a fraction, and the notation $\frac{dy}{dx}$ preserves the intuition of fractions. The small *differences* δx and δy, and the infinitesimal *differences* dx and dy, are the origins of the word 'differentiation'.

Setting out using $\frac{dy}{dx}$ Notation: The remaining worked exercises in this section show how the new notation is used in calculations on the geometry of a curve.

WORKED EXERCISE:
(a) Differentiate $y = 4 - x^2$.
(b) Find the gradient of the tangent at the point $P(-1, 3)$ on the curve.
(c) Find the point on the curve where the gradient of the tangent is 6.

SOLUTION:
(a) $y = 4 - x^2$
$\frac{dy}{dx} = -2x$ (Just apply the usual rule for differentiating.)

(b) At the point $P(-1, 3)$, the gradient of the tangent is
$\frac{dy}{dx} = (-2) \times (-1)$ (Substitute $x = -1$ into the formula for $\frac{dy}{dx}$.)
$= 2.$

(c) Put $\frac{dy}{dx} = 6.$
Then $-2x = 6$
$x = -3.$
When $x = -3$, $y = 4 - 9$
$= -5,$
so the gradient is 6 at the point $(-3, -5)$.

Tangents and Normals to a Curve: Let P be any point lying on a curve $y = f(x)$. Using the derivative, it is a simple procedure to find the equations of the tangent and normal to a curve at any point.

10 TANGENTS AND NORMALS TO A CURVE:

- The *tangent* at P is the line through P whose gradient is the derivative at P.
- The *normal* at P is the line through P perpendicular to the tangent at P.
- Equations of tangents and normals are easily found using point–gradient form.

WORKED EXERCISE:

(a) Find the gradient and angle of inclination of the tangent to $y = (x+1)^2$ at $A(0,1)$.
(b) Find the gradient and angle of inclination of the normal at A, and draw a sketch.

SOLUTION:

(a) Expanding, $y = x^2 + 2x + 1$,

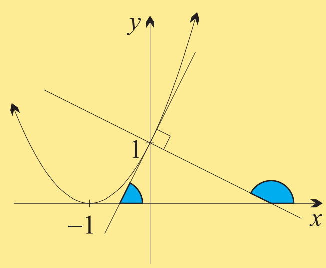
so $\dfrac{dy}{dx} = 2x + 2$.

When $x = 0$, $\dfrac{dy}{dx} = 0 + 2$,

so the tangent at $A(0,1)$ has gradient 2.

Solving $\tan\alpha = 2$, where $0° \le \alpha < 180°$,

the angle of inclination is about $63°26$.



(b) The normal has gradient $-\frac{1}{2}$ (opposite of the reciprocal).

Solving $\tan\beta = -\frac{1}{2}$, where $0° \le \beta < 180°$,

its angle of inclination is about $153°26$.

WORKED EXERCISE:

(a) Find the equations of the tangents to $y = x^2 + x + 1$ at $P(1,3)$ and $Q(-1,1)$.
(b) At what point do these tangents intersect?

SOLUTION:

(a) Differentiating, $\dfrac{dy}{dx} = 2x + 1$.

When $x = 1$, $\dfrac{dy}{dx} = 3$,

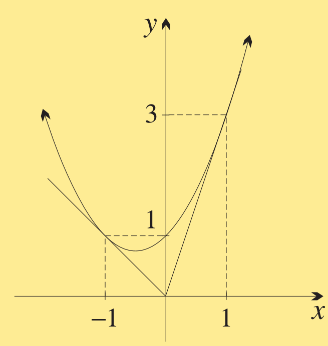
so, using point–gradient form, the tangent at $P(1,3)$ is

$$y - y_1 = m(x - x_1)$$
$$y - 3 = 3(x - 1)$$
$$y = 3x.$$

When $x = -1$, $\dfrac{dy}{dx} = -1$,

so the tangent at $Q(-1,1)$ is $y - y_1 = m(x - x_1)$

$$y - 1 = -1(x + 1)$$
$$y = -x.$$



(b) Both $y = 3x$ and $y = -x$ pass through the origin,

so the origin is the point of intersection of the two tangents.

WORKED EXERCISE:

(a) Find the equation of the tangent to $y = x^3$ at $A(1, 1)$.
(b) Find the equation of the normal to the curve at $A(1, 1)$.
(c) Find the coordinates of the y-intercepts T and N of the tangent and normal.
(d) Sketch the situation and find the area of $\triangle ANT$.

SOLUTION:

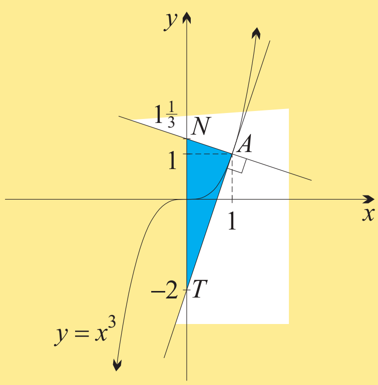
(a) Differentiating, $\dfrac{dy}{dx} = 3x^2$.

When $x = 1$, $\dfrac{dy}{dx} = 3$,

so the tangent at A has gradient 3.

Using point–gradient form, the tangent is

$$y = y_1 = m(x - x_1)$$
$$y - 1 = 3(x - 1)$$
$$y = 3x - 2.$$



(b) The normal has gradient $-\frac{1}{3}$.

Using point–gradient form, the normal is

$$y = y_1 = m(x - x_1)$$
$$y - 1 = -\tfrac{1}{3}(x - 1)$$
$$y = -\tfrac{1}{3}x + 1\tfrac{1}{3}.$$

(c) Both lines are already in gradient–intercept form, and their y-intercepts are therefore $T(0, -2)$ and $N(0, 1\frac{1}{3})$.

(d) The triangle ANT has base NT and perpendicular height 1,

where $NT = 2 + 1\frac{1}{3}$
$= 3\frac{1}{3}$,

so area $\triangle ANT = \frac{1}{2} \times 3\frac{1}{3} \times 1$
$= 1\frac{2}{3}$ square units.

The Word 'Calculus': 'Calculus' is a Latin word meaning 'stone'. An *abacus* consists of *stones* sliding on bars and was once commonly used to help with arithmetic — this is the origin of the modern word 'calculate'. The word 'calculus' can refer to any systematic method of calculation, but is usually reserved for the twin theories of differentiation and integration studied in this course.

Exercise 9D

1. Find the derivative $\dfrac{dy}{dx}$ of each function. Then find the value of $\dfrac{dy}{dx}$ when $x = 10$.

(a) $y = x^2 + 7x - 10$
(b) $y = x^3 + 3x^2 + 6x + 8$
(c) $y = x^4 + x^2 + 8x$
(d) $y = \frac{1}{3}x^3 - \frac{1}{4}x^2 + x$
(e) $y = 4x - 7$
(f) $y = 7$

2. Find the derivative of each function. You will need to expand the brackets first.

(a) $y = (x + 1)(x - 1)$
(b) $y = x^2(3 - x^3)$
(c) $y = (x - 1)(x - 2)$

3. (a) Differentiate $y = x^5 + x^3 + x$.
(b) Find the gradients of the tangent and normal to the curve at the point where $x = 0$.
(c) Find the gradients of the tangent and normal to the curve at the point where $x = -1$.

4. (a) Differentiate $y = x^3 - 2x$ and find the values of $\frac{dy}{dx}$ when $x = 1$ and when $x = 2$.
(b) Find, correct to the nearest degree where appropriate, the angles of inclination of the tangents at the points where $x = 1$ and $x = 2$. [HINT: You will need the formula gradient $= \tan\alpha$, where α is the angle of inclination of the tangent.]

5. Differentiate each function. Then solve $\frac{dy}{dx} = 0$ and hence find the coordinates of any points on each curve where the tangent is horizontal.
(a) $y = 3 - 2x + x^2$
(b) $y = x^4 + 18x^2$

6. Differentiate $y = x^2 + x$. Then solve $\frac{dy}{dx} = 7$ and hence find the coordinates of any points on $y = x^2 + x$ where the tangent has gradient 7.

7. Differentiate $y = x^3 - 1$. Then solve $\frac{dy}{dx} = 12$ and hence find the coordinates of any points on $y = x^3 - 1$ where the tangent has gradient 12.

8. (a) Differentiate $y = x^2 - 3x$, then find the gradient of the tangent at the point $P(4, 4)$.
(b) Use point–gradient form to find the equation of the tangent to the curve at P.
(c) Write down the gradient of the normal at P and hence find its equation.

DEVELOPMENT

9. Differentiate $y = x^2 - 8x + 15$. Hence find the equations of the tangent and normal at:
(a) $A(1, 8)$
(b) $B(6, 3)$
(c) $C(0, 15)$
(d) $D(4, -1)$

10. In each part, find the derivative $\frac{dy}{dx}$ of the function. Then find the equations of the tangent and normal to the curve at the point indicated.
(a) $y = x^2 - 6x$ at $O(0, 0)$
(b) $y = x^3 - 4x$ at $P(2, 0)$
(c) $y = x^2 - x^4$ at $Q(-1, 0)$
(d) $y = x^3 - 3x + 2$ at $R(1, 0)$

11. (a) Find the equations of the tangent and normal to $y = x^2$ at the point $H(2, 4)$.
(b) Find the points A and B where the tangent and normal respectively meet the y-axis.
(c) Sketch the situation, then find the length AB and the area of $\triangle ABH$.

12. (a) Find the equations of the tangent and normal to $y = 9 - x^2$ at the point $K(1, 8)$.
(b) Find the points A and B where the tangent and normal respectively meet the x-axis.
(c) Sketch the situation, then find the length AB and the area of $\triangle ABK$.

13. (a) Show that the line $y = 3$ meets the parabola $y = 4 - x^2$ at $D(1, 3)$ and $E(-1, 3)$.
(b) Find the equations of the tangents to $y = 4 - x^2$ at D and E.
(c) Find the point where these tangents intersect. Sketch the situation.

14. (a) Differentiate $y = 4 - x^2$, and hence find the coordinates of the points A and B on the curve where the tangent has gradient 2 and -2 respectively. Sketch the situation.
(b) Find the equations of the tangents at A and B, and find their point of intersection.
(c) Find the equations of the normals at A and B, and find their point of intersection.

15. The tangent and normal to $f(x) = x^3$ at $U(1, 1)$ meet the y-axis at P and Q respectively. Find their equations, the coordinates of P and Q, and the area of $\triangle QUP$.

16. Differentiate each function with respect to x. The pronumerals a, b and c are constants.

(a) $y = ax^2 + bx + c$ (b) $y = 3ax^4 + 4bx^3$

CHALLENGE

17. Differentiate $y = x^2 + bx + c$, and hence find b and c, given that:

(a) the parabola passes through the origin, and the tangent there has gradient 7,

(b) the parabola has y-intercept -3 and gradient -2 there,

(c) the parabola is tangent to the x-axis at the point $(5, 0)$,

(d) when $x = 3$ the gradient is 5, and $x = 2$ is a zero,

(e) the parabola is tangent to $3x + y - 5 = 0$ at the point $T(3, -1)$,

(f) the line $3x + y - 5 = 0$ is a normal at the point $T(3, -1)$.

18. Find all values of x for which the tangent at x has positive gradient.

(a) $y = 4x + 2$ (b) $y = 9 - x^2$ (c) $y = x^3 - 3x$ (d) $y = x^4 - 2x^2$

9 E Differentiating Powers with Negative Indices

The formula for the derivative of x^n also applies when n is negative. Using the formula requires the ability to convert quickly between a power with a negative index and a reciprocal of a power. For example:

$$\frac{1}{x} = x^{-1} \qquad \text{and} \qquad 5x^{-2} = \frac{5}{x^2} \qquad \text{and} \qquad \frac{1}{3x^5} = \tfrac{1}{3}x^{-5}$$

The rule is unchanged:

11 DIFFERENTIATING POWERS WITH NEGATIVE INDICES:

'Take the index as a factor and reduce the index by 1'.

For example, if $y = x^{-4}$, then $\dfrac{dy}{dx} = -4x^{-5}$.

WORKED EXERCISE:

Differentiate each function, giving your answer without negative indices.

(a) $f(x) = \dfrac{1}{x}$ (b) $f(x) = \dfrac{5}{x^2}$ (c) $f(x) = \dfrac{1}{3x^5}$

SOLUTION:

(a)
$$\begin{aligned} f(x) &= \frac{1}{x} \\ &= x^{-1} \\ f'(x) &= -x^{-2} \\ &= -\frac{1}{x^2} \end{aligned}$$

(b)
$$\begin{aligned} f(x) &= \frac{5}{x^2} \\ &= 5x^{-2} \\ f'(x) &= -10x^{-3} \\ &= -\frac{10}{x^3} \end{aligned}$$

(c)
$$\begin{aligned} f(x) &= \frac{1}{3x^5} \\ &= \tfrac{1}{3}x^{-5} \\ f'(x) &= -\tfrac{5}{3}x^{-6} \\ &= -\frac{5}{3x^6} \end{aligned}$$

WORKED EXERCISE:

(a) Differentiate $f(x) = \dfrac{16}{x^3} - \dfrac{24}{x^2}$, giving your answer as a single fraction.

(b) Hence find the point on $y = f(x)$ where the tangent is horizontal.

SOLUTION:

(a) $f(x) = 16x^{-3} - 24x^{-2}$ (Write the functions with negative indices.)

$f'(x) = -48x^{-4} + 48x^{-3}$ (Apply the usual rule for differentiation.)

$= -\dfrac{48}{x^4} + \dfrac{48}{x^3}$ (Go back to fraction notation.)

$= \dfrac{-48 + 48x}{x^4}$ (Combine the fractions, using a common denominator.)

$= \dfrac{48(x-1)}{x^4}$ (Complete the factoring of the numerator.)

(b) Hence $f'(x) = 0$ when $\dfrac{48(x-1)}{x^4} = 0$

that is, when $x = 1$.

Since $f(1) = 16 - 24 = -8$, the tangent is horizontal at the point $(1, -8)$.

Dividing Through by the Denominator: This technique is often sufficient to differentiate an expression involving fractions.

WORKED EXERCISE:

Differentiate each function after dividing through by the denominator. Leave your answer to part (b) with a negative index.

(a) $y = \dfrac{x^3 + x^2 + x}{x}$

(b) $y = \dfrac{5 - x^2 + 5x^4}{x^2}$

SOLUTION:

(a) $y = \dfrac{x^3 + x^2 + x}{x}$

$= x^2 + x + 1$

$\dfrac{dy}{dx} = 2x + 1$

(b) $y = \dfrac{5 - x^2 + 5x^4}{x^2}$

$= 5x^{-2} - 1 + 5x^2$

$\dfrac{dy}{dx} = -10x^{-3} + 10x$

Exercise 9E

1. Use the rule that the derivative of x^n is nx^{n-1} to write down the derivative $f'(x)$ of:

(a) $f(x) = x^{-1}$
(b) $f(x) = x^{-5}$
(c) $f(x) = 3x^{-1}$
(d) $f(x) = 5x^{-2}$
(e) $f(x) = -\frac{4}{3}x^{-3}$
(f) $f(x) = 2x^{-2} + \frac{1}{2}x^{-8}$

2. Rewrite each function using a negative power of x, then differentiate it.

(a) $f(x) = \dfrac{1}{x}$
(b) $f(x) = \dfrac{1}{x^2}$
(c) $f(x) = \dfrac{1}{x^4}$
(d) $f(x) = \dfrac{3}{x}$

3. Find the derivative $\dfrac{dy}{dx}$ of each function. Then find the value of $\dfrac{dy}{dx}$ when $x = 1$.

(a) $y = x^4 - x^2 + 1$
(b) $y = \frac{1}{3}x^6 - \frac{1}{2}x^4 + x^2$
(c) $y = \frac{1}{10}x^5 - \frac{1}{6}x^3 + \frac{1}{2}x$
(d) $y = (x+1)(x-1)$
(e) $y = x^2(3 - x^3)$
(f) $y = (2x-1)(x-2)$
(g) $y = 4x^{-1}$
(h) $y = \frac{2}{3}x^{-3}$
(i) $y = x^{-1} - x^{-2}$

4. For each function, first divide through by the denominator, then find the derivative. Leave your answer with negative indices where appropriate.

(a) $y = \dfrac{3x^4 - 5x^2}{x}$ (c) $y = \dfrac{5x^6 + 4x^5}{3x^3}$ (e) $y = \dfrac{1 + 7x}{x^3}$

(b) $y = \dfrac{x^4 - 4x^2}{x^2}$ (d) $y = \dfrac{3x^2 - 1}{x}$ (f) $y = \dfrac{3x^5 - 5x^3 + x}{x^2}$

DEVELOPMENT

5. Rewrite each function using negative powers of x, then differentiate it. Give each final answer in fractional form without negative indices.

(a) $f(x) = \dfrac{1}{x^6} - \dfrac{1}{x^8}$ (c) $f(x) = \dfrac{5}{x^3}$ (e) $f(x) = -\dfrac{7}{x}$ (g) $f(x) = -\dfrac{7}{3x}$

(b) $f(x) = \dfrac{1}{3x}$ (d) $f(x) = \dfrac{1}{5x^4}$ (f) $f(x) = \dfrac{7}{2x}$ (h) $f(x) = -\dfrac{3}{5x^5}$

6. (a) Differentiate $f(x) = \dfrac{1}{x}$, and hence find $f'(3)$ and $f'(\frac{1}{3})$.

(b) Find the two points on $y = \dfrac{1}{x}$ where the tangent has gradient -1.

(c) Find the two points on $y = \dfrac{1}{x}$ where the tangent has gradient -4.

(d) Are there any points on $y = \dfrac{1}{x}$ where the tangent has zero gradient?

(e) Are there any points on $y = \dfrac{1}{x}$ where the tangent has negative gradient?

7. (a) Differentiate $f(x) = -\dfrac{3}{x}$, and hence find $f'(2)$ and $f'(6)$.

(b) Find the two points on the curve where the tangent has gradient 3.

8. (a) Differentiate $f(x) = \dfrac{12}{x}$, and hence find $f'(2)$ and $f'(6)$.

(b) Hence find the tangent and normal to the curve at $M(2, 6)$ and at $N(6, 2)$.

(c) Find the points on the curve where the tangent has gradient -12. Sketch the situation.

9. Differentiate $y = \dfrac{a}{x} - \dfrac{b}{cx^2}$, where a, b and c are constants.

CHALLENGE

10. [Differentiating the function $f(x) = \dfrac{1}{x}$ by first principles]

(a) Show that $\dfrac{f(x+h) - f(x)}{h} = \dfrac{-1}{x(x+h)}$.

(b) Take the limit as $h \to 0$ to show that $f'(x) = -\dfrac{1}{x^2}$.

11. (a) Show that the tangent to $y = x^2 + 15x + 36$ at the point P where $x = a$ has equation $y = (2a + 15)x - a^2 + 36$.

(b) By substituting $O(0, 0)$ into the equation and solving it for a, find the equations of any tangents passing through the origin.

9 F Differentiating Powers with Fractional Indices

The formula for the derivative of x^n applies also when n is a fraction. When using the formula, you will often need to convert between fractional index notation and surd notation:

$$\sqrt{x} = x^{\frac{1}{2}} \qquad \text{and} \qquad \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}} \qquad \text{and} \qquad x^{-1\frac{1}{2}} = \frac{1}{x\sqrt{x}}$$

The rule is unchanged:

12 DIFFERENTIATING POWERS WITH FRACTIONAL INDICES:

'Take the index as a factor and reduce the index by 1'.

For example, if $y = x^{\frac{5}{2}}$, then $\dfrac{dy}{dx} = \frac{5}{2}x^{\frac{3}{2}}$.

WORKED EXERCISE:

Differentiate each function, giving the answer without fractional indices.

(a) $y = 12\sqrt{x}$ (b) $y = \sqrt{16x}$ (c) $y = \dfrac{3}{\sqrt{x}}$

SOLUTION:

(a)
$$\begin{aligned} y &= 12\sqrt{x} \\ &= 12x^{\frac{1}{2}} \\ f'(x) &= 6x^{-\frac{1}{2}} \\ &= \frac{6}{\sqrt{x}} \end{aligned}$$

(b)
$$\begin{aligned} f(x) &= \sqrt{16x} \\ &= 4x^{\frac{1}{2}} \\ f'(x) &= 4 \times \tfrac{1}{2} \times x^{-\frac{1}{2}} \sqrt{7} \\ &= \frac{2}{\sqrt{x}} \end{aligned}$$

(c)
$$\begin{aligned} f(x) &= \frac{3}{\sqrt{x}} \\ &= 3x^{-\frac{1}{2}} \\ f'(x) &= -\tfrac{1}{2} \times 3 \times x^{-1\frac{1}{2}} \\ &= -\frac{3}{2x\sqrt{x}} \end{aligned}$$

WORKED EXERCISE:

Express each function as a sum of powers of x and hence differentiate it.

(a) $y = \dfrac{10x - 6}{\sqrt{x}}$ (b) $y = \dfrac{3x + 4\sqrt{x}}{x}$

SOLUTION:

(a)
$$\begin{aligned} y &= \frac{10x^1}{x^{\frac{1}{2}}} - \frac{6}{x^{\frac{1}{2}}} \\ &= 10x^{\frac{1}{2}} - 6x^{-\frac{1}{2}} \\ y' &= 5x^{-\frac{1}{2}} + 3x^{-1\frac{1}{2}} \end{aligned}$$

(b)
$$\begin{aligned} y &= 3 + \frac{4x^{\frac{1}{2}}}{x^1} \\ &= 3 + 4x^{-\frac{1}{2}} \\ y' &= -2x^{-1\frac{1}{2}} \end{aligned}$$

Exercise 9F

1. Use the rule that the derivative of x^n is nx^{n-1} to write down the derivative $\dfrac{dy}{dx}$ of:

(a) $y = x^{\frac{1}{2}}$ (d) $y = 6x^{\frac{2}{3}}$ (g) $y = 7x^{2\frac{1}{3}}$

(b) $y = x^{-\frac{1}{2}}$ (e) $y = 12x^{-\frac{1}{3}}$ (h) $y = 5x^{-\frac{2}{3}}$

(c) $y = x^{1\frac{1}{2}}$ (f) $y = 4x^{\frac{1}{4}} + 8x^{-\frac{1}{4}}$ (i) $y = -10x^{-0{\cdot}6}$

2. Write each function as a power with a fractional index and then differentiate it.

(a) $y = \sqrt{x}$ (b) $y = \sqrt[3]{x}$ (c) $y = \sqrt[4]{x}$ (d) $y = 10\sqrt{x}$

3. (a) Write $y = x\sqrt{x}$ as a single power of x, and hence differentiate it.
(b) Write $y = x^2\sqrt{x}$ as a single power of x, and hence differentiate it.
(c) Write $y = \dfrac{1}{\sqrt{x}}$ as a single power of x, and hence differentiate it.
(d) Write $y = \dfrac{1}{x\sqrt{x}}$ as a single power of x, and hence differentiate it.

DEVELOPMENT

4. Write each function as a power with a fractional index and then differentiate it.
(a) $y = 24\sqrt{x}$ (d) $y = 30\sqrt[3]{x^2}$ (g) $y = 2x\sqrt{x}$ (j) $y = \dfrac{1}{\sqrt{x}}$
(b) $y = 24\sqrt[3]{x}$ (e) $y = \sqrt{64x}$ (h) $y = 12x\sqrt{x}$ (k) $y = \dfrac{6}{\sqrt{x}}$
(c) $y = \sqrt[3]{x^2}$ (f) $y = \sqrt{25x}$ (i) $y = 4x^2\sqrt{x}$ (l) $y = \dfrac{5}{x\sqrt{x}}$

5. (a) Differentiate $y = \sqrt{x}$ and evaluate $\dfrac{dy}{dx}$ at $x = 1$ and at $x = 4$.
(b) Find the equations of the tangents to $y = \sqrt{x}$ at the points where $x = 1$ and $x = 4$.
(c) Find the equations of the normals to $y = \sqrt{x}$ at the points where $x = 1$ and $x = 4$.

6. (a) Show that the derivative of $y = \dfrac{4}{\sqrt{x}}$ is $\dfrac{dy}{dx} = -\dfrac{2}{x\sqrt{x}}$.
(b) Find the equations of the tangent and normal to the curve at the point where $x = 4$.
(c) Explain why the domain is $x > 0$ and why every tangent has negative gradient.

7. (a) Differentiate $y = 2\sqrt{x}$ and find the gradients of the tangent and normal at $P(4, 4)$.
(b) Find the equations of the tangent and normal at P.
(c) Find the points A and B where the tangent and normal respectively meet the x-axis.
(d) Find the length AB and hence find the area of $\triangle PAB$.

8. Find any points on these curves where the tangent has gradient -1.
(a) $y = \dfrac{1}{x}$ (b) $y = \frac{1}{2}x^{-2}$ (c) $y = -\sqrt{x}$ (d) $y = \frac{1}{4}x^4 - \frac{1}{2}x^2 - x$

9. Divide each function through by the denominator, using index notation, then differentiate.
(a) $y = \dfrac{x^2 + 6x\sqrt{x} + x}{x}$ (b) $y = \dfrac{3x - 3x\sqrt{x} - 8x^2}{x}$ (c) $y = \dfrac{3x - 2\sqrt{x}}{\sqrt{x}}$

CHALLENGE

10. [Differentiating $f(x) = \sqrt{x}$ by first principles]
(a) Write down $\dfrac{f(x+h) - f(x)}{h}$ for the function $f(x) = \sqrt{x}$, then multiply the top and bottom of it by $\sqrt{x+h} + \sqrt{x}$. Hence show that $\dfrac{f(x+h) - f(x)}{h} = \dfrac{1}{\sqrt{x+h} + \sqrt{x}}$.
(b) Take the limit as $h \to 0$ to prove that $f'(x) = \dfrac{1}{2\sqrt{x}}$.

11. (a) Find the gradient of the tangent to $y = x^2 - 10x + 9$ at the point P where $x = a$.
(b) Find the value of y at $x = a$, and hence show that the tangent at P has equation $y = (2a - 10)x - a^2 + 9$.
(c) By substituting $(0, 0)$ into the tangent, find the value of a when the tangent passes through the origin, and hence find the equations of the tangents through the origin.

9 G The Chain Rule

Sections 9G, 9H and 9I develop three methods that extend the rules for differentiation to cover compound functions of various types.

A Chain of Functions: The semicircle function $y = \sqrt{25 - x^2}$ is the *composition* of two functions — 'square and subtract from 25', followed by 'take the positive square root'. We can represent the situation by a *chain of functions*:

x				u				y
0	$\longrightarrow$	Square and subtract from 25	$\longrightarrow$	25	$\longrightarrow$	Take the positive square root	$\longrightarrow$	5
3	$\longrightarrow$		$\longrightarrow$	16	$\longrightarrow$		$\longrightarrow$	4
-4	$\longrightarrow$		$\longrightarrow$	9	$\longrightarrow$		$\longrightarrow$	3
x	$\longrightarrow$		$\longrightarrow$	$25 - x^2$	$\longrightarrow$		$\longrightarrow$	$\sqrt{25 - x^2}$

The middle column is the output of the first function 'subtract the square from 25', and is then the input of the second function 'take the positive square root'. This decomposition of the original function $y = \sqrt{25 - x^2}$ into the chain of functions may be expressed as follows:

'Let $u = 25 - x^2$. Then $y = \sqrt{u}$.'

The Chain Rule: Suppose then that y is a function of u, where u is a function of x.

$$\begin{aligned}\frac{dy}{dx} &= \lim_{\delta x \to 0} \frac{\delta y}{\delta x} \\ &= \lim_{\delta x \to 0} \left(\frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x} \right) \qquad \text{(multiplying top and bottom by } \delta u\text{)} \\ &= \lim_{\delta u \to 0} \frac{\delta y}{\delta u} \times \lim_{\delta x \to 0} \frac{\delta u}{\delta x} \qquad \text{(because } \delta u \to 0 \text{ as } \delta x \to 0\text{)} \\ &= \frac{dy}{du} \times \frac{du}{dx}.\end{aligned}$$

Although the proof uses limits, the usual attitude to this rule is that 'the du's cancel out'. The chain rule should be remembered in this form:

13 THE CHAIN RULE: Suppose that y is a function of u, where u is a function of x.

Then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

WORKED EXERCISE:

Use the chain rule to differentiate each function.

(a) $(x^2 + 1)^6$ (b) $7(3x + 4)^5$ (c) $(ax + b)^n$

NOTE: The working in the right-hand column is the recommended setting out of the calculation. The calculation should begin with that working, because the first step is the decomposition of the function into a chain of two functions.

SOLUTION:

(a) Let $y = (x^2 + 1)^6$.

Then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$= 6(x^2 + 1)^5 \times 2x$

$= 12x(x^2 + 1)^5$.

Let $u = x^2 + 1$.

Then $y = u^6$.

Hence $\dfrac{du}{dx} = 2x$

and $\dfrac{dy}{du} = 6u^5$.

(b) Let $y = 7(3x+4)^5$.

Then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$= 35(3x+4)^4 \times 3$

$= 105(3x+4)^4$.

Let $u = 3x + 4$.

Then $y = 7u^5$.

Hence $\dfrac{du}{dx} = 3$

and $\dfrac{dy}{du} = 35u^4$.

(c) Let $y = (ax+b)^n$.

Then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$= n(ax+b)^{n-1} \times a$

$= an(ax+b)^{n-1}$.

Let $u = ax + b$.

Then $y = u^n$.

Hence $\dfrac{du}{dx} = a$

and $\dfrac{dy}{du} = nu^{n-1}$.

Powers of a Linear Function: Part (c) of the previous worked exercise should be remembered as a formula for differentiating any linear function of x raised to a power.

14 POWERS OF A LINEAR FUNCTION: $\dfrac{d}{dx}(ax+b)^n = an(ax+b)^{n-1}$

WORKED EXERCISE:

Differentiate: (a) $y = (4x-1)^7$ (b) $y = \dfrac{1}{3x-8}$ (c) $y = \sqrt{5x+3}$

SOLUTION:

(a) $\dfrac{dy}{dx} = 28(4x-1)^6$ (Here $a = 4$, $b = -1$ and $n = 7$.)

(b) $y = (3x-8)^{-1}$, (Convert to index form.)

$\dfrac{dy}{dx} = -3(3x-8)^{-2}$ (Here $a = 3$, $b = -8$ and $n = -1$.)

(c) $y = (5x+3)^{\frac{1}{2}}$, (Convert to index form.)

$\dfrac{dy}{dx} = 5 \times \frac{1}{2} \times (5x+3)^{-\frac{1}{2}}$ (Here $a = 5$, $b = 3$ and $n = \frac{1}{2}$.)

$= \dfrac{5}{2\sqrt{5x+3}}$

Exercise 9G

1. Use the chain rule $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$ to differentiate each function. In each example, first identify u as a function of x, and y as a function of u.

(a) $y = (3x+7)^4$
(b) $y = (5x-9)^6$
(c) $y = (5-4x)^7$
(d) $y = (1-x)^4$
(e) $y = (x^2+1)^{12}$
(f) $y = (x^2-2)^7$
(g) $y = (5-x^2)^3$
(h) $y = (3x^2+7)^7$
(i) $y = (5-2x^2)^4$
(j) $y = (x^4+1)^4$
(k) $y = (3x^3-7)^5$
(l) $y = (5-8x^5)^{10}$

2. Use the chain rule to differentiate these functions.

(a) $y = (7x+2)^{-1}$
(b) $y = 3(x-1)^{-2}$
(c) $y = (x^3-12)^{-4}$
(d) $y = (5x^2-2)^{-3}$
(e) $y = 8(7-x^2)^4$
(f) $y = -3(x^3+x+1)^6$
(g) $y = (1-x-x^2-x^3)^4$
(h) $y = (x^3-x^2)^{-5}$
(i) $y = (x^2+3x+1)^{-9}$

3. Use the standard form $\frac{d}{dx}(ax+b)^n = an(ax+b)^{n-1}$ to differentiate:

(a) $y=(5x-7)^5$
(b) $y=(7x+3)^7$
(c) $y=9(5x+3)^4$
(d) $y=(4-3x)^7$
(e) $y=11(3-x)^2$
(f) $y=(4x-5)^{-7}$
(g) $y=2(3x+7)^{-5}$
(h) $y=(10-3x)^{-4}$
(i) $y=3(5-7x)^{-4}$

4. (a) Differentiate $y=(x-3)^2$ by expanding the RHS and differentiating each term.
(b) Differentiate $y=(x-3)^2$, using the chain rule.

5. (a) Differentiate $y=3(1-2x)^2$ by expanding the RHS and differentiating each term.
(b) Differentiate $y=3(1-2x)^2$, using the chain rule.

6. Rewrite each function with negative index notation, then differentiate it using the formula $\frac{d}{dx}(ax+b)^n = an(ax+b)^{n-1}$. Give each answer as a fraction without negative indices.

(a) $y=\frac{1}{2x+7}$
(b) $y=\frac{1}{2-x}$
(c) $y=\frac{1}{3+5x}$
(d) $y=\frac{7}{4-3x}$
(e) $y=\frac{4}{(3x-1)^5}$
(f) $y=-\frac{5}{(x+1)^3}$

DEVELOPMENT

7. (a) Find the derivative of $y=(5x-4)^4$.
(b) Find the values of y and $\frac{dy}{dx}$ at the point P where $x=1$.
(c) Hence find the equations of the tangent and normal at P.

8. Find the equation of the tangent to each curve at the point P where $x=1$.
(a) $y=(x^2+1)^3$
(b) $y=(3x-2)^{-1}$
(c) $y=(x^2+1)^{-1}$

9. Differentiate each function using the chain rule. Then solve $\frac{dy}{dx}=0$ to find the x-coordinate and y-coordinate of any points where the tangent is horizontal.

(a) $y=(x-5)^4$
(b) $y=(x^2-1)^3$
(c) $y=(2x+x^2)^5$
(d) $y=\frac{1}{5x+2}$
(e) $y=4+(x-5)^6$
(f) $y=\frac{1}{1+x^2}$

10. Find the x-coordinates of any points on $y=(4x-7)^3$ where the tangent is:
(a) parallel to $y=108x+7$,
(b) perpendicular to $y=-\frac{1}{12}x+2$.

11. Differentiate by the chain rule, after converting to index form first if necessary:

(a) $y=(5x+4)^{\frac{1}{2}}$
(b) $y=\sqrt{3-2x}$
(c) $y=7\sqrt{x^2+1}$
(d) $y=\sqrt{9-x^2}$
(e) $y=-\sqrt{a^2-x^2}$
(f) $y=\sqrt{x+4}$
(g) $y=\sqrt{4-3x}$
(h) $y=\sqrt{mx+b}$
(i) $y=(5-x)^{-\frac{1}{2}}$

12. Differentiate each function, then find the tangent and normal at the point where $x=2$.
(a) $y=\sqrt{2x+5}$
(b) $y=\sqrt{x^2-3}$

13. Differentiate each function, then find the x-coordinate and y-coordinate of any points where the tangent is horizontal.
(a) $y=\sqrt{3-2x}$
(b) $y=\sqrt{x^2-2x+5}$
(c) $y=\sqrt{x^2-2x}$

14. (a) Find the derivative of the semicircle $y = \sqrt{169 - x^2}$, and sketch the semicircle.
(b) Show that the tangent at $P(12, 5)$ is $12x + 5y = 169$.
(c) Verify that the tangent at P is perpendicular to the radius OP.

CHALLENGE

15. Differentiate:

(a) $y = \left(\sqrt{x} - 3\right)^{11}$
(b) $y = 3\sqrt{4 - \frac{1}{2}x}$
(c) $y = \dfrac{3}{1 - x\sqrt{2}}$
(d) $y = (5 - x)^{-\frac{1}{2}}$
(e) $y = \dfrac{-a}{\sqrt{1 + ax}}$
(f) $y = \dfrac{b}{\sqrt{c - x}}$
(g) $y = -4\left(x + \dfrac{1}{x}\right)^4$
(h) $y = \left(\sqrt{x} + \dfrac{1}{\sqrt{x}}\right)^6$

16. (a) Differentiate $y = (x - a)^3$, then find the value of a if $\dfrac{dy}{dx} = 12$ when $x = 6$.
(b) Differentiate $y = \dfrac{1}{x + a}$, then find the value of a if $\dfrac{dy}{dx} = -1$ when $x = 6$.

17. Differentiate $y = a(x + b)^2 - 8$, then find a and b if the parabola:
(a) passes through the origin with gradient 16,
(b) has tangent $y = 2x$ at the point $P(4, 8)$.

18. (a) Find the equation of the tangent to $y = \dfrac{1}{x - 4}$ at the point L where $x = b$.
(b) Hence find the equation of the tangent to the curve passing through:
(i) the origin,
(ii) $W(6, 0)$.

9 H The Product Rule

The product rule extends the methods for differentiation to functions that are products of two simpler functions. For example,

$$y = x(x - 10)^4$$

is the product of x and $(x - 10)^4$.

Statement of the Product Rule: The proof of the product rule is given in the appendix at the end of this chapter.

15 THE PRODUCT RULE: Suppose that the function

$$y = u \times v$$

is the *product* of two functions u and v, each of which is a function of x. Then

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx} \qquad \text{or} \qquad y' = vu' + uv'.$$

The second form uses the convention of the dash $'$ to represent differentiation with respect to x. That is, $y' = \dfrac{dy}{dx}$ and $u' = \dfrac{du}{dx}$ and $v' = \dfrac{dv}{dx}$.

Using the Product Rule: The working in the right-hand column is the recommended setting out. The first step is to decompose the function into the product of two simpler functions.

WORKED EXERCISE:
Differentiate each function, writing the result in fully factored form. Then state for what value(s) of x the derivative is zero.

(a) $x(x-10)^4$ (b) $x^2(3x+2)^3$

SOLUTION:

(a) Let $y = x(x-10)^4$.

$$\text{Then } \frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$
$$= (x-10)^4 \times 1 + x \times 4(x-10)^3$$
$$= (x-10)^3(x-10+4x)$$
$$= (x-10)^3(5x-10)$$
$$= 5(x-10)^3(x-2),$$

so the derivative is zero for $x = 10$ and for $x = 2$.

Let $u = x$
and $v = (x-10)^4$.
Then $\dfrac{du}{dx} = 1$
and $\dfrac{dv}{dx} = 4(x-10)^3$.

(b) Let $y = x^2(3x+2)^3$.

$$\text{Then } y' = vu' + uv'$$
$$= 2x \times (3x+2)^3 + x^2 \times 9(3x+2)^2$$
$$= x(3x+2)^2(6x+4+9x)$$
$$= x(3x+2)^2(15x+4),$$

so the derivative is zero for $x = 0$, $x = -\frac{2}{3}$ and for $x = -\frac{4}{15}$.

Let $u = x^2$
and $v = (3x+2)^3$.
Then $u' = 2x$
and $v' = 9(3x+2)^2$.

NOTE: The product rule can seem difficult to use with the algebraic functions under consideration at present, because the calculations can easily become quite involved. The rule will seem more straightforward later in the context of exponential and trigonometric functions.

Exercise 9H

1. (a) Differentiate $y = x^3(x-2)$ by expanding the product and differentiating each term.
(b) Differentiate $y = x^3(x-2)$, using the product rule with $u = x^3$ and $v = x-2$. Use the setting out shown in the worked exercises above.

2. (a) Differentiate $y = (2x+1)(x-5)$ by expanding and then differentiating each term.
(b) Differentiate $y = (2x+1)(x-5)$, using the product rule with $u = 2x+1$ and $v = x-5$.

3. (a) Differentiate $y = (x^2-3)(x^2+3)$ by expanding and then differentiating each term.
(b) Differentiate $y = (x^2-3)(x^2+3)$, using the product rule with $u = x^2-3$ and $v = x^2+3$.

4. (a) Given that $u = x^4$ and $v = (2x-1)^5$, find $\dfrac{du}{dx}$ and $\dfrac{dv}{dx}$.
(b) Use the product rule and the results of part (a) to show that if $y = x^4(2x-1)^5$, then $\dfrac{dy}{dx} = 4x^3(2x-1)^5 + 10x^4(2x-1)^4$.
(c) By taking out the common factor $2x^3(2x-1)^4$, show that $\dfrac{dy}{dx} = 2x^3(2x-1)^4(9x-2)$.
(d) Hence find the x-coordinates of the points on the curve where the tangent is horizontal.

5. Differentiate these functions by the product rule, giving your answers in factored form.
(a) $y = x(3x+5)^3$ (b) $y = x^2(x-1)^3$ (c) $y = x^4(1-5x)^6$

6. Show that the derivative of $y = x(1-x)^6$ is $\dfrac{dy}{dx} = (7x-1)(x-1)^5$, then find the tangent and normal to the curve at the origin.

7. (a) Differentiate the function $y = x^3(1-x)^5$.
(b) Hence show that the curve has horizontal tangents at $x = 0$, $x = 1$ and $x = \frac{3}{8}$.

DEVELOPMENT

8. Differentiate these functions using the product rule, identifying the factors u and v in each example. Express the answers in fully factored form, and state the values of x for which the derivative is zero.
(a) $y = x(x-1)^4$ (d) $y = x(3-2x)^5$ (g) $y = x^5(1-x)^7$
(b) $y = x(x+5)^5$ (e) $y = x^3(x+1)^4$ (h) $y = (x-1)(x-2)^3$
(c) $y = x(4-3x)^5$ (f) $y = x^3(3x-2)^4$ (i) $y = (x+2)(x+5)^6$

9. (a) Show that the derivative of $y = (2x-1)^3(x-2)^4$ is $\dfrac{dy}{dx} = 2(2x-1)^2(x-2)^3(7x-8)$.
(b) Hence find the tangent and normal to the curve at the point $A(1,1)$.

10. Show that the derivative of $y = (x+1)(x+3)^3$ is $\dfrac{dy}{dx} = 2(2x+3)(x+3)^2$, then find the tangent and normal to the curve at $A(-1,0)$.

11. (a) Differentiate $y = (x^2+1)^5$, using the chain rule. Then use the product rule to differentiate $y = x(x^2+1)^5$.
(b) Differentiate $y = (1-x^2)^4$. Hence differentiate $y = 2x^3(1-x^2)^4$.
(c) Differentiate $y = (x^2+x+1)^3$, then differentiate $y = -2(x^2+x+1)^3x$.
(d) Similarly, differentiate $y = x(4-9x^4)^4$.

12. (a) Differentiate $y = (x^2-10)^3x^4$, using the chain rule to differentiate the first factor.
(b) Hence find the points on the curve where the tangent is horizontal.

13. Differentiate each function, using the product rule. Then combine terms, using a common denominator, and factor the numerator completely. State the values of x for which the derivative is zero.
(a) $y = 6x\sqrt{x+1}$ (b) $y = -4x\sqrt{1-2x}$ (c) $y = 10x^2\sqrt{2x-1}$

CHALLENGE

14. Find the derivative of each function below, giving the answer in fully factored form. Then state the values of x for which the derivative is zero.
(a) $y = (x+1)^3(x+2)^4$ (b) $y = (2x-3)^4(2x+3)^5$ (c) $y = x\sqrt{1-x^2}$

15. (a) Differentiate $y = a(x-1)(x-5)$, using the product rule, then sketch the curve.
(b) Show that the tangents at the x-intercepts $(1,0)$ and $(5,0)$ have opposite gradients.
(c) Find the equations of these tangents and their point M of intersection.
(d) Find the point V where the tangent is horizontal.
(e) Show that M is vertically below V and twice as far from the x-axis.

9 I The Quotient Rule

The quotient rule extends the formulae for differentiation to functions that are quotients of two simpler functions. For example,

$$y = \frac{2x+1}{2x-1} \quad \text{is the quotient of the two functions } 2x+1 \text{ and } 2x-1.$$

Statement of the Quotient Rule: The proof of the quotient rule is also given in the appendix at the end of this chapter.

16 THE QUOTIENT RULE: Suppose that the function

$$y = \frac{u}{v}$$

is the *quotient* of two functions u and v, each of which is a function of x. Then

$$\frac{dy}{dx} = \frac{v\,\dfrac{du}{dx} - u\,\dfrac{dv}{dx}}{v^2} \qquad \text{or} \qquad y' = \frac{vu' - uv'}{v^2}.$$

WORKED EXERCISE:

Differentiate each function, stating any values of x where the derivative is zero.

(a) $\dfrac{2x+1}{2x-1}$ (b) $\dfrac{x}{x^2+1}$

SOLUTION:

(a) Let $y = \dfrac{2x+1}{2x-1}$.

Then
$$\frac{dy}{dx} = \frac{v\,\dfrac{du}{dx} - u\,\dfrac{dv}{dx}}{v^2}$$
$$= \frac{2(2x-1) - 2(2x+1)}{(2x-1)^2}$$
$$= \frac{-4}{(2x-1)^2}, \text{ which is never zero.}$$

Let $u = 2x+1$
and $v = 2x-1$.
Then $\dfrac{du}{dx} = 2$
and $\dfrac{dv}{dx} = 2$.

(b) Let $y = \dfrac{x}{x^2+1}$.

Then
$$y' = \frac{vu' - uv'}{v^2}$$
$$= \frac{(x^2+1) - 2x^2}{(x^2+1)^2}$$
$$= \frac{1 - x^2}{(x^2+1)^2}$$
$$= \frac{(1-x)(1+x)}{(x^2+1)^2}, \text{ which is zero when } x = 1 \text{ or } x = -1.$$

Let $u = x$
and $v = x^2+1$.
Then $u' = 1$
and $v' = 2x$.

NOTE: Both these functions could have been differentiated using the product rule after writing them as $(2x+1)(2x-1)^{-1}$ and $x(x^2+1)^{-1}$. The quotient rule, however, makes the calculation much easier.

Exercise 9I

1. (a) Given that $u = 2x + 3$ and $v = 3x + 2$, find u' and v'.

(b) Hence use the quotient rule $y' = \dfrac{vu' - uv'}{v^2}$ to show that the derivative of the function $y = \dfrac{2x+3}{3x+2}$ is $y' = -\dfrac{5}{(3x+2)^2}$.

2. Differentiate each function, using the quotient rule $y' = \dfrac{vu' - uv'}{v^2}$. Identify u and v first and use the setting out shown in the worked exercises on the previous page.

(a) $y = \dfrac{x}{x+1}$ (c) $y = \dfrac{x}{1-3x}$ (e) $y = \dfrac{x+2}{x-2}$ (g) $y = \dfrac{3x-2}{2x-3}$

(b) $y = \dfrac{2x}{x+2}$ (d) $y = \dfrac{x+1}{x-1}$ (f) $y = \dfrac{x-2}{x+2}$ (h) $y = \dfrac{5-4x}{5+4x}$

3. Differentiate each function, using the quotient rule. Express your answer in fully factored form, then state any values of x for which the tangent is horizontal.

(a) $y = \dfrac{x^2}{x+1}$ (c) $y = \dfrac{x^2}{1-x}$ (e) $y = \dfrac{x^2-1}{x^2+1}$

(b) $y = \dfrac{x}{3-x^2}$ (d) $y = \dfrac{x}{1-x^2}$ (f) $y = \dfrac{x^2-9}{x^2-4}$

4. Differentiate $y = \dfrac{1}{3x-2}$ in two different ways.

(a) Use the chain rule with $u = 3x - 2$ and $y = \dfrac{1}{u}$. This is the better method.

(b) Use the quotient rule with $u = 1$ and $v = 3x - 2$. This method is longer.

DEVELOPMENT

5. (a) Show that the derivative of $y = \dfrac{x}{5-3x}$ is $\dfrac{dy}{dx} = \dfrac{5}{(5-3x)^2}$.

(b) Hence find the gradient of the tangent at $K(2, -2)$ and its angle of inclination.

(c) Find the equations of the tangent and normal at K.

6. (a) Show that the derivative of $y = \dfrac{x^2-4}{x-1}$ is $\dfrac{dy}{dx} = \dfrac{x^2-2x+4}{(x-1)^2}$.

(b) Hence find the gradient of the tangent at $L(4, 4)$ and its angle of inclination.

(c) Find the equations of the tangent and normal at L.

7. (a) Find the equation of the tangent to $y = \dfrac{x}{x+1}$ at the origin $O(0, 0)$.

(b) Show that the tangent to $y = \dfrac{x}{x+1}$ at the point $P(1, \tfrac{1}{2})$ is $y = \tfrac{1}{4}x + \tfrac{1}{4}$.

(c) Find the points A and B where the tangent in part (b) crosses the x-axis and y-axis respectively.

(d) Find the area of the triangle ABO, where O is the origin.

(e) Use simultaneous equations to find where the tangents at O and P intersect.

8. (a) Differentiate $y = \dfrac{x^2}{x+1}$ and hence find the value of c if $y' = 0$ at $x = c$.

(b) Differentiate $y = \dfrac{x^2+k}{x^2-k}$ and hence find the value of k if $y' = 1$ at $x = -3$.

CHALLENGE

9. Differentiate, using the most appropriate method. Factor each answer completely.

(a) $(3x-7)^4$ (e) $x(4-x)^3$ (i) $(x^3+5)^2$ (m) $x\sqrt{x}+x^2\sqrt{x}$

(b) $\dfrac{x^2+3x-2}{x}$ (f) $\dfrac{3-x}{3+x}$ (j) $\dfrac{x^2+x+1}{2\sqrt{x}}$ (n) $\left(x-\dfrac{1}{x}\right)^2$

(c) $(2x+3)(2x-3)$ (g) $(x^4-1)^5$ (k) $\frac{2}{3}x^2(x^3-1)$ (o) $x^3(x-1)^8$

(d) $\dfrac{1}{x^2-9}$ (h) $\dfrac{1}{\sqrt{2-x}}$ (l) $\dfrac{x}{x+5}$ (p) $\left(\sqrt{x}+\dfrac{1}{\sqrt{x}}\right)^2$

10. (a) Show that the derivative of $y=\dfrac{x-\alpha}{x-\beta}$ is $y'=\dfrac{\alpha-\beta}{(x-\beta)^2}$ (where α and β are constants).

(b) Explain why all tangents have positive gradient when $\alpha>\beta$, and all tangents have negative gradient when $\alpha<\beta$.

9 J Limits and Continuity

A tangent can only be drawn at a point on a curve if the curve has no break at the point and no sharp corner. Sections 9J and 9K will make these ideas more precise.

This will require some further work with limits, which were used in Section 9B to develop first-principles differentiation.

Some Rules for Limits: Here is the informal definition of a limit that we are using in this course.

17 DEFINITION OF THE LIMIT OF A FUNCTION:

$\lim_{x\to a} f(x)=\ell$ means that $f(x)$ is 'as close as we like' to ℓ when x is near a.

Here are some of the assumptions we have been making about limits.

18

LIMIT OF A SUM: $\lim_{x\to a}\big(f(x)+g(x)\big)=\lim_{x\to a} f(x)\ +\ \lim_{x\to a} g(x)$,

LIMIT OF A MULTIPLE: $\lim_{x\to a} kf(x)=k\times\lim_{x\to a} f(x)$,

LIMIT OF A PRODUCT: $\lim_{x\to a} f(x)g(x)=\lim_{x\to a} f(x)\times\lim_{x\to a} g(x)$,

LIMIT OF A QUOTIENT: $\lim_{x\to a}\dfrac{f(x)}{g(x)}=\dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, provided $\lim_{x\to a} g(x)\neq 0$.

WORKED EXERCISE:

(a) Find $\lim_{x\to 0}\dfrac{x}{x^2+x}$. (b) Find $\lim_{x\to 1}\dfrac{x^2-1}{x-1}$.

SOLUTION:

(a) $\lim_{x\to 0}\dfrac{x}{x^2+x}$

$=\lim_{x\to 0}\dfrac{x}{x(x+1)}$

$=\lim_{x\to 0}\dfrac{1}{x+1}$, since $x\neq 0$,

$=1$

(b) $\lim_{x\to 1}\dfrac{x^2-1}{x-1}$

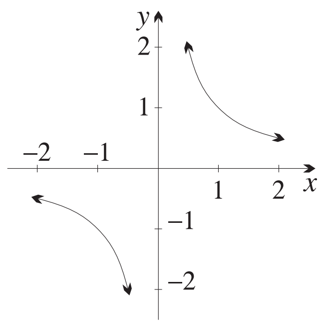
$=\lim_{x\to 1}\dfrac{(x-1)(x+1)}{x-1}$

$=\lim_{x\to 1}(x+1)$, since $x\neq 1$,

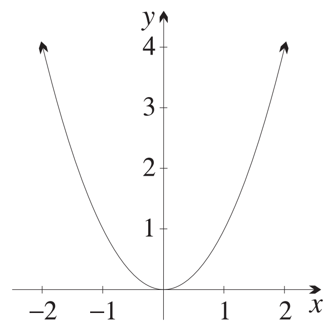
$=2$

Continuity at a Point — Informal Definition: As discussed in Chapter Four, continuity at a point means that there is no break in the curve around that point.

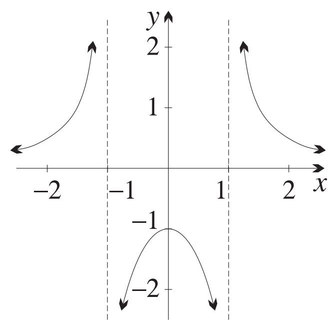
19 CONTINUITY AT A POINT — INFORMAL DEFINITION:
- A function $f(x)$ is called *continuous at* $x = a$ if the graph of $y = f(x)$ can be drawn through the point where $x = a$ without any break.
- If there is a break in the curve, we say that there is a *discontinuity* at $x = a$.



EXAMPLE: $y = \dfrac{1}{x}$ has a discontinuity at $x = 0$, and is continuous everywhere else.



EXAMPLE: $y = x^2$ is continuous for all values of x.

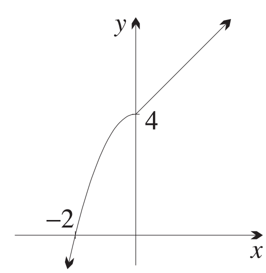


EXAMPLE: $y = \dfrac{1}{x^2 - 1}$ has discontinuities at $x = 1$ and at $x = -1$, and is continuous everywhere else.

Piecewise-Defined Functions: A *piecewise-defined function* has different definitions in different parts of its domain. Here is an example:

$$f(x) = \begin{cases} 4 - x^2, & \text{for } x \le 0, \\ 4 + x, & \text{for } x > 0. \end{cases}$$

Clearly the two pieces of this graph join up at the point $(0, 4)$, making the function *continuous* at $x = 0$.



The Limits where the Pieces Join: A more formal way of talking about this situation involves using limits to analyse the behaviour of $f(x)$ on each side of $x = 0$. Two limits are needed.

The first is the limit on the left, where x is near zero and less than zero:

$$\lim_{x \to 0^-} f(x), \quad \text{meaning 'the limit as } x \text{ approaches 0 from the negative side'.}$$

The second is the limit on the right of $x = 0$, where x is near zero and greater than zero:

$$\lim_{x \to 0^+} f(x), \quad \text{meaning 'the limit as } x \text{ approaches 0 from the positive side'.}$$

Here are the calculations needed for these two limits, as well as the value of $f(x)$ at $x = 0$:

$$\begin{aligned} \lim_{x \to 0^-} f(x) &= \lim_{x \to 0^-} (4 - x^2) \\ &= 4 \end{aligned} \qquad \begin{aligned} \lim_{x \to 0^+} f(x) &= \lim_{x \to 0^+} (4 + x) \\ &= 4 \end{aligned} \qquad \begin{aligned} f(0) &= 4 - 0^2 \\ &= 4 \end{aligned}$$

The formal reason why $f(x)$ is continuous at $x = 0$ is that these three values all exist and are all equal.

Continuity at a Point — Formal Definition: Here then is a somewhat stricter definition of continuity at a point, using the machinery of limits.

20 CONTINUITY AT A POINT — FORMAL DEFINITION:
A function $f(x)$ is called *continuous* at $x = a$ if

$$\lim_{x \to a^-} f(x) \quad \text{and} \quad \lim_{x \to a^+} f(x) \quad \text{and} \quad f(a)$$

all exist and are all equal.

WORKED EXERCISE:

(a) Find $\lim_{x \to 1^+} f(x)$, $\lim_{x \to 1^-} f(x)$ and $f(1)$ for the function $f(x) = \begin{cases} 2^x, & \text{for } x < 1, \\ x^2, & \text{for } x \geq 1. \end{cases}$

(b) Sketch the function, and state whether the function is continuous at $x = 1$.

(c) State the domain and range of the function.

SOLUTION:

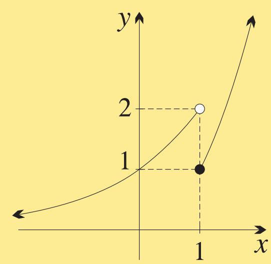
(a) $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 2^x = 2$,

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x^2 = 1$,

$f(1) = 1$.

(b) Hence the function is not continuous at $x = 1$.

(c) The domain is all real x, and the range is $y > 0$.



An Assumption of Continuity: It is intuitively obvious that a function like $y = x^2$ is continuous for every value of x. A full treatment of this is not possible in this course. Instead, we will make a general assumption of continuity, loosely stated as follows.

21 ASSUMPTION: The functions in this course are continuous for every value of x in their domain, except where there is an obvious problem.

Exercise 9J

NOTE: In every graph, every curve must end with a closed circle if the endpoint is included, an open circle if it is not, or an arrow if it continues forever. After working on these limit questions, you should revise differentiation from first principles in Exercise 9B.

1. (a) On one set of axes, sketch $y = 1 + x^2$ and $y = 1 - x$, showing all intercepts.

(b) Hence sketch the curve defined piecewise by $y = \begin{cases} 1 + x^2, & \text{for } x \geq 0 \\ 1 - x, & \text{for } x < 0. \end{cases}$

(c) Is $f(x)$ continuous at $x = 0$?

(d) Write down the domain and range of $f(x)$.

2. (a) On one set of axes, sketch $y = x - 1$ and $y = 2 - x$, showing all intercepts.

(b) Hence sketch the curve defined piecewise by $y = \begin{cases} x - 1, & \text{for } x > 1 \\ 2 - x, & \text{for } x \leq 1. \end{cases}$

(c) Is $f(x)$ continuous at $x = 1$?

(d) Write down the domain and range of $f(x)$.

3. (a) Explain why the function $y = \dfrac{x^2 - 25}{x - 5}$ is undefined at $x = 5$.

(b) Factor the numerator in $y = \dfrac{x^2 - 25}{x - 5}$, then simplify the function for $x \neq 5$.

(c) Hence find $\displaystyle\lim_{x \to 5} \frac{x^2 - 25}{x - 5}$.

4. By first factoring top and bottom and cancelling any common factors, find:

(a) $\displaystyle\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$ (c) $\displaystyle\lim_{h \to 0} \frac{h^3 - 9h^2 + h}{h}$ (e) $\displaystyle\lim_{x \to 0} \frac{x^3 + 6x}{x^2 - 3x}$

(b) $\displaystyle\lim_{x \to -2} \frac{x^2 - 4}{x + 2}$ (d) $\displaystyle\lim_{h \to -3} \frac{h^2 - 9}{h^2 + 7h + 12}$ (f) $\displaystyle\lim_{x \to 0} \frac{x^4 - 4x^2}{x^2 - 2x}$

DEVELOPMENT

5. For each function below, find $\displaystyle\lim_{x \to 2^-} f(x)$, $\displaystyle\lim_{x \to 2^+} f(x)$ and $f(2)$, and draw a conclusion about continuity at $x = 2$. Then sketch the curve and state the domain and range.

(a) $f(x) = \begin{cases} x^3, & \text{for } x \le 2, \\ 10 - x, & \text{for } x > 2. \end{cases}$

(b) $f(x) = \begin{cases} 3^x, & \text{for } x < 2, \\ 13 - x^2, & \text{for } x > 2, \\ 4, & \text{for } x = 2. \end{cases}$

(c) $f(x) = \begin{cases} \frac{1}{x}, & \text{for } 0 < x < 2, \\ 1 - \frac{1}{4}x, & \text{for } x > 2, \\ \frac{1}{2}, & \text{for } x = 2. \end{cases}$

(d) $f(x) = \begin{cases} x, & \text{for } x < 2, \\ 2 - x, & \text{for } x > 2, \\ 2, & \text{for } x = 2. \end{cases}$

6. Factor the top and bottom of each function and cancel, noting first the value(s) of x where the function is undefined. Then sketch the curve and state its domain and range.

(a) $y = \dfrac{x^2 + 2x + 1}{x + 1}$ (b) $y = \dfrac{x^4 - x^2}{x^2 - 1}$ (c) $y = \dfrac{x - 3}{x^2 - 4x + 3}$ (d) $y = \dfrac{3x + 3}{x + 1}$

7. (a) Find: (i) $\displaystyle\lim_{h \to 0} \frac{h^2 + 2xh}{h}$ (ii) $\displaystyle\lim_{h \to 0} \frac{h^2 + 2xh + 5h}{h}$ (iii) $\displaystyle\lim_{h \to 0} \frac{4h - h^2 - 2xh}{h}$

(b) Use part (a) and the definition $f'(x) = \displaystyle\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ of the derivative to find, by first principles, the derivatives of:

(i) $f(x) = x^2$, (ii) $f(x) = x^2 + 5x$, (iii) $f(x) = 1 + 4x - x^2$.

CHALLENGE

8. (a) Show that the secant joining the points $P(x, f(x))$ and $Q\big(x+h, f(x+h)\big)$ on the curve $y = x^2 - x + 1$ has gradient $2x + h - 1$.

(b) Take $\displaystyle\lim_{h \to 0}$ (gradient of PQ), and hence show that $f'(x) = 2x - 1$.

9. Factor the denominator to find all zeroes and discontinuities of each function.

(a) $y = \dfrac{x}{x - 3}$ (b) $y = \dfrac{x}{x^2 - 6x - 7}$ (c) $y = \dfrac{x^3 - x}{x^3 - 9x}$

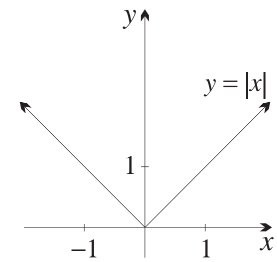
10. (a) Draw up a table of values for $y = \dfrac{|x|}{x}$ and sketch the curve.

(b) Write down $\displaystyle\lim_{x \to 0^+} y$, $\displaystyle\lim_{x \to 0^-} y$ and $y(0)$, and discuss the continuity at $x = 0$.

(c) State the domain and range of the function.

9 K Differentiability

A tangent can only be drawn at a point P on a curve if the curve is *smooth* at that point, meaning that the curve can be drawn through P without any sharp change of direction.



For example, the curve $y = |x|$ sketched opposite has a sharp corner at the origin, where it changes gradient abruptly from -1 to 1. A tangent cannot be drawn there, and the function therefore has no derivative at $x = 0$.

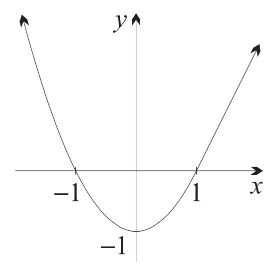
22 DIFFERENTIABILITY AT A POINT:
- Informally, a function $f(x)$ is called *differentiable* (or *smooth*) at $x = a$ if the graph passes through the point where $x = a$ without any sharp corner.
- More formally, a function $f(x)$ is called *differentiable* at $x = a$ if the derivative $f'(a)$ exists there.

Thus $y = |x|$ is *continuous* at $x = 0$, but is not *differentiable* there.

Clearly a function that is not even continuous at some value $x = a$ cannot be differentiable there, because there is no way of drawing a tangent at a place where there is a break in the curve.

23 IF IT IS NOT CONTINUOUS, IT CAN'T BE DIFFERENTIABLE:
If $f(x)$ is not continuous at $x = a$, then it is certainly not differentiable there.

Piecewise-Defined Functions: The sketch opposite shows



$$f(x) = \begin{cases} x^2 - 1, & \text{for } x < 1, \\ 2x - 2, & \text{for } x \geq 1, \end{cases} \quad \text{so } f'(x) = \begin{cases} 2x, & \text{for } x < 1, \\ 2, & \text{for } x > 1. \end{cases}$$

The graph is continuous, because the two pieces join at $P(1, 0)$:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1) = 0.$$

But in this case, when the two pieces join, they do so with the same gradient. The combined curve is therefore *smooth* or *differentiable* at the point $P(1, 0)$. The reason for this is that the gradients on the left and right of $x = 1$ converge to the same limit of 2:

$$\lim_{x \to 1^-} f'(x) = \lim_{x \to 1^-} 2x = 2, \quad \text{and} \quad \lim_{x \to 1^+} f'(x) = \lim_{x \to 1^+} 2 = 2.$$

Thus the function does have a well-defined derivative of 2 when $x = 1$, and the curve is differentiable there, with a well-defined tangent at the point $P(1, 0)$.

24 DIFFERENTIABILITY: To test a piecewise-defined function for differentiability at a join between pieces at $x = a$:
1. Test whether the function is continuous at $x = a$.
2. Test whether $\lim_{x \to a^-} f'(x)$ and $\lim_{x \to a^+} f'(x)$ exist and are equal.

WORKED EXERCISE:

Test each function for continuity and then for differentiability at $x = 0$. Then sketch the curve.

(a) $f(x) = \begin{cases} x^2 - 1, & \text{for } x \le 0, \\ x^2 + 1, & \text{for } x > 0. \end{cases}$ (b) $f(x) = \begin{cases} x, & \text{for } x \le 0, \\ x - x^2, & \text{for } x > 0. \end{cases}$

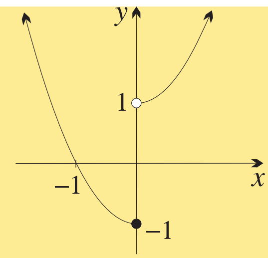
SOLUTION:

(a) First, $\lim\limits_{x \to 0^-} f(x) = -1$,

and $\lim\limits_{x \to 0^+} f(x) = 1$,

so the function is not even continuous at $x = 0$,

let alone differentiable there.



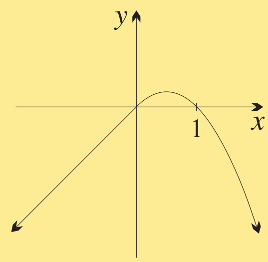
(b) First, $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^-} f(x) = f(0) = 0$,

so the function is continuous at $x = 0$.

Secondly, $f'(x) = \begin{cases} 1, & \text{for } x < 0, \\ 1 - 2x, & \text{for } x > 0, \end{cases}$

so $\lim\limits_{x \to 0^+} f'(x) = \lim\limits_{x \to 0^-} f'(x) = 1$,

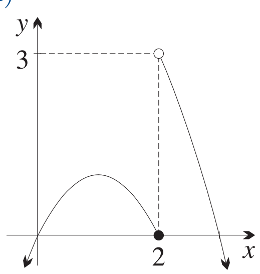
and the function is differentiable at $x = 0$, with $f'(0) = 1$.



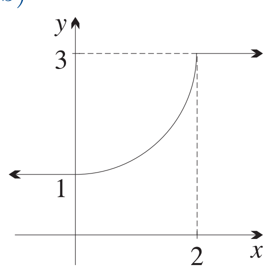
Exercise 9K

1. State whether each function $f(x)$ is continuous at $x = 0$ and at $x = 2$, and whether it is differentiable (smooth) there.

(a)



(b)



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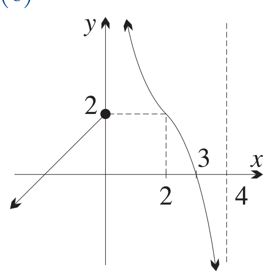

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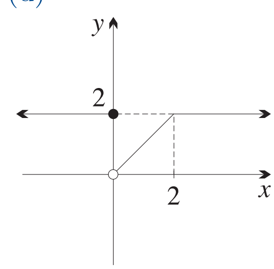


2. (a) Sketch the graph of the function $f(x) = \begin{cases} x^2, & \text{for } x \le 1, \\ 2x - 1, & \text{for } x > 1. \end{cases}$

(b) Show that the function is continuous at $x = 1$.

(c) Explain why $f'(x) = \begin{cases} 2x, & \text{for } x < 1, \\ 2, & \text{for } x > 1. \end{cases}$

(d) Hence explain why there is a tangent at $x = 1$, state its gradient, and state $f'(1)$.

DEVELOPMENT

3. Test each function for continuity at $x = 1$. If the function is continuous there, find $\lim\limits_{x \to 1^+} f'(x)$ and $\lim\limits_{x \to 1^-} f'(x)$ to check for differentiability at $x = 1$. Then sketch the graph.

(a) $f(x) = \begin{cases} (x+1)^2, & \text{for } x \le 1, \\ 4x - 2, & \text{for } x > 1. \end{cases}$

(b) $f(x) = \begin{cases} 3 - 2x, & \text{for } x < 1, \\ \dfrac{1}{x}, & \text{for } x \ge 1. \end{cases}$

(c) $f(x) = \begin{cases} 2 - x^2, & \text{for } x \le 1, \\ (x-2)^2, & \text{for } x > 1. \end{cases}$

(d) $f(x) = \begin{cases} (x-1)^3, & \text{for } x \le 1, \\ (x-1)^2, & \text{for } x > 1. \end{cases}$

4. Sketch each function. State any values of x where it is not continuous or not differentiable.
(a) $y = |x+2|$ (b) $y = |x| + 2$ (c) $y = 3 - |x-3|$

5. Sketch the graph of the function $y = \begin{cases} x^3 - x, & \text{for } -1 \le x \le 1, \\ x^2 - 1, & \text{for } x > 1 \text{ or } x < -1. \end{cases}$
State any values of x where the function is not continuous or not differentiable.

CHALLENGE

6. (a) Sketch $y = \begin{cases} \sqrt{1-(x+1)^2}\,, & \text{for } -2 \le x \le 0, \\ \sqrt{1-(x-1)^2}\,, & \text{for } 0 < x \le 2. \end{cases}$
[HINT: Each part is a semicircle, which can easily be seen by squaring it.]
(b) Repeat part (a) for $y = \begin{cases} \sqrt{1-(x+1)^2}\,, & \text{for } -2 \le x \le 0, \\ -\sqrt{1-(x-1)^2}\,, & \text{for } 0 < x \le 2. \end{cases}$

9L Chapter Review Exercise

1. Use the definition of the derivative $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ to find the derivative of each function by first principles.
(a) $f(x) = x^2 + 5x$ (b) $f(x) = 6 - x^2$ (c) $f(x) = 3x^2 - 2x + 7$

2. Write down the derivative of each function. You will need to expand any brackets.
(a) $y = x^3 - 2x^2 + 3x - 4$ (d) $y = (x+3)(x-2)$ (g) $y = 4x^3 - 4x^{-3}$
(b) $y = x^6 - 4x^4$ (e) $y = (2x-1)(2-3x)$ (h) $y = 3x^{\frac{1}{2}} + 3x^{-\frac{1}{2}}$
(c) $y = 3x^2(x - 2x^3)$ (f) $y = 3x^{-2} - 2x^{-1}$ (i) $y = x^{-2}(x^2 - x + 1)$

3. Rewrite each function in index notation and then differentiate it. Write each final answer without negative or fractional indices.
(a) $y = \dfrac{3}{x}$ (c) $y = 7\sqrt{x}$ (e) $y = -3x\sqrt{x}$
(b) $y = \dfrac{1}{6x^2}$ (d) $y = \sqrt{144x}$ (f) $y = \dfrac{6}{\sqrt{x}}$

4. Divide each function through by the numerator and then differentiate it. Give each final answer without negative or fractional indices.
(a) $y = \dfrac{3x^4 - 2x^3}{x^2}$ (c) $y = \dfrac{5x^3 - 7}{x}$ (e) $y = \dfrac{4x + 5\sqrt{x}}{\sqrt{x}}$
(b) $y = \dfrac{x^3 - x^2 + 7x}{2x}$ (d) $y = \dfrac{x^2 + 2x + 1}{x^2}$ (f) $y = \dfrac{2x^2\sqrt{x} + 3x\sqrt{x}}{x}$

5. Use the formula $\dfrac{d}{dx}(ax+b)^n = an(ax+b)^{n-1}$ to differentiate:
(a) $y = (3x+7)^3$ (c) $y = \dfrac{1}{5x-1}$ (e) $y = \sqrt{5x+1}$
(b) $y = (5-2x)^2$ (d) $y = \dfrac{1}{(2-7x)^2}$ (f) $y = \dfrac{1}{\sqrt{1-x}}$

6. Use the chain rule to differentiate:
(a) $y = (7x^2 - 1)^3$ (c) $y = (1 + x - x^2)^8$ (e) $y = \sqrt{9 - x^2}$
(b) $y = (1 + x^3)^{-5}$ (d) $y = \dfrac{1}{(x^2-1)^3}$ (f) $y = \dfrac{1}{\sqrt{9-x^2}}$

7. Differentiate each function, using the product or quotient rule. Factor each answer completely.

(a) $y = x^9(x+1)^7$
(b) $y = \dfrac{x^2}{1-x}$
(c) $y = x^2(4x^2+1)^4$
(d) $y = \dfrac{2x-3}{2x+3}$
(e) $y = (x+1)^5(x-1)^4$
(f) $y = \dfrac{x^2+5}{x-2}$

8. Differentiate $y = x^2 + 3x + 2$. Hence find the gradient and the angle of inclination, correct to the nearest minute when appropriate, of the tangents at the points where:

(a) $x = 0$
(b) $x = -1$
(c) $x = -2$

9. (a) Find the equations of the tangent and normal to $y = x^3 - 3x$ at the origin.
(b) Find the equation of the tangent and normal to the curve at the point $P(1, -2)$.
(c) By solving $f'(x) = 0$, find the points on the curve where the tangent is horizontal.
(d) Find the points on the curve where the tangent has gradient 9.

10. (a) Find the equations of the tangent and normal to $y = x^2 - 5x$ at the point $P(2, -6)$.
(b) The tangent and normal at P meet the x-axis at A and B respectively. Find the coordinates of A and B.
(c) Find the length of the interval AB and the area of the triangle PAB.

11. Find the tangents to the curve $y = x^4 - 4x^3 + 4x^2 + x$ at the origin and at the point where $x = 2$, and show that these two lines are the same.

12. Find any points on each curve where the tangent has the given angle of inclination.

(a) $y = \frac{1}{3}x^3 - 7$, $45°$
(b) $y = x^2 + \frac{1}{3}x^3$, $135°$

13. (a) Find the points on $y = x^3 - 4x$ where the tangents are parallel to $\ell : x + y + 2 = 0$.
(b) Find the equations of these two tangents, and show that ℓ is one of them.

14. By factoring top and bottom and cancelling, find these limits.

(a) $\displaystyle\lim_{x \to 0} \frac{x^2 - 12x}{x^2 + 6x}$
(b) $\displaystyle\lim_{x \to 3} \frac{x^2 - 3x}{x^2 - 9}$
(c) $\displaystyle\lim_{x \to -2} \frac{x^2 + 5x + 6}{x^2 + 6x + 8}$

15. (a) Sketch the curve $y = \begin{cases} x^2, & \text{for } x \le 0, \\ x^2 + 1, & \text{for } x > 0. \end{cases}$
(b) Find $\displaystyle\lim_{x \to 0^-} f(x)$, $\displaystyle\lim_{x \to 0^+} f(x)$ and $f(0)$. Is the curve continuous at $x = 0$?
(c) Write down the domain and range of $f(x)$.

16. (a) Sketch the curve $y = \begin{cases} x^2 - 4, & \text{for } x < 2, \\ 4x - 8, & \text{for } x \ge 2. \end{cases}$
(b) Find $\displaystyle\lim_{x \to 2^-} f(x)$, $\displaystyle\lim_{x \to 2^+} f(x)$ and $f(2)$. Is the curve continuous at $x = 2$?
(c) Find $\displaystyle\lim_{x \to 2^-} f'(x)$ and $\displaystyle\lim_{x \to 2^+} f'(x)$. Is the curve differentiable at $x = 2$?
(d) What are the domain and range of $f(x)$?

Appendix — Proving Some Rules for Differentiation

This appendix explains some of the proofs that were needed in the course of this chapter. First, the rule for differentiating powers of x is very simple:

THEOREM: Let $f(x) = x^n$, where n is any real number. Then $f'(x) = nx^{n-1}$.

Unfortunately, the full proof is quite complicated, and the theorem is only proven below in the case where n is a positive integer. (Challenge questions in Exercise 9E and 9F develop the proofs for $n = -1$ and $n = \frac{1}{2}$ respectively.)

We first need to develop a factoring of the difference of nth powers.

Factoring the Difference of nth Powers: The series

$$u^{n-1} + u^{n-2}x + u^{n-3}x^2 + \cdots + x^{n-1}, \quad \text{where } u \neq x,$$

is a GP consisting of n terms with first term $a = u^{n-1}$ and ratio $r = u^{-1}x$.
Hence, using the formula for the sum of the first n terms of a GP,

$$\begin{aligned} u^{n-1} + u^{n-2}x + u^{n-3}x^2 + \cdots + x^{n-1} &= \frac{a(1-r^n)}{1-r} \\ &= \frac{u^{n-1}(1-u^{-n}x^n)}{1-u^{-1}x} \\ &= \frac{u^{n-1} - u^{-1}x^n}{1-u^{-1}x}. \end{aligned}$$

Multiplying numerator and denominator by u gives the result needed below:

$$u^{n-1} + u^{n-2}x + u^{n-3}x^2 + \cdots + x^{n-1} = \frac{u^n - x^n}{u - x}.$$

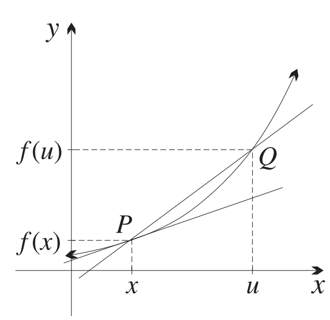
An Alternative Notation for the Derivative: It is easier here to work with an alternative limiting formula for the derivative to that given on page 234.

The diagram is the same, with the tangent at a point P and a secant PQ.
Let the point P has coordinates $\big(x, f(x)\big)$, as before,
but let Q have x-coordinate u and y-coordinate $f(u)$, rather than x and $f(x+h)$.

Then the secant PQ has gradient $\dfrac{f(u) - f(x)}{u - x}$,

and taking the limit as $u \to x$ gives the alternative formula

$$f'(x) = \lim_{u \to x} \frac{f(u) - f(x)}{u - x}.$$



The Derivative of x^n when n is a positive integer:

Let $f(x) = x^n$, where n is a positive integer.

We can now prove that $f'(x) = nx^{n-1}$.

$$\begin{aligned} \text{First,} \quad \frac{f(u) - f(x)}{u - x} &= \frac{u^n - x^n}{u - x} \quad \text{where } u \neq x, \\ &= u^{n-1} + u^{n-2}x + u^{n-3}x^2 + \cdots + x^{n-1}, \quad \text{as shown above.} \end{aligned}$$

$$\begin{aligned} \text{Taking the limit as } u \to x, \quad f'(x) &= x^{n-1} + x^{n-1} + \cdots + x^{n-1} \quad \text{(There are } n \text{ terms.)} \\ &= nx^{n-1}, \quad \text{as required.} \end{aligned}$$

Linear Combinations of Functions: Two results need to be proven:

DERIVATIVE OF A SUM: If $f(x) = g(x) + h(x)$, then $f'(x) = g'(x) + h'(x)$.

DERIVATIVE OF A MULTIPLE: If $f(x) = ag(x)$, then $f'(x) = ag'(x)$.

PROOF:

For the first:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{g(x+h)+h(x+h)-g(x)-h(x)}{h} \\ &= \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} + \lim_{h\to 0} \frac{h(x+h)-h(x)}{h} \\ &= g'(x)+h'(x). \end{aligned}$$

For the second:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{ag(x+h)-ag(x)}{h} \\ &= a \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\ &= ag'(x). \end{aligned}$$

Proof of the Product Rule: The product rule states that:

THEOREM: Let $y = uv$, where u and v are functions of x. Then

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}.$$

PROOF: Suppose that x changes to $x + \delta x$.
Suppose that as a result, u changes to $u + \delta u$ and v changes to $v + \delta v$, and that finally y changes to $y + \delta y$.

Then $\quad y = uv,$

and $\quad y + \delta y = (u + \delta u)(v + \delta v)$

$\quad = uv + v\,\delta u + u\,\delta v + \delta u\,\delta v,$

so $\quad \delta y = v\,\delta u + u\,\delta v + \delta u\,\delta v.$

Hence, dividing by δx, $\quad \dfrac{\delta y}{\delta x} = v\dfrac{\delta u}{\delta x} + u\dfrac{\delta v}{\delta x} + \dfrac{\delta u}{\delta x} \times \dfrac{\delta v}{\delta x} \times \delta x,$

and taking limits as $\delta x \to 0$, $\quad \dfrac{dy}{dx} = v\dfrac{du}{dx} + u\dfrac{dv}{dx} + 0,$ as required.

Proof of Quotient Rule: The quotient rule states that:

THEOREM: Let $y = \dfrac{u}{v}$, where u and v are functions of x. Then

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

PROOF: Differentiate uv^{-1} using the product rule.

Let $\quad y = uv^{-1}.$

$$\begin{aligned} \text{Then } \frac{dy}{dx} &= V\frac{dU}{dx} + U\frac{dV}{dx} \\ &= v^{-1}\frac{du}{dx} - uv^{-2}\frac{dv}{dx} \\ &= \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2} \end{aligned}$$

$\left(\text{after multiplying by } \dfrac{v^2}{v^2}\right).$

Let $\quad U = u$

and $\quad V = v^{-1}.$

Then $\dfrac{dU}{dx} = \dfrac{du}{dx}$

and by the chain rule,

$$\begin{aligned} \frac{dV}{dx} &= \frac{dV}{dv} \times \frac{dv}{dx} \\ &= -v^{-2}\frac{dv}{dx}. \end{aligned}$$

CHAPTER **TEN**

The Quadratic Function

The previous chapter on differentiation established that the derivative of any quadratic function is a linear function. For example,

$$\frac{d}{dx}(x^2 - 5x + 6) = 2x - 5.$$

Thus quadratics are the next most elementary functions to study after the linear functions of Chapter Six. This relationship between linear and quadratic functions is the underlying reason why quadratics arise so often in mathematics.

10 A Factoring and the Graph

A *quadratic function* is a function that can be written in the form

$$f(x) = ax^2 + bx + c, \text{ where } a, b \text{ and } c \text{ are constants, and } a \neq 0.$$

A *quadratic equation* is an equation that can be written in the form

$$ax^2 + bx + c = 0, \text{ where } a, b \text{ and } c \text{ are constants, and } a \neq 0.$$

The requirement that $a \neq 0$ means that the term in x^2 cannot vanish. Thus linear functions and equations are not regarded as special cases of quadratics.

The word 'quadratic' comes from the Latin word *quadratus*, meaning 'square', and reminds us that quadratics tend to arise as the areas of regions in the plane.

Zeroes and Roots: The solutions of a quadratic equation are called the *roots* of the equation, and the x-intercepts of a quadratic function are called the *zeroes* of the function. This distinction is often not strictly observed, however, because questions about quadratic functions and their graphs are closely related to questions about quadratic equations.

The Four Questions about the Graph of a Quadratic: The graph of a quadratic function $y = ax^2 + bx + c$ is a *parabola*. Before sketching the parabola, four questions need to be answered.

1

FOUR QUESTIONS ABOUT THE PARABOLA $y = ax^2 + bx + c$:

1. Which way up is the parabola? ANSWER: Look at the sign of a.
2. What is the y-intercept? ANSWER: Put $x = 0$, and then $y = c$.
3. What are the axis of symmetry and the vertex?
4. What are the x-intercepts, if there are any?

The first two questions are easy to answer, but the next two questions need careful working. Sections 10A, 10B and 10C will review in succession the three standard approaches to them:

- Section 10A: Factoring
- Section 10B: Completing the square
- Section 10C: Formulae

Factoring and the Zeroes: Most quadratics cannot easily be factored, but when factoring is possible, this is usually the quickest approach to sketching the curve. The zeroes are found using the following principle:

2 FACTORING AND THE ZEROES:

- If $A \times B = 0$, then $A = 0$ or $B = 0$.
- Thus to find the zeroes of a quadratic, put each factor in turn equal to zero.

For example, the zeroes of $y = (x - 3)(x - 5)$ are given by

$$x - 3 = 0 \qquad \text{or} \qquad x - 5 = 0,$$

so the zeroes are $x = 3$ and $x = 5$.

Finding the Axis of Symmetry and Vertex from the Zeroes: The axis of symmetry is the vertical line midway between the x-intercepts. Thus its x-intercept is the average of the zeroes.

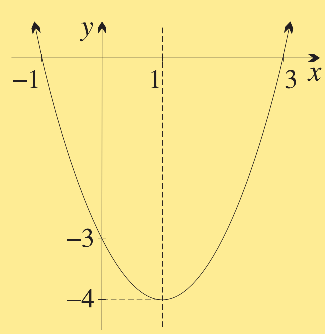
3 ZEROES AND THE AXIS OF SYMMETRY AND VERTEX:

- If a quadratic has zeroes α and β, its axis is the line $x = \frac{1}{2}(\alpha + \beta)$.
- Substitution into the quadratic gives the y-coordinate of the vertex.

For example, we saw that $y = (x - 3)(x - 5)$ has zeroes $x = 3$ and $x = 5$.
Averaging these zeroes, the axis of symmetry is the line $x = 4$.
Substituting $x = 4$ gives $y = -1$, so the vertex is $(4, -1)$.

WORKED EXERCISE:
Sketch the curve $y = x^2 - 2x - 3$.

SOLUTION:



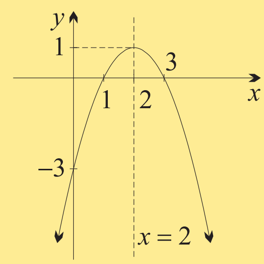
1. Since $a > 0$, the curve is concave up.
2. When $x = 0$, $\quad y = -3$,
   so the y-intercept is -3.
3. Factoring, $\quad y = (x + 1)(x - 3)$.
   When $y = 0$, $\quad x + 1 = 0$ or $x - 3 = 0$,
   so the zeroes are $\quad x = -1$ and $x = 3$.
4. Taking the average, the axis of symmetry is
   $$x = \tfrac{1}{2}(-1 + 3)$$
   $$x = 1.$$
   When $x = 1$, $y = -4$, so the vertex is $(1, -4)$.

WORKED EXERCISE:

Sketch the curve $y = -x^2 + 4x - 3$.

SOLUTION:

1. Since $a < 0$, the curve is concave down.
2. When $x = 0$, $y = -3$.
3. Factoring, $y = -(x^2 - 4x + 3)$
   $= -(x - 1)(x - 3)$.
   When $y = 0$, $x - 1 = 0$ or $x - 3 = 0$,
   so the zeroes are $x = 1$ and $x = 3$.
4. Taking the average, the axis of symmetry is
   $x = \frac{1}{2}(1 + 3)$
   $x = 2$.
   When $x = 2$, $y = 1$, so the vertex is $(2, 1)$.



Quadratic Inequations: As discussed in Chapter Four, a quadratic inequation is best solved from a sketch of the quadratic function. The function is positive when the graph is above the x-axis, and is negative when the graph is below the x-axis.

WORKED EXERCISE:

Use the graphs drawn in the previous two worked exercises to solve these two inequations.

(a) $x^2 - 2x - 3 \le 0$ (b) $x^2 + 3 > 4x$

SOLUTION:

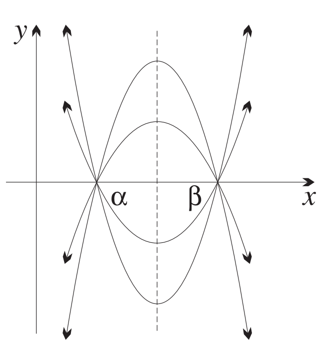
(a) The inequation is $x^2 - 2x - 3 \le 0$.
This means that the first graph must be below or on the x-axis,
so the solution is $-1 \le x \le 3$.

(b) The inequation is $x^2 + 3 > 4x$
Rearranging, $-x^2 + 4x - 3 < 0$
This means that the second graph must be below the x-axis,
so the solution is $x < 1$ or $x > 3$.

Quadratics with Given Zeroes: If a quadratic $f(x)$ has zeroes at $x = \alpha$ and $x = \beta$, then its equation must have the form

$$f(x) = a(x - \alpha)(x - \beta)$$

where a is the coefficient of x^2. By taking different values of a, this equation forms a *family* of quadratics, all with the same x-intercepts, as sketched opposite. In general:



4 THE FAMILY OF QUADRATICS WITH GIVEN ZEROES:
The quadratics with zeroes at $x = \alpha$ and $x = \beta$ form a family of parabolas with equation

$$y = a(x - \alpha)(x - \beta).$$

WORKED EXERCISE:

(a) Write down the family of quadratics with zeroes $x = -2$ and $x = 4$.

(b) Then find the equation of such a quadratic if:

(i) the y-intercept is 16, (ii) the curve passes through $(5, 1)$.

SOLUTION:

(a) The family of quadratics with zeroes -2 and 4 is $y = a(x+2)(x-4)$.

(b) (i) Substituting the point $(0, 16)$ gives $16 = a \times 2 \times (-4)$

$a = -2$,

so the quadratic is $y = -2(x+2)(x-4)$.

(ii) Substituting the point $(5, 1)$, gives $1 = a \times 7 \times 1$

$a = \frac{1}{7}$

so the quadratic is $y = \frac{1}{7}(x+2)(x-4)$.

Monic Quadratics: Factoring a quadratic is easier when the coefficient of x^2 is 1. Such quadratics are called *monic*.

MONIC QUADRATICS:

5

- A quadratic is called *monic* if the coefficient of x^2 is 1.
  For example, $y = x^2 - 8x + 15$ is monic, but $y = -x^2 + 8x - 15$ is non-monic.
- The only monic quadratic whose zeroes are α and β is $(x-\alpha)(x-\beta)$.
  For example, $y = (x-3)(x-5)$ is the only monic quadratic with zeroes 3 and 5.

Exercise 10A

1. Sketch the graph of each function. Your graph must show all the intercepts with the x-axis and y-axis and the coordinates of the vertex.

(a) $y = x(x+4)$ (b) $y = x(x-6)$ (c) $y = (x-1)(x+1)$ (d) $y = (x-4)(x+4)$ (e) $y = (x-1)(x+7)$ (f) $y = (x-3)(x+11)$

2. Use the graphs drawn in parts (a)–(f) of the previous question to solve:

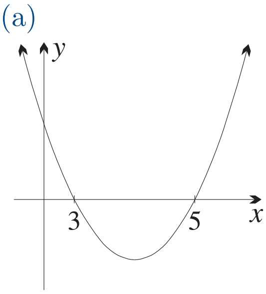
(a) $x(x+4) < 0$ (b) $x(x-6) > 0$ (c) $(x-1)(x+1) \le 0$ (d) $(x-4)(x+4) \ge 0$ (e) $(x-1)(x+7) < 0$ (f) $(x-3)(x+11) \ge 0$

3. Write down, in factored form, the equation of the monic quadratic function with zeroes:

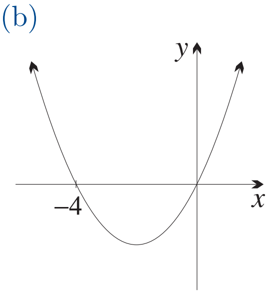
(a) 4 and 6 (b) 3 and 8 (c) -3 and 5 (d) -6 and -1

4. Write down, in factored form, the equation of each quadratic function sketched below, given that the coefficient of x^2 is either 1 or -1.

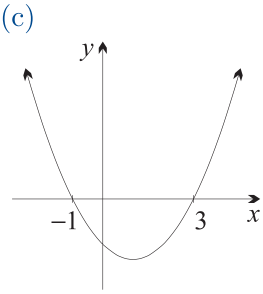
(a)



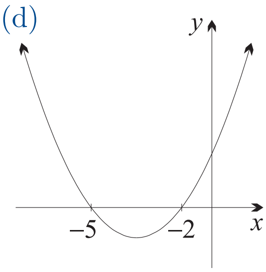
(b)



(c)



(d)



DEVELOPMENT

5. Sketch the graph of each function, showing all intercepts with the axes and the coordinates of the vertex.
(a) $y = x(6 - x)$ (b) $y = -(x - 2)(x + 2)$ (c) $y = (x + 2)(8 - x)$

6. Use factoring to find the zeroes of each quadratic function. Hence sketch each graph, showing all intercepts and the coordinates of the vertex.
(a) $y = x^2 - 2x$ (c) $y = x^2 + 4x - 5$ (e) $y = x^2 + 10x + 21$
(b) $y = x^2 - 81$ (d) $y = x^2 - 8x + 15$ (f) $y = x^2 + 8x - 33$

7. Use the graphs drawn in parts (c)–(e) of the previous question to solve:
(a) $x^2 + 4x - 5 > 0$ (b) $x^2 - 8x + 15 \geq 0$ (c) $x^2 + 10x + 21 \leq 0$

8. Use factoring to find the zeroes of each quadratic function. Hence sketch each graph, showing all intercepts and the coordinates of the vertex.
(a) $y = x - x^2$ (c) $y = 16 - x^2$ (e) $y = 8 - 2x - x^2$
(b) $y = 8x - x^2$ (d) $y = 1 - x^2$ (f) $y = 6x - 5 - x^2$

9. Use the graphs drawn in parts (d)–(f) of the previous question to solve:
(a) $1 - x^2 > 0$ (b) $8 - 2x - x^2 > 0$ (c) $6x - 5 - x^2 < 0$

10. Solve each inequation by sketching the graph of an appropriate quadratic.
(a) $x(x - 1) < 0$ (c) $x^2 - 5x + 6 \leq 0$ (e) $x^2 + x < 20$
(b) $x^2 + 7x \geq 0$ (d) $x^2 > 9x$ (f) $x^2 + 25 \leq 10x$

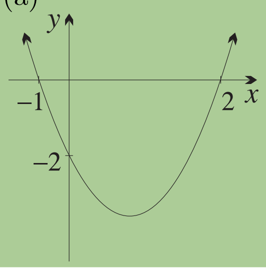
11. Use factoring to sketch the graphs of the following non-monic quadratic functions, clearly indicating the vertex and the intercepts with the axes. You may want to use the fraction button on your calculator to evaluate the y-coordinate of the vertex.
(a) $y = 3x^2 + x - 2$ (b) $y = 2x^2 + 7x + 5$ (c) $y = 2x^2 + 5x - 3$

CHALLENGE

12. Use factoring to find the zeroes of each quadratic function. Sketch a graph of the function, clearly indicating all intercepts and the coordinates of the vertex.
(a) $y = 2x^2 - 18$ (c) $y = 3x^2 + x - 4$ (e) $y = 3x^2 - 10x - 8$
(b) $y = 9x^2 - 25$ (d) $y = 3x - 1 - 2x^2$ (f) $y = 7x - 3 - 4x^2$

13. The general form of a quadratic with zeroes $x = 2$ and $x = 8$ is $y = a(x - 2)(x - 8)$. Find the equation of such a quadratic for which:
(a) the coefficient of x^2 is 3,
(b) the constant term is -3,
(c) the y-intercept is -16,
(d) the coefficient of x is 25,
(e) the vertex is $(5, -12)$,
(f) the curve passes through $(1, -20)$.

14. Find the equations of the quadratic functions sketched below.

(a)


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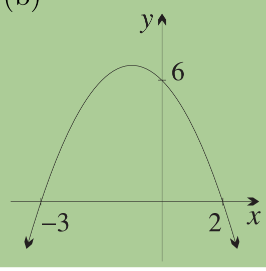
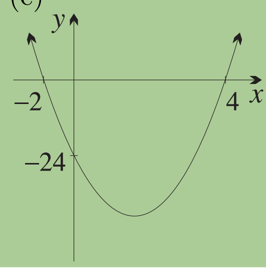
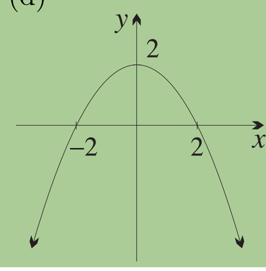


(c)


(d)


10 B Completing the Square and the Graph

Completing the square is the most general method of dealing with quadratics. It works in every case, in contrast with factoring, which really only works in exceptional cases.

Formulae for the zeroes and vertex can be developed by completing the square in a general quadratic, but the method remains necessary in many situations and must be known.

Completing the Square in a Monic Quadratic: The quadratic $y = x^2 + 6x + 5$ is not a perfect square, but it can be made into the sum of a perfect square and a constant by adding and subtracting $3^2 = 9$:

$$\begin{aligned} y &= x^2 + 6x + 5 \\ &= (x^2 + 6x + 9) + 5 - 9 && \text{(Add and subtract 9.)} \\ &= (x+3)^2 - 4 && \text{(This is because } (x+3)^2 = x^2 + 6x + 9.\text{)} \end{aligned}$$

The procedure that has been followed here is:

- Take the coefficient 6 of $6x$, halve it to get 3, then square to get 9.
- Add and subtract 9 to produce a perfect square plus a constant.

The reason why it all works is that $(x + 3)^2 = x^2 + 6x + 9$.

6 COMPLETING THE SQUARE IN A MONIC QUADRATIC:

- Take the coefficient of x, halve it, then square the result.
- Add and subtract this number to produce a perfect square plus a constant.

WORKED EXERCISE:

Complete the square in each quadratic.

(a) $y = x^2 - 4x - 5$ (b) $y = x^2 + x + 1$

SOLUTION:

(a) Here $y = x^2 - 4x - 5$.

The coefficient of x is 4. Halve it to get 2, then square to get $2^2 = 4$.

$$\begin{aligned} \text{Hence } y &= (x^2 - 4x + 4) - 4 - 5 && \text{(Add and subtract 4.)} \\ &= (x-2)^2 - 9. && \text{(This is because } (x-2)^2 = x^2 - 4x + 4.\text{)} \end{aligned}$$

(b) Here $y = x^2 + x + 1$.

The coefficient of x is 1. Halve it to get $\frac{1}{2}$, then square to get $(\frac{1}{2})^2 = \frac{1}{4}$.

$$\begin{aligned} \text{Hence } y &= (x^2 + x + \tfrac{1}{4}) - \tfrac{1}{4} + 1 && \text{(Add and subtract } \tfrac{1}{4}.\text{)} \\ &= (x + \tfrac{1}{2})^2 + \tfrac{3}{4}. && \text{(This is because } (x + \tfrac{1}{2})^2 = x^2 + x + \tfrac{1}{4}.\text{)} \end{aligned}$$

Sketching the Graph from the Completed Square: The work in Chapter Three on transformations of graphs tells us that $y = a(x-h)^2 + k$ is just $y = ax^2$ shifted h units to the right and k units upwards. Hence its vertex must be at (h, k).

7 DRAWING THE GRAPH OF A QUADRATIC FROM THE COMPLETED SQUARE:

- The curve $y = a(x - h)^2 + k$ is the translation of $y = ax^2$ to a parabola with vertex at (h, k).
- The zeroes can be found from the completed-square form by solving $y = 0$.

If the zeroes exist, the quadratic can then be written in factored form.

WORKED EXERCISE:

Use the previous completed squares to sketch the graphs of these quadratics. If possible, express each quadratic in factored form.

(a) $y = x^2 - 4x - 5$ (b) $y = x^2 + x + 1$

SOLUTION:

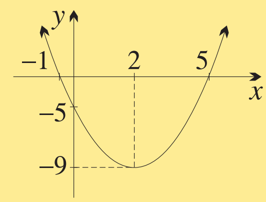
(a) $y = x^2 - 4x - 5$ is concave up with y-intercept -5.

Since $y = (x-2)^2 - 9$, the vertex is $(2, -9)$.

Put $y = 0$, then $(x-2)^2 = 9$

$$x - 2 = 3 \text{ or } x - 2 = -3$$
$$x = 5 \text{ or } -1.$$

Using these zeroes, $y = (x-5)(x+1)$.



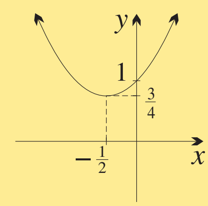
(b) $y = x^2 + x + 1$ is concave up with y-intercept 1.

Since $y = (x + \frac{1}{2})^2 + \frac{3}{4}$, the vertex is $(-\frac{1}{2}, \frac{3}{4})$.

Put $y = 0$, then $(x + \frac{1}{2})^2 = -\frac{3}{4}$.

Since negative numbers do not have square roots,

this equation has no solutions, and there are no x-intercepts.



NOTE: The axis of symmetry and the vertex can be read directly off the completed-square form — the zeroes can then be calculated. This contrasts with factoring, where the zeroes are found first and the vertex can then be calculated.

Completing the Square in a Non-Monic Quadratic: For *non-monic quadratics*, where the coefficient of x^2 is not 1, move the coefficient outside large working brackets before completing the square. The sketching of the graph then proceeds as before.

8 COMPLETING THE SQUARE IN A NON-MONIC QUADRATIC:
- Move the coefficient of x outside large working brackets.
- Complete the square of the monic quadratic inside the brackets as usual.

WORKED EXERCISE:

Complete the square in each quadratic.

(a) $y = 2x^2 - 12x + 16$ (b) $y = -x^2 + 8x - 15$

SOLUTION:

(a) $y = 2x^2 - 12x + 16$

$= 2(x^2 - 6x + 8)$ (Move the 2 outside large working brackets.)

$= 2\Big((x^2 - 6x + 9) - 9 + 8\Big)$ (Add and subtract 9 inside the outer brackets.)

$= 2\Big((x-3)^2 - 1\Big)$ (Complete the square inside the outer brackets.)

$= 2(x-3)^2 - 2$ (Finally, remove the outer brackets.)

(b) $y = -x^2 + 8x - 15$

$= -(x^2 - 8x + 15)$ (Move the minus outside working brackets.)

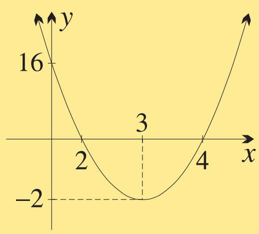
$= -\Big((x^2 - 8x + 16) + 15 - 16\Big)$ (Add and subtract 16 inside the outer brackets.)

$= -\Big((x-4)^2 - 1\Big)$ (Complete the square inside the outer brackets.)

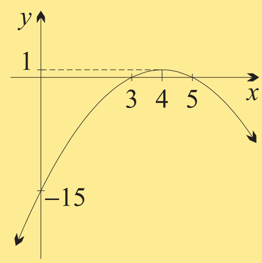
$= -(x-4)^2 + 1$ (Finally, remove the outer brackets.)

WORKED EXERCISE:
Use the previous completed squares to sketch the graphs of these quadratics. If possible, express each quadratic in factored form.
(a) $y = 2x^2 - 12x + 16$ (b) $y = -x^2 + 8x - 15$

SOLUTION:

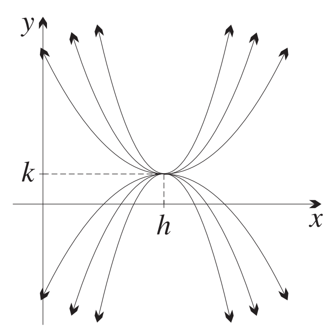


(a) $y = 2x^2 - 12x + 16$ is concave up with y-intercept 16.
Since $y = 2(x-3)^2 - 2$, the vertex is $(3, -2)$.
Put $y = 0$, then $2(x-3)^2 = 2$
$$(x-3)^2 = 1$$
$$x - 3 = 1 \text{ or } x - 3 = -1$$
$$x = 4 \text{ or } 2\,.$$
Using these zeroes, $y = 2(x-4)(x-2)$.



(b) $y = -x^2 + 8x - 15$ is concave down with y-intercept -15.
Since $y = -(x-4)^2 + 1$, the vertex is $(4, 1)$.
Put $y = 0$, then $(x-4)^2 = 1$
$$x - 4 = 1 \text{ or } x - 4 = -1$$
$$x = 5 \text{ or } 3.$$
Using these zeroes, $y = -(x-3)(x-5)$.

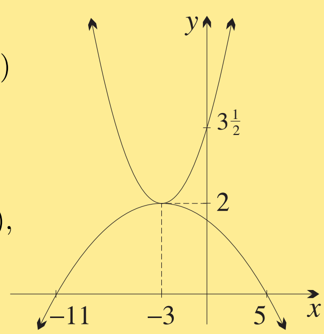
The Family of Quadratics with a Common Vertex: A quadratic with vertex (h, k) must have an equation of the form
$$y = a(x-h)^2 + k,$$
for some value of a. This equation gives a family of parabolas with vertex (h, k), as different values of a are taken, as in the sketch opposite.



9 THE FAMILY OF QUADRATICS WITH A COMMON VERTEX:
The quadratics with vertex (h, k) form a family of parabolas with equation
$$y = a(x-h)^2 + k.$$

WORKED EXERCISE:
Write down the family of quadratics with vertex $(-3, 2)$. Then find the equation of such a quadratic:
(a) if $x = 5$ is one of its zeroes, (b) if the coefficient of x is equal to 1.

SOLUTION:



The family of quadratics with vertex $(-3, 2)$ is
$$y = a(x+3)^2 + 2. \qquad (*)$$
(a) Substituting $(5, 0)$ into $(*)$ gives $0 = a \times 64 + 2$,
so $a = -\frac{1}{32}$, and the quadratic is $y = -\frac{1}{32}(x+3)^2 + 2$.

(b) Expanding $(*)$, $y = ax^2 + 6ax + (9a + 2)$,
so $6a = 1$.
Hence $a = \frac{1}{6}$ and the quadratic is $y = \frac{1}{6}(x+3)^2 + 2$.

Exercise 10B

1. Complete the square in each quadratic.
(a) $y = x^2 - 4x + 5$ (b) $y = x^2 + 6x + 11$ (c) $y = x^2 - 2x + 8$

2. Explain how to shift the graph of $y = x^2$ in order to graph each function.
(a) $y = x^2 + 1$ (c) $y = (x-4)^2$ (e) $y = (x-1)^2 + 4$ (g) $y = (x+7)^2 + 2$
(b) $y = x^2 - 3$ (d) $y = (x+2)^2$ (f) $y = (x-3)^2 - 2$ (h) $y = (x+4)^2 - 1$

3. Sketch the graph of each function by carefully following the steps in Box 7 above.
(a) $y = (x-3)^2 - 1$ (b) $y = (x+5)^2 - 9$ (c) $y = (x-1)^2 - 4$

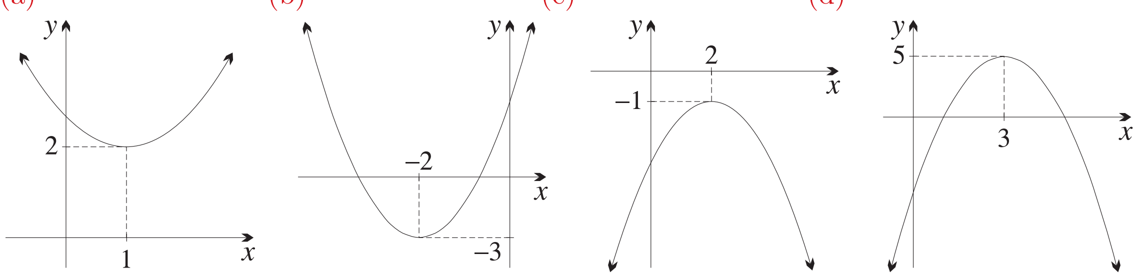
4. Repeat the previous question for these quadratics, giving the x-intercepts in surd form.
(a) $y = (x+1)^2 - 3$ (b) $y = (x-4)^2 - 7$ (c) $y = (x-3)^2 - 2$

DEVELOPMENT

5. Complete the square in each quadratic. Then sketch the graph of each function, showing the vertex and the intercepts with the axes.
(a) $y = x^2 - 2x$ (c) $y = x^2 - 2x - 5$ (e) $y = x^2 + x + 1$
(b) $y = x^2 - 4x + 3$ (d) $y = x^2 + 6x - 7$ (f) $y = x^2 - 3x + 1$

6. Sketch the graph of each function by carefully following the steps in Box 7 above.
(a) $y = -(x+2)^2 + 16$ (b) $y = -(x-4)^2 + 1$ (c) $y = -(x-7)^2 + 25$

7. Write down the equation of each of the quadratic functions sketched below, given that in each case, the coefficient of x^2 is either 1 or -1.
(a) (b) (c) (d)



8. Write down the equation of the monic quadratic with vertex:
(a) $(2, 5)$ (b) $(0, -3)$ (c) $(-1, 7)$ (d) $(3, -11)$

CHALLENGE

9. Complete the square in each quadratic. (Notice that in each case, the coefficient of x^2 is not 1.) Then sketch each curve, indicating the vertex and the intercepts with the axes.
(a) $y = -x^2 - 2x$ (d) $y = 2x^2 - 4x + 3$ (g) $y = -5x^2 - 20x - 23$
(b) $y = -x^2 + 4x + 1$ (e) $y = 4x^2 - 16x + 13$ (h) $y = 2x^2 + 5x - 12$
(c) $y = -x^2 + 5x - 6$ (f) $y = -3x^2 + 6x + 3$ (i) $y = 3x^2 + 2x - 8$

10. Explain why $y = a(x+4)^2 + 2$ is the general form of a quadratic whose vertex is $(-4, 2)$. Then find the equation of such a quadratic for which:
(a) the quadratic is monic,
(b) the coefficient of x^2 is 3,
(c) one of the zeroes is $x = 3$,
(d) the y-intercept is 16,
(e) the curve passes through the origin,
(f) the curve passes through $(1, 20)$.

11. Complete the square in the general quadratic $y = ax^2 + bx + c$. Hence write down the vertex of its graph and find the zeroes and the y-intercept.

10 C The Quadratic Formulae and the Graph

Completing the square in the general quadratic yields formulae for the axis of symmetry and for the zeroes of a quadratic function. These formulae are extremely useful, and will allow the theory of quadratics to be advanced considerably.

Completing the Square in the General Quadratic: Here is the completion of the square in the general quadratic $y = ax^2 + bx + c$:

$$\begin{aligned} y &= a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) && \text{(Move the } a \text{ outside large working brackets.)} \\ &= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a}\right) && \text{(Half the coefficient of } x \text{ is } \frac{b}{2a}.\text{)} \\ &= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a}. \end{aligned}$$

Hence the axis of symmetry is $x = -\dfrac{b}{2a}$, and the vertex is $\left(-\dfrac{b}{2a}, -\dfrac{b^2 - 4ac}{4a}\right)$.
You should remember only the formula for the axis of symmetry, and find the y-coordinate of the vertex by substituting back into the quadratic.

10 THE AXIS OF SYMMETRY OF $y = ax^2 + bx + c$:

- The axis of symmetry is the line $x = -\dfrac{b}{2a}$.
- Substitute into the quadratic to find the y-coordinate of the vertex.

To find the formula for the zeroes, put $y = 0$ into the completed square above:

$$\begin{aligned} a\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a} \\ \left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} \\ x + \frac{b}{2a} &= \frac{\sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x + \frac{b}{2a} = -\frac{\sqrt{b^2 - 4ac}}{2a} \\ x &= \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \end{aligned}$$

The quantity $b^2 - 4ac$ will soon become very important in the theory of quadratics. It is called the *discriminant* and is given the symbol Δ (Greek capital delta).

11 THE ZEROES (y-INTERCEPTS) OF THE QUADRATIC $y = ax^2 + bx + c$:

$$x = \frac{-b + \sqrt{\Delta}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{\Delta}}{2a}, \quad \text{where } \Delta = b^2 - 4ac.$$

- Always calculate the discriminant first when finding the zeroes of a quadratic.
- If $\Delta < 0$, there are no zeroes, because negatives don't have square roots.
- If $\Delta = 0$, there is only one zero, because 0 is the only square root of 0.

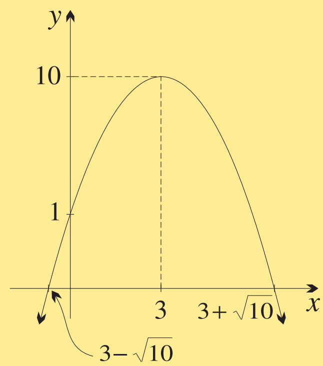
NOTE: The vertex of a parabola can be written in the form $\left(-\dfrac{b}{2a}, -\dfrac{\Delta}{4a}\right)$.
You may prefer to memorise this formula for the vertex rather than finding the axis of symmetry and substituting to find the y-coordinate.

WORKED EXERCISE:
Use the quadratic formulae to sketch each quadratic. Give any irrational zeroes first in simplified surd form, then approximated correct to four significant figures. If possible, write each quadratic in factored form.

(a) $y = -x^2 + 6x + 1$ (b) $y = 3x^2 - 6x + 4$

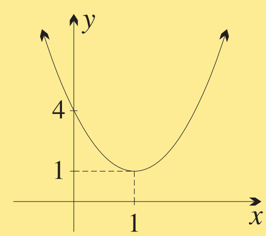
SOLUTION:

(a) The curve $y = -x^2 + 6x + 1$ is concave down, with y-intercept 1.
The two formulae are now applied with $a = -1$, $b = 6$ and $c = 1$.
First, the axis of symmetry is $x = -\frac{b}{2a}$
$x = 3.$
When $x = 3$, $y = 10$, so the vertex is $(3, 10)$.



Secondly, $\Delta = b^2 - 4ac$
$= 40$
$= 4 \times 10,$
so $y = 0$ when $x = \dfrac{-b + \sqrt{\Delta}}{2a}$ or $\dfrac{-b - \sqrt{\Delta}}{2a}$
$= \dfrac{-6 + 2\sqrt{10}}{-2}$ or $\dfrac{-6 - 2\sqrt{10}}{-2}$
$= 3 - \sqrt{10}$ or $3 + \sqrt{10}$
$\doteqdot -0{\cdot}1623$ or $6{\cdot}162.$
Hence also $y = -\left(x - 3 + \sqrt{10}\right)\left(x - 3 - \sqrt{10}\right).$ (See Box 4 above.)

(b) The curve $y = 3x^2 - 6x + 4$ is concave up, with y-intercept 4.
Apply the formulae with $a = 3$, $b = -6$ and $c = 4$.
First, the axis of symmetry is $x = 1$,
and substituting $x = 1$, the vertex is $(1, 1)$.
Secondly, $\Delta = 36 - 48$
which is negative, so there are no zeroes.



Exercise 10C

1. Find the discriminant $\Delta = b^2 - 4ac$ of each quadratic. Then find the zeroes, using the formula $x = \dfrac{-b + \sqrt{\Delta}}{2a}$ or $\dfrac{-b - \sqrt{\Delta}}{2a}$.

(a) $y = x^2 + 3x + 2$
(b) $y = x^2 + 6x + 9$
(c) $y = -x^2 + 9x - 18$
(d) $y = 2x^2 + 5x - 3$
(e) $y = -2x^2 + 5x - 2$
(f) $y = 3x^2 - 5x + 2$

2. Find the discriminant $\Delta = b^2 - 4ac$ of each quadratic, then find the zeroes. Give the zeroes first in surd form, then correct to four significant figures.

(a) $y = x^2 + 2x - 2$
(b) $y = x^2 - 4x + 1$
(c) $y = x^2 + 3x - 2$
(d) $y = -x^2 - 2x + 4$
(e) $y = 3x^2 - 2x - 2$
(f) $y = 2x^2 + 4x - 1$

DEVELOPMENT

3. Sketch a graph of each quadratic function by carefully following through the steps outlined in Boxes 10 and 11 and the worked exercises above.

(a) $y = x^2 + 6x + 5$
(b) $y = x^2 + 4x + 4$
(c) $y = -x^2 + 2x + 24$
(d) $y = -x^2 + 2x + 1$
(e) $y = x^2 + 4x - 1$
(f) $y = 2x^2 + 2x - 1$

4. Use the graphs in parts (a)–(d) of the previous question to solve:
(a) $x^2 + 6x + 5 < 0$ (b) $x^2 + 4x + 4 > 0$ (c) $-x^2 + 2x + 24 \le 0$ (d) $-x^2 + 2x + 1 \ge 0$

5. Use the quadratic formula to find the zeroes of each quadratic function. Hence write the function in the form $y = (x - \alpha)(x - \beta)$.
(a) $y = x^2 + x - 20$ (b) $y = x^2 + 5x + 4$ (c) $y = x^2 - 9x + 14$

6. Solve each pair of equations simultaneously.
(a) $y = x^2 - 4x + 3$ and $y = x + 3$,
(b) $y = 2x^2 + 7x - 4$ and $y = 3x - 6$,
(c) $y = -x^2 + x - 3$ and $y = 2x + 1$,
(d) $y = -2x^2 + 5x - 1$ and $y = 3 - x$.

7. Find the discriminant $\Delta = b^2 - 4ac$ of each quadratic. Hence state how many zeroes the quadratic has.
(a) $y = x^2 + 2x - 3$ (b) $y = x^2 + 3x + 1$ (c) $y = 9x^2 - 6x + 1$ (d) $y = -2x^2 + 5x - 7$

CHALLENGE

8. By substituting the axis of symmetry $x = -\dfrac{b}{2a}$ into the equation of the general quadratic $y = ax^2 + bx + c$, show that the vertex has y-coordinate $-\dfrac{\Delta}{4a}$. Use this formula to check the vertices that you obtained in question 3 above.

10 D Equations Reducible to Quadratics

Many equations can be solved using substitutions that reduce them to quadratic equations. For example, in the following worked exercise, the degree-4 equation $x^4 - 13x^2 + 36 = 0$ becomes a quadratic equation with the substitution $u = x^2$.

WORKED EXERCISE:
By making substitutions that will reduce them to quadratic equations, solve:
(a) $x^4 - 13x^2 + 36 = 0$ (b) $9^x - 26 \times 3^x - 27 = 0$

SOLUTION:
(a) Let $u = x^2$.
Then $x^4 = u^2$.
Hence $u^2 - 13u + 36 = 0$
$(u - 9)(u - 4) = 0$
$u = 9$ or $u = 4$.
So $x^2 = 9$ or $x^2 = 4$
$x = 3, -3, 2$ or -2.

(b) Let $u = 3^x$.
Then $9^x = u^2$ (since $9 = 3^2$).
Hence $u^2 - 26u - 27 = 0$
$(u - 27)(u + 1) = 0$
$u = 27$ or $u = -1$.
So $3^x = 27$ or $3^x = -1$
$x = 3$. (3^x cannot be negative.)

Exercise 10D

1. (a) Solve $u^2 - 13u + 36 = 0$. (b) Let $u = x^2$, and hence solve $x^4 - 13x^2 + 36 = 0$.

2. (a) Solve $u^2 - 29u + 100 = 0$. (b) Let $u = x^2$, and hence solve $x^4 - 29x^2 + 100 = 0$.

3. (a) Solve $u^2 - 9u + 8 = 0$. (b) Let $u = x^3$, and hence solve $x^6 - 9x^3 + 8 = 0$.

4. (a) Solve $u + \dfrac{27}{u} = 28$. (b) Let $u = x^3$, and hence solve $x^3 + \dfrac{27}{x^3} = 28$.

5. (a) Solve $u^2 - 18u + 72 = 0$.
(b) Let $u = x^2 - x$, and hence solve $(x^2 - x)^2 - 18(x^2 - x) + 72 = 0$.

6. (a) Solve $u + 8 = \dfrac{48}{u}$.
(b) Let $u = x^2 - 4x$, and hence solve $(x^2 - 4x) + 8 = \dfrac{48}{x^2 - 4x}$.

DEVELOPMENT

7. (a) Solve $3u^2 - 10u + 8 = 0$. (b) Let $u = x^2$, and hence solve $3x^4 - 10x^2 + 8 = 0$.

8. (a) Solve $16u + \dfrac{16}{u} = 257$. (b) Let $u = x^2$, and hence solve $16x^2 + \dfrac{16}{x^2} = 257$.

9. By making a suitable substitution, reduce each equation to a quadratic and solve it.
(a) $3x^4 - 13x^2 + 4 = 0$
(b) $x^6 - 26x^3 - 27 = 0$
(c) $(x + 1)^4 - (x + 1)^2 - 12 = 0$
(d) $x^2 + x - \dfrac{2}{x^2 + x} = 1$

10. Solve each equation.
(a) $3^{2x} - 12 \times 3^x + 27 = 0$ [Let $u = 3^x$.]
(b) $5^{2x} - 6 \times 5^x + 5 = 0$ [Let $u = 5^x$.]
(c) $4^x - 12 \times 2^x + 32 = 0$ [Let $u = 2^x$.]
(d) $2 \times 2^{2x} - 9 \times 2^x + 4 = 0$ [Let $u = 2^x$.]

11. Solve each pair of simultaneous equations.
(a) $x^2 + y^2 = 10$, $x + 2y = 7$
(b) $x^2 + y^2 - 2y = 7$, $x - y = 3$
(c) $x^2 + y^2 - 2x + 6y - 35 = 0$, $2x + 3y = 5$

CHALLENGE

12. Use the given substitution to solve each equation for $0° \le x \le 360°$.
(a) $2\sin^2 x - 3\sin x + 1 = 0$ [Let $u = \sin x$.]
(b) $\tan^2 x - 2\tan x + 1 = 0$ [Let $u = \tan x$.]
(c) $2\sin^2 x = 3(\cos x + 1)$ [Let $u = \cos x$.]
(d) $\sec^2 x + 2\tan x = 0$ [Let $u = \tan x$.]

13. Solve each equation by reducing it to a quadratic equation.
(a) $2\left(x + \dfrac{1}{x}\right)^2 + \left(x + \dfrac{1}{x}\right) - 15 = 0$ [Let $u = x + \dfrac{1}{x}$.]
(b) $x(x+1)(x+2)(x+3) = 35$ [Expand $(x+1)(x+2)$ and $x(x+3)$ and let $u = x^2 + 3x$.]

10 E Problems on Maximisation and Minimisation

We come now to an entirely new type of problem that involves finding the maximum or minimum value of a function, and the value of x for which it occurs. This section will only be able to deal with quadratic functions, but in Chapter Twelve, calculus will be used to deal with far more general functions.

Maximisation and Minimisation: The maximum or minimum of a quadratic must occur at the vertex. The value of a distinguishes between maximum and minimum.

12 MAXIMISATION AND MINIMISATION USING THE AXIS OF SYMMETRY:
1. Find the axis of symmetry and substitute it to find the vertex.
2. The sign of a distinguishes between maximum and minimum:
   - When $a > 0$, the graph is concave up and so must have a minimum.
   - When $a < 0$, the graph is concave down and so must have a maximum.

WORKED EXERCISE:

Use the formula $x = -\dfrac{b}{2a}$ to find the axis of symmetry of each quadratic function. Then find its maximum or minimum value and the value of x at which it occurs.

(a) $y = x^2 - 4x + 7$ (b) $y = 3 - 8x - x^2$

SOLUTION:

(a) For $y = x^2 - 4x + 7$, the axis of symmetry is $x = -\left(\dfrac{-4}{2}\right)$

$x = 2.$

When $x = 2$, $y = 4 - 8 + 7$

$= 3.$

Since $a > 0$, the curve is concave up, so y has a minimum of 3 when $x = 2$.

(b) For $y = 3 - 8x - x^2$, the axis of symmetry is $x = -\left(\dfrac{-8}{-2}\right)$

$x = -4.$

When $x = -4$, $y = 3 + 32 - 16$

$= 19.$

Since $a < 0$, the curve is concave down, so y has a maximum of 19 when $x = -4$.

Maximisation and Minimisation Using Factoring: When a quadratic has already been factored, it is easier to find the axis of symmetry by taking the average of the roots rather than expanding the quadratic and using the formula $x = -\dfrac{b}{2a}$.

WORKED EXERCISE:

Find the maximum value of $y = x(20 - x)$.

SOLUTION:

The zeroes are $x = 0$ and $x = 20$, so the axis of symmetry is $x = 10$.
The graph is concave down, because the coefficient of x^2 is negative,
so the value at $x = 10$ is a maximum.
Substituting, the maximum value is $y = 10 \times 10 = 100$.

Solving Problems on Maxima and Minima: When a maximisation problem is presented in words, a function needs to be created. This requires introducing two variables. One variable (conventionally called y) will be the quantity to be maximised. The other variable (conventionally called x) will be the quantity that can be changed.

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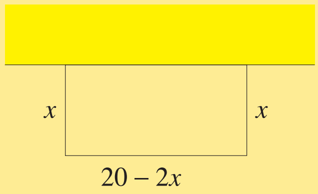
PROBLEMS ON MAXIMA AND MINIMA: After drawing a picture:

1. If no variables have been named, introduce two variables:
   'Let y (or whatever) be the variable to be maximised or minimised.
   Let x (or whatever) be the variable than can be changed.'
2. Express y as a function of x.
3. Find the maximum or minimum value of y and the value of x at which it occurs.
4. Write a careful conclusion in words.

WORKED EXERCISE:
Farmer Brown builds a rectangular chookyard, using an existing wall as one fence. If she has 20 metres of fencing, find the maximum area of the chookyard and the length of the section of the fence that is parallel to the wall.

SOLUTION:
Let x be the length in metres perpendicular to the wall.
Let A be the area of the chookyard.
The length parallel to the wall is $20 - 2x$ metres,
so $$A = x(20 - 2x).$$
Since the zeroes are 0 and 10, the axis is $x = 5$,
and when $x = 5$, $A = 50$.
Hence the maximum area is 50 square metres,
and occurs when the section of the fence parallel to the wall is 10 metres long.



Exercise **10E**

1. Find the minimum value of each quadratic function.
(a) $y = x^2 - 6x + 5$
(b) $y = x^2 - 2x + 5$
(c) $y = x^2 - 6x + 7$
(d) $y = x^2 - 10x + 16$
(e) $y = x^2 + 7$
(f) $y = x^2 + 4x$

2. Find the maximum value of each quadratic function.
(a) $y = -x^2 - 4x - 5$
(b) $y = -x^2 + 2x + 7$
(c) $y = -x^2 - 6x - 3$
(d) $y = 9 - x^2$
(e) $y = 4x - x^2$
(f) $y = 8 - 2x - x^2$

3. Find the maximum or minimum value of each quadratic function.
(a) $y = (x - 3)(x - 7)$
(b) $y = (x + 2)(x + 8)$
(c) $y = -(x - 1)(x - 7)$

4. Two numbers have a sum of 4.
(a) Let the numbers be x and $4 - x$, and show that their product is $P = 4x - x^2$.
(b) Find the value of x at which P will be a maximum, and hence find the maximum value of P.

5. Two numbers have a sum of 30.
(a) Let the numbers be x and $30 - x$, and show that their product is $P = 30x - x^2$.
(b) Find the value of x at which P will be a maximum, and hence find the maximum value of P.

6. Two numbers have a sum of 6.
(a) Let the two numbers be x and $6 - x$, and show that the sum of the squares of the two numbers is $S = 2x^2 - 12x + 36$.
(b) Find the value of x at which S is a minimum, and hence find the least value of S.

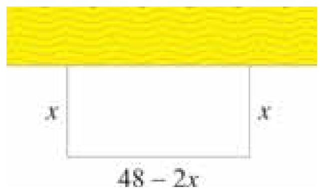
7. A stone is thrown upwards so that at any time t seconds after throwing, the height of the stone is $h = 100 + 10t - 5t^2$ metres.
(a) Find the time at which the stone reaches its maximum height.
(b) Hence find the maximum height reached.

8. A manufacturer finds that the cost C, in thousands of dollars, of manufacturing his product is given by $C = 2x^2 - 8x + 15$, where x is the number of machines.
(a) Find how many machines he should operate to minimise the cost of production.
(b) Hence find the minimum cost of production.

DEVELOPMENT

9. Find the maximum or minimum value of each quadratic function.
(a) $y = x^2 - 3x + 2$
(b) $y = x^2 - 5x + 6$
(c) $y = -x^2 + 3x - 1$
(d) $y = -2x^2 + x + 1$
(e) $y = -3x^2 + 3x - 2$
(f) $y = 4x^2 - 2x + 3$

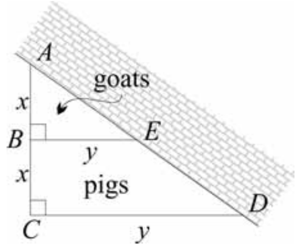
10. A farmer uses 48 metres of fencing to enclose three sides of a rectangular paddock. A river bank forms the fourth side.
(a) Let the paddock have width x metres. Show that it will have length $(48 - 2x)$ metres.
(b) Hence show that the area of the paddock is given by $A = (48x - 2x^2)$ square metres.
(c) Hence find the dimensions of the paddock that will maximise its area.



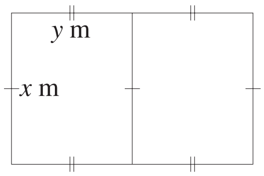
11. A rectangle has a perimeter of 16 metres.
(a) If x is the length of one side, show that the length of the other side is $8 - x$.
(b) Find an expression for the area A of the rectangle in terms of x.
(c) Hence find the maximum value of the area A.

12. Use the method given in the previous question to find the maximum area of a rectangle that has a perimeter of 40 metres.

13. Farmer Brown has 60 metres of fencing available and wishes to construct enclosures for her goats and pigs against an existing wall, as shown in the diagram.
(a) Using the pronumerals given, show that $y = \dfrac{60 - 2x}{3}$.
(b) Hence show that the combined enclosed area A is given by $A = 40x - \frac{4}{3}x^2$.
(c) Find the value of x that makes the area A a maximum.
(d) Find the maximum possible combined area.



14. 1600 metres of fencing is to be used to enclose a rectangular area and to divide it into two equal areas as shown.
(a) Using the pronumerals given, show that $y = \dfrac{1600 - 3x}{4}$.
(b) Hence show that the combined enclosed area A is given by $A = 800x - \frac{3}{2}x^2$.
(c) Hence find the values of x and y for which the area enclosed is greatest.



15. A square $PQRS$ has side length 5 cm. The points A and B lie on the sides PQ and SP respectively such that $PA = BP = x$.
(a) Draw a diagram illustrating this information and shade the quadrilateral $BARS$.
(b) Show that the area of the quadrilateral $BARS$ is given by $\frac{1}{2}(25 + 5x - x^2)$.
[HINT: Subtract the areas of triangles PAB and AQR from the area of the square.]
(c) Hence find the maximum area of the quadrilateral $BARS$.

CHALLENGE

16. A Tasmanian orchardist notices that an apple tree will produce 300 apples per year if 16 trees are planted in every standard-sized field. For every additional tree planted in the standard-sized field, she finds that the yield per tree decreases by 10 apples per year.

(a) Show that if she plants an additional x trees in every standard-sized field, then the total number of apples produced will be $N = -10x^2 + 140x + 4800$.

(b) How many trees should be planted in each field in order to maximise the number of apples that are produced?

17. The total cost of producing x items per day is $\frac{1}{3}x^2 + 45x + 27$ dollars, and the price per item at which each may be sold is $60 - \frac{1}{2}x$ dollars.

(a) Given that the daily profit P can be found by calculating the difference between the revenue and cost of production, show that $P = x(60 - \frac{1}{2}x) - (\frac{1}{3}x^2 + 45x + 27)$.

(b) Hence find the number of items that should be produced in order to maximise the daily profit, and find the maximum daily profit.

18. A piece of wire of length 80 cm is to be cut into two sections. One section is to be bent into a square, and the other into a rectangle 4 times as long as it is wide.

(a) Let x be the side length of the square and y be the width of the rectangle. Show that $2x + 5y = 40$.

(b) Show that the sum A of the area of the square and the area of the rectangle is given by $A = \frac{41}{4}y^2 - 100y + 400$.

(c) Find the lengths of both sections of wire if A is to be a minimum.

10 F The Theory of the Discriminant

In Section 10C, we established that the zeroes of the general quadratic function $y = ax^2 + bx + c$ are

$$x = \frac{-b + \sqrt{\Delta}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{\Delta}}{2a}, \quad \text{where } \Delta = b^2 - 4ac.$$

In this section, the theory of the discriminant Δ is developed further. It is an essential tool for analysing quadratic functions.

The Discriminant Discriminates: The formula above may seem to mean that every quadratic function has two zeroes. The square root in the formula, however, makes the situation more complicated, because:

- negative numbers have no square roots,
- zero has just one square root, namely zero,
- only positive numbers have two square roots.

Thus the number of zeroes depends on the *sign* of the discriminant.

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THE DISCRIMINANT AND THE NUMBER OF ZEROES:

- If $\Delta > 0$, there are two zeroes, $x = \dfrac{-b + \sqrt{\Delta}}{2a}$ and $x = \dfrac{-b - \sqrt{\Delta}}{2a}$.
- If $\Delta = 0$, there is only one zero, $x = -\dfrac{b}{2a}$.
- If $\Delta < 0$, there are no zeroes.

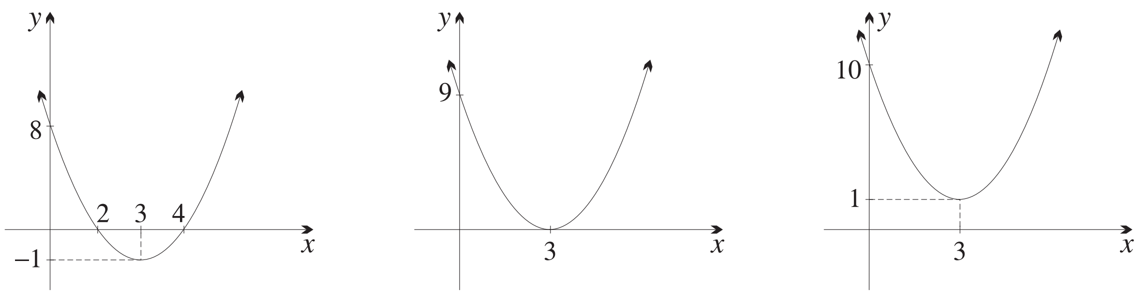
Unreal Zeroes and Double Zeroes: Another form of language is often used here.

- When $\Delta > 0$, there are two zeroes and they are *distinct* (different).
- When $\Delta = 0$, one can think of the situation as two zeroes coinciding, and we say that *there are two equal zeroes*, or that the zero is a *double zero*.
- When $\Delta < 0$, we can say that *there are two unreal zeroes* (meaning two zeroes that are not real numbers), rather than saying that there are no zeroes.

This adjustment of the language allows us to say that every quadratic has two zeroes, and the question then is whether those zeroes are real or unreal, and whether they are equal or distinct.

15 THE LANGUAGE OF DOUBLE ZEROES AND UNREAL ZEROES:

- If $\Delta > 0$, there are two distinct real zeroes.
- If $\Delta = 0$, there is one real *double zero* (or *two equal zeroes*).
- If $\Delta < 0$, there are no zeroes (or two distinct *unreal* zeroes).



$$y = x^2 - 6x + 8 = (x-3)^2 - 1,\quad \Delta = 6^2 - 4 \times 8 = 4$$

$$y = x^2 - 6x + 9 = (x-3)^2,\quad \Delta = 6^2 - 4 \times 9 = 0$$

$$y = x^2 - 6x + 10 = (x-3)^2 + 1,\quad \Delta = 6^2 - 4 \times 10 = -4$$

These three graphs all have axis of symmetry $x = 3$, but different constant terms.

- The first has two real and distinct zeroes, 2 and 4.
- In the second, the two zeroes coincide to give the one *double zero*, $x = 3$.
- The third has no zeroes (or as we shall sometimes say, it has two *unreal* zeroes).

Quadratics that are Perfect Squares: The middle graph above is an example of a quadratic that is a *perfect square*:

$$x^2 - 6x + 9 = (x-3)^2$$

Notice that the discriminant is zero, so that the quadratic has only one zero:

$$\Delta = 6^2 - 4 \times 9 = 0.$$

In general, when $\Delta = 0$ and the two zeroes coincide, the quadratic meets the x-axis in only one point, and the x-axis is a tangent to the curve. The quadratic can then be expressed as a multiple of a perfect square.

16 THE DISCRIMINANT AND PERFECT SQUARES:

When the discriminant $\Delta = 0$:

- the quadratic is a multiple of a perfect square, $y = a(x-\alpha)^2$, and
- the x-axis is a tangent to the parabola at the double zero $x = \alpha$.

Are the Zeroes Rational or Irrational: Suppose now that all the three coefficients in $y = ax^2 + bx + c$ are rational numbers. Then because we need to take the square root of Δ, the zeroes will also be *rational numbers* if Δ is a square. Otherwise the zeroes will involve a surd and be *irrational numbers*. Thus the discriminant allows another distinction to be made about the zeroes:

17 THE DISCRIMINANT AND RATIONAL ZEROES: Suppose that a, b and c are all rational.
- If Δ is a square, then the zeroes are rational.
- If Δ is positive but not a square, then the zeroes are irrational.

WORKED EXERCISE:

Use the discriminant to describe the zeroes of each quadratic. If the quadratic is a multiple of a perfect square, express it in this form.

(a) $y = 5x^2 - 2x - 3$ (b) $y = 3x^2 - 12x + 12$ (c) $y = 8 + 3x - 2x^2$

SOLUTION:

(a) $y = 5x^2 - 2x - 3$,
$\Delta = 4 + 4 \times 15$
$= 64$,
so there are two real zeroes, and they are rational.

(b) $y = 3x^2 - 12x + 12$,
$\Delta = 144 - 4 \times 36$
$= 0$,
so there is one zero, and it is rational.
Also $y = 3(x-2)^2$.

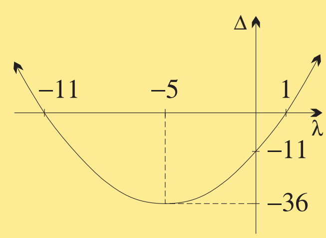
(c) $y = 8 + 3x - 2x^2$,
$\Delta = 9 + 4 \times 16$
$= 73$,
so there are two real zeroes, and they are irrational.

WORKED EXERCISE:

For what values of λ does $x^2 - (\lambda + 5)x + 9 = 0$ have: (a) equal roots, (b) no roots?

SOLUTION:

Here $\Delta = (\lambda + 5)^2 - 36$.



(a) $\Delta = 0$ when $(\lambda + 5)^2 = 36$
$\lambda + 5 = 6$ or $\lambda + 5 = -6$
$\lambda = 1$ or -11,
so there are equal roots when $\lambda = 1$ and when $\lambda = -11$.

(b) There are no roots when Δ is negative,
so from the graph of Δ as a function of λ, there are no roots for $-11 < \lambda < 1$.

Exercise 10F

1. The disciminants of six quadratic equations with rational coefficients are given below. Are the roots of each equation real or unreal? If the roots are real, state whether they are rational or irrational, and whether they are equal or unequal.

(a) $\Delta = 7$ (c) $\Delta = 0$ (e) $\Delta = -3$
(b) $\Delta = -9$ (d) $\Delta = 64$ (f) $\Delta = 16$

2. Find the discriminant Δ of each equation. Hence state how many roots there are, and whether or not they are rational.

(a) $x^2 - 4x + 3 = 0$ (d) $x^2 + 3x + 4 = 0$ (g) $2x^2 - 3x + 5 = 0$
(b) $x^2 - 6x + 9 = 0$ (e) $-x^2 - 4x + 2 = 0$ (h) $6x^2 + 11x - 10 = 0$
(c) $x^2 + 2x - 7 = 0$ (f) $4x^2 + 4x + 1 = 0$ (i) $9x^2 - 1 = 0$

3. Find the discriminant Δ. (It will be an expression involving g.) By solving $\Delta = 0$, find the values of g for which each quadratic function has exactly one zero.

(a) $y = x^2 + 10x + g$
(b) $y = gx^2 - 4x + 1$
(c) $y = 2x^2 - 3x + (g + 1)$
(d) $y = (g - 2)x^2 + 6x + 1$

4. Find the discriminant Δ. (It will be an expression involving k.) By solving $\Delta \geq 0$, find the values of k for which the roots of each quadratic equation are real numbers.

(a) $x^2 + 2x + k = 0$
(b) $kx^2 - 8x + 2 = 0$
(c) $3x^2 - 4x + (k + 1) = 0$
(d) $(2k - 1)x^2 - 5x + 2 = 0$

5. Find the discriminant Δ. (It will be an expression involving ℓ.) By solving $\Delta < 0$, find the values of ℓ for which each quadratic equation has no real roots.

(a) $x^2 + 6x + \ell = 0$
(b) $\ell x^2 - 10x + 1 = 0$
(c) $2x^2 - 5x + (\ell + 3) = 0$
(d) $(\ell - 1)x^2 + 6x + 3 = 0$

DEVELOPMENT

6. By evaluating the discriminant Δ and solving $\Delta = 0$, find the values of g for which each quadratic function has exactly one zero.

(a) $y = gx^2 - gx + 1$
(b) $y = gx^2 + 7x + g$
(c) $y = 4x^2 + 4gx + (6g + 7)$
(d) $y = 9x^2 - 2(g + 1)x + 1$

7. By evaluating the discriminant Δ and solving $\Delta \geq 0$, find the values of k for which each equation has real roots. [HINT: In each case it will be necessary to solve a quadratic inequality. This should always be done by sketching a graph.]

(a) $x^2 + kx + 4 = 0$
(b) $x^2 - 3kx + 9 = 0$
(c) $4x^2 - (6 + k)x + 1 = 0$
(d) $9x^2 + (k - 6)x + 1 = 0$

8. By evaluating the discriminant Δ and solving $\Delta < 0$, find the values of ℓ for which each function has no real zeroes. [HINT: Again, it will be necessary in each case to solve a quadratic inequality. This should always be done by sketching a graph.]

(a) $y = x^2 + \ell x + 4$
(b) $y = \ell x^2 + 6x + \ell$
(c) $y = x^2 + (\ell + 1)x + 4$
(d) $y = 9x^2 - 4(\ell - 1)x - \ell$
(e) $y = \ell x^2 - 4\ell x - (\ell - 5)$
(f) $y = (\ell - 3)x^2 + 2\ell x + (\ell + 2)$

9. (a) Show that the x-coordinates of the points of intersection of the circle $x^2 + y^2 = 4$ and the line $y = x + 1$ satisfy the equation $2x^2 + 2x - 3 = 0$.

(b) Evaluate the discriminant Δ and explain why this shows that there are two points of intersection.

10. Using the method outlined in the previous question, determine how many times the line and circle intersect in each case.

(a) $x^2 + y^2 = 9$, $y = 2 - x$
(b) $x^2 + y^2 = 1$, $y = x + 2$
(c) $x^2 + y^2 = 5$, $y = -2x + 5$
(d) $(x - 3)^2 + y^2 = 4$, $y = x - 4$

11. [HINT: In each case you will need to rearrange the equation into the form $ax^2 + bx + c = 0$ before finding Δ.] Find the values of m for which the roots of the equation:

(a) $x^2 + x = m$ do not exist,
(b) $1 - 3x - mx^2 = 0$ are real and distinct,
(c) $2x^2 + 4x + 5 = 3x + m$ are real and equal,
(d) $x(x - 2m) = m - 2x - 3$ are unreal.

CHALLENGE

12. (a) By substituting the line into the curve, show that $y = x + b$ and $y = 2x^2 - 7x + 4$ intersect when $2x^2 - 8x + (4 - b) = 0$.
(b) By solving $\Delta = 0$, find the value of b for which the line is a tangent to the curve.

13. Find Δ for each equation. By writing Δ as a perfect square, show that each equation has rational roots for all rational values of m and n.
(a) $4x^2 + (m - 4)x - m = 0$
(b) $(m - 1)x^2 + mx + 1 = 0$
(c) $mx^2 + (2m + n)x + 2n = 0$
(d) $2mx^2 - (4m + 1)x + 2 = 0$

14. Prove that the roots of each equation are real and distinct for all real values of λ. [HINT: Find Δ and write it in such a way that it is obviously positive.]
(a) $x^2 + \lambda x - 1 = 0$
(b) $3x^2 + 2\lambda x - 4 = 0$
(c) $\lambda x^2 - (\lambda + 4)x + 2 = 0$
(d) $x^2 + (\lambda + 1)x + (\lambda - 2) = 0$

15. (a) Show that the quadratic equation $(a^2 + b^2)x^2 + 2b(a + c)x + (b^2 + c^2) = 0$, where a, b and c are real constants, has real roots when $(b^2 - ac)^2 \le 0$.
(b) State this condition in a simpler form.

10 G Definite and Indefinite Quadratics

The last section showed how the discriminant discriminates between quadratics with two, one or no zeroes. The distinction between quadratics with no zeroes and those with one zero or two is sufficiently important for special words to be used to describe them.

Definite and Indefinite Quadratics: Let $f(x) = ax^2 + bx + c$ be a quadratic.

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POSITIVE DEFINITE, NEGATIVE DEFINITE AND INDEFINITE:

- $f(x)$ is called *positive definite* if $f(x)$ is positive for all values of x,
  and *negative definite* if $f(x)$ is negative for all values of x.
- $f(x)$ is called *definite* if it is positive definite or negative definite.
- $f(x)$ is called *indefinite* if it is not definite.

These definitions may be clearer when expressed in terms of zeroes:

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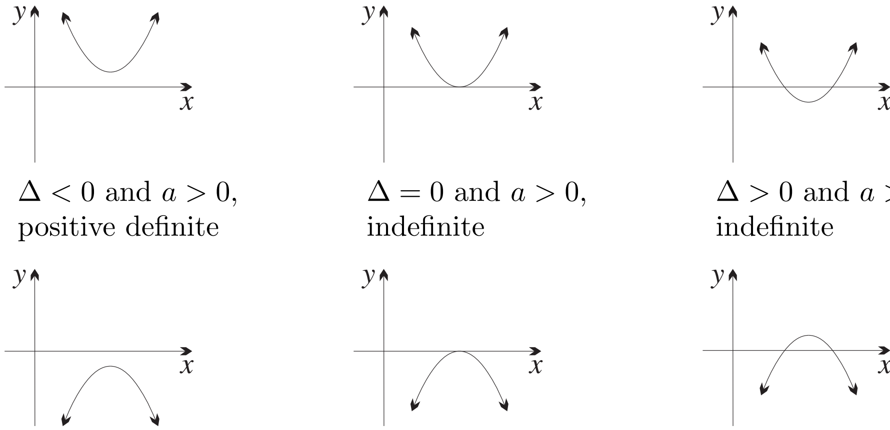
DEFINITE AND INDEFINITE AND ZEROES:

- A quadratic is *definite* if it has no zeroes,
  being *positive definite* if it is always positive,
  and *negative definite* if it is always negative.
- A quadratic is *indefinite* if it has at least one zero.

The word 'definite' means 'we can be definite about the sign of $f(x)$ whatever the value of x'. An 'indefinite' quadratic takes different signs for different values of x (or is zero at least once).

The Six Cases: There are three possibilities for Δ — negative, zero and positive — and two possibilities for a — positive and negative. This makes six possible cases altogether, and these cases are graphed below.



$\Delta < 0$ and $a > 0$, positive definite

$\Delta = 0$ and $a > 0$, indefinite

$\Delta > 0$ and $a > 0$, indefinite

$\Delta < 0$ and $a < 0$, negative definite

$\Delta = 0$ and $a < 0$, indefinite

$\Delta > 0$ and $a < 0$, indefinite

WORKED EXERCISE:

For what values of k is $y = 3x^2 + 6x + k$ positive definite?

SOLUTION:

Here $a = 3$ is positive, and $\Delta = b^2 - 4ac$
$= 36 - 12k.$
$= 12(3 - k).$

Hence $\Delta < 0$ when $k > 3$, and so y is positive definite for $k > 3$.

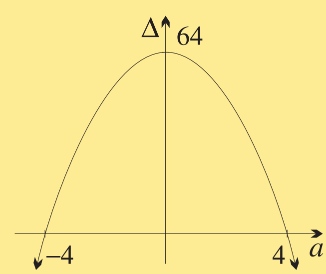
WORKED EXERCISE:

For what values of a is the quadratic $f(x) = ax^2 + 8x + a$ positive definite, negative definite and indefinite?

SOLUTION: Here $\Delta = 64 - 4a^2$
$= 4(16 - a^2),$

so $\Delta \geq 0$ when $-4 \leq a \leq 4$,

and $\Delta < 0$ when $a < -4$ and when $a > 4$.



Hence $f(x)$ is indefinite for $-4 \leq a \leq 4$
(but $a \neq 0$, because when $a = 0$ it is not a quadratic).
Also, $f(x)$ is positive definite for $a > 4$,
and $f(x)$ is negative definite for $a < -4$.

Indefinite Quadratics and Factoring: An indefinite quadratic can always be factored — if it has two zeroes α and β, it can be written as $a(x - \alpha)(x - \beta)$, and if it has one double zero α, it can be written as $a(x - \alpha)^2$.

A definite quadratic, however, cannot be factored, because it has no zeroes.

20 INDEFINITE QUADRATICS AND FACTORING:
A quadratic can be factored into real factors if and only if it is indefinite.

Exercise 10G

1. By examining the discriminant and the coefficient of x^2, show that:
(a) $y = x^2 + 3x + 7$ is positive definite,
(b) $y = x^2 + 3x + 1$ is indefinite,
(c) $y = -x^2 + 5x - 9$ is negative definite,
(d) $y = -x^2 + 7x - 9$ is indefinite.

2. Evaluate the discriminant, then look carefully at the coefficient of x^2. Hence determine whether each function is positive definite, negative definite or indefinite.
(a) $y = 2x^2 - 5x + 7$
(b) $y = x^2 - 4x + 4$
(c) $y = 5x - x^2 - 9$
(d) $y = -x^2 + 7x - 3$
(e) $y = 25 - 20x + 4x^2$
(f) $y = 3x + 2x^2 + 11$

3. Find the values of k for which each expression is:
(i) positive definite (put $\Delta < 0$ and note that $a > 0$), (ii) indefinite (put $\Delta \geq 0$).
(a) $x^2 - 4x + k$
(b) $2x^2 - 5x + 4k$

4. Find the values of m for which each expression is:
(i) negative definite (put $\Delta < 0$ and note that $a < 0$), (ii) indefinite (put $\Delta \geq 0$).
(a) $-x^2 + 6x + m$
(b) $-2x^2 + 3x - m$

DEVELOPMENT

NOTE: In the following questions you will need to solve a quadratic inequality. This should always be done by sketching a graph.

5. Find the values of k for which each expression is:
(i) positive definite (put $\Delta < 0$ and note that $a > 0$), (ii) indefinite (put $\Delta \geq 0$).
(a) $2x^2 - kx + 8$
(b) $3x^2 + (12 - k)x + 12$
(c) $x^2 - 2(k - 3)x + (k - 1)$

6. Find the values of m for which each expressions is:
(i) negative definite (put $\Delta < 0$ and note that $a < 0$), (ii) indefinite (put $\Delta \geq 0$).
(a) $-x^2 + mx - 4$
(b) $-x^2 + (m - 2)x - 25$
(c) $-4x^2 + 4(m + 1)x - (4m + 1)$

7. (a) Show that the parabola $y = x^2 - 2x + 3$ and the line $y = 2 - 3x$ intersect when $x^2 + x + 1 = 0$.
(b) Show that $x^2 + x + 1 = 0$ is positive definite, and hence that the line and parabola never intersect.

8. Find the values of ℓ that will make each quadratic a perfect square.
(a) $x^2 - 2\ell x + 16$
(b) $2\ell x^2 + 2\ell x + 1$
(c) $(5\ell - 1)x^2 - 4x + (2\ell - 1)$
(d) $(4\ell + 1)x^2 - 6\ell x + 4$

CHALLENGE

9. Sketch a possible graph of the quadratic function $y = ax^2 + bx + c$ if:
(a) $a > 0$, $b > 0$, $c > 0$ and $b^2 - 4ac > 0$
(b) $a < 0$, $c < 0$ and $b = 0$
(c) $a > 0$, $b < 0$, $c > 0$ and $b^2 = 4ac$
(d) $a > 0$, $b < 0$ and $b^2 - 4ac < 0$

10. State, in terms of a, b and c, the conditions necessary for $ax^2 + 2bx + 3c$ to be:
(a) positive definite,
(b) negative definite,
(c) indefinite.

10 H Sum and Product of Roots

Many problems on quadratics depend on the sum and product of the roots, rather than on the roots themelves. For example, the axis of symmetry is found by taking the average of the roots, which is half the sum of the roots.

The formulae for the sum and product of the roots are very straightforward, and do not involve the surds that often appear in the roots themselves.

Forming a Quadratic Equation with Given Roots: Suppose that we are asked to form a quadratic equation with roots α and β. The simplest such equation is

$$(x-\alpha)(x-\beta)=0.$$

Expanding this out, $x^2-\alpha x-\beta x+\alpha\beta=0,$
and collecting like terms, $x^2-(\alpha+\beta)x+\alpha\beta=0.$

21 FORMING A QUADRATIC EQUATION WITH ROOTS α AND β:

- One such equation is $(x-\alpha)(x-\beta)=0,$
  which in expanded form is $x^2-(\alpha+\beta)x+\alpha\beta=0.$
- Multiply out any fractions to simplify the equation.

WORKED EXERCISE:
Form quadratic equations with integer coefficients that have roots:
(a) 7 and 10, (b) $3\frac{1}{2}$ and $-2\frac{1}{3}$. (c) $\sqrt{7}$ and $-\sqrt{7}$

SOLUTION:
(a) One such equation is $(x-7)(x-10)=0,$
and expanding, $x^2-17x+70=0.$

(b) One such equation is $\left(x-3\frac{1}{2}\right)\left(x+2\frac{1}{3}\right)=0.$
Multiplication by $6=2\times 3$ will clear both fractions:
$\boxed{\times 6}$ $(2x-7)(3x+7)=0,$
and expanding, $6x^2-7x-49=0.$

(c) One such equation is $\left(x-\sqrt{7}\right)\left(x+\sqrt{7}\right)=0,$
and expanding, $x^2-7=0.$

Formulae for the Sum and Product of Roots: Let $ax^2+bx+c=0$ have roots α and β.
Dividing through by a gives $x^2+\dfrac{b}{a}x+\dfrac{c}{a}=0$, so by Box 21 above:

22 SUM AND PRODUCT OF ROOTS:

If α and β are the roots of $ax^2+bx+c=0$, then

$$\alpha+\beta=-\frac{b}{a} \qquad \text{and} \qquad \alpha\beta=\frac{c}{a}.$$

WORKED EXERCISE:

If α and β are the roots of the equation $2x^2 - 6x - 1 = 0$, find:

(a) $\alpha + \beta$ (b) $\alpha\beta$ (c) $\alpha^2\beta + \alpha\beta^2$ (d) $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$ (e) $\alpha^2 + \beta^2$

SOLUTION:

Here $a = 2$, $b = -6$ and $c = -1$, and we apply the formulae in Box 22.

(a) $\alpha + \beta = -\dfrac{b}{a}$ (The first formula.)
$= -\left(\dfrac{-6}{2}\right)$
$= 3$

(b) $\alpha\beta = \dfrac{c}{a}$ (The second formula.)
$\dfrac{-1}{2}$
$= -\frac{1}{2}$

The results of parts (a) and (b) are now applied to obtain further expressions.

(c) $\alpha^2\beta + \alpha\beta^2 = \alpha\beta(\alpha + \beta)$
$= -\frac{1}{2} \times 3$
$= -1\frac{1}{2}$

(d) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} = \dfrac{\beta + \alpha}{\alpha\beta}$
$= 3 \div (-\frac{1}{2})$
$= -6$

(e) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$
$= 3^2 - 2 \times (-\frac{1}{2})$
$= 9 + 1$
$= 10$

WORKED EXERCISE:

(a) Prove that $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$.

(b) Let $y = x^2 - 9x + 2$ have zeroes α and β. Use the identity in part (a) to find:
(i) $(\alpha - \beta)^2$ (ii) $|\alpha - \beta|$

SOLUTION:

(a) $(\alpha + \beta)^2 - 4\alpha\beta = \alpha^2 + 2\alpha\beta + \beta^2 - 4\alpha\beta$
$= \alpha^2 - 2\alpha\beta + \beta^2$
$= (\alpha - \beta)^2$

(b) (i) In the given function, $\alpha + \beta = 9$ and $\alpha\beta = 2$, so $(\alpha - \beta)^2 = 9^2 - 4 \times 2$
$= 73.$

(ii) Hence from part (i), taking square roots, $|\alpha - \beta| = \sqrt{73}.$

Problems Where a Relation Between the Roots is Known: Some problems involve finding a pronumeral coefficient of a quadratic, given details about the roots.

WORKED EXERCISE:

Find m, given that one of the roots of $x^2 + mx + 18 = 0$ is twice the other.

SOLUTION:

Let the roots be α and 2α. [NOTE: This is the essential step here.]

Then using the product of roots, $\alpha \times 2\alpha = 18$
$\alpha^2 = 9$
$\alpha = 3$ or -3.

Now using the sum of roots, $\alpha + 2\alpha = -m$
$m = -3\alpha$
$m = -9$ or 9.

Exercise 10H

1. Form quadratic equations that have roots:
   (a) 4 and 7, (b) -5 and 2, (c) -9 and -8, (d) 0 and 12.

2. (a) Use the formulae $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$ to write down the sum $\alpha + \beta$ and the product $\alpha\beta$ of the roots of $x^2 + 7x + 10 = 0$.
   (b) Solve $x^2 + 7x + 10 = 0$ by factoring, and check your results.

3. Without solving the equation, write down the sum $\alpha + \beta$ and the product $\alpha\beta$ of the roots of each quadratic equation.
   (a) $x^2 - 2x + 5 = 0$
   (b) $x^2 + x - 6 = 0$
   (c) $x^2 + x = 0$
   (d) $2x^2 + 3x - 1 = 0$
   (e) $3x^2 - 6x + 1 = 0$
   (f) $-x^2 + 3x - 6 = 0$

4. Rearrange each equation into the form $ax^2 + bx + c = 0$. Then write down the sum $\alpha + \beta$ and the product $\alpha\beta$ of the roots.
   (a) $x^2 = 3x - 2$
   (b) $5 = 2x - x^2$
   (c) $3x^2 - 4 = x$
   (d) $4 + 5x^2 = -5x$
   (e) $3x^2 + 2x = 4(x + 1)$
   (f) $(x + 2)^2 = 5 - x$

DEVELOPMENT

5. If α and β are the zeroes of $y = x^2 - 4x - 5$, find the values of the following without actually finding α and β.
   (a) $\alpha + \beta$ (b) $\alpha\beta$ (c) $2\alpha + 2\beta$ (d) $\frac{1}{\alpha} + \frac{1}{\beta}$

6. Let α and β be the roots of $x^2 - 7x + 12 = 0$. Without actually solving the equation, find:
   (a) $\alpha + \beta$ (b) $\alpha\beta$ (c) $\alpha^2\beta^3 + \alpha^3\beta^2$ (d) $\frac{2}{\alpha} + \frac{2}{\beta}$

7. If α and β are the zeroes of $y = x^2 - 3x + 2$, find the values of the following without actually finding α and β. In part (g) you will need the identity $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$.
   (a) $\alpha + \beta$
   (b) $\alpha\beta$
   (c) $7\alpha + 7\beta$
   (d) $\alpha^2\beta + \alpha\beta^2$
   (e) $(\alpha + 3)(\beta + 3)$
   (f) $\frac{1}{\alpha} + \frac{1}{\beta}$
   (g) $\alpha^2 + \beta^2$
   (h) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$

8. Let α and β be the roots of $2x^2 - 5x + 1 = 0$. Without actually solving the equation, find:
   (a) $\alpha + \beta$
   (b) $\alpha\beta$
   (c) $(\alpha - 1)(\beta - 1)$
   (d) $\alpha^{-1} + \beta^{-1}$
   (e) $\alpha^3\beta^2 + \alpha^2\beta^3$
   (f) $\frac{2}{\alpha} + \frac{2}{\beta}$
   (g) $\alpha^2 + \beta^2$
   (h) $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$

9. (a) Show that $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$.
   (b) Without actually solving the equation, find the sum $\alpha + \beta$ and the product $\alpha\beta$ of the roots of each equation below. Then find $(\alpha - \beta)^2$.
   (i) $x^2 - 3x + 1 = 0$ (ii) $x^2 + 5x - 7 = 0$
   (c) Hence use the fact that $|\alpha - \beta| = \sqrt{(\alpha - \beta)^2}$ to find $|\alpha - \beta|$ for each equation.

10. A quadratic equation with roots α and β has the form $x^2 - (\alpha + \beta)x + \alpha\beta = 0$. Find the quadratic equations that have the following roots:
    (a) 1 and 3
    (b) -2 and 6
    (c) -1 and -4
    (d) $\frac{1}{2}$ and $\frac{3}{2}$
    (e) $2 + \sqrt{3}$ and $2 - \sqrt{3}$
    (f) $-1 - \sqrt{5}$ and $-1 + \sqrt{5}$

11. Find m if α and β are the roots of $(2m-1)x^2 + (1+m)x + 1 = 0$ and:
(a) $\alpha + \beta = 0$ (b) $\alpha\beta = 1$ (c) $\alpha = 2$ (d) $\alpha + \beta = 2\alpha\beta$

12. Find the values of g for which the function $y = 2x^2 - (3g-1)x + (2g-5)$ has:
(a) one zero equal to 0, [HINT: Let the zeroes be 0 and α.]
(b) the sum of the zeroes equal to their product, [HINT: Put $\alpha + \beta = \alpha\beta$.]
(c) zeroes that are reciprocals of each other, [HINT: Let the zeroes be α and $\dfrac{1}{\alpha}$.]
(d) zeroes equal in magnitude but opposite in sign. [HINT: Let the zeroes be α and $-\alpha$.]

CHALLENGE

13. (a) Find the value of m if one root of the equation $x^2 + 6x + m = 0$ is twice the other. [HINT: Let the roots be α and 2α and show that $3\alpha = -6$ and $2\alpha^2 = m$.]
(b) Find the value of ℓ if one zero of $y = x^2 - 2\ell x + (\ell + 3)$ is three times the other.
(c) The roots of the quadratic equation $x^2 - mx + n = 0$ differ by 1. Without actually solving the equation, prove that $m^2 = 4n + 1$.

14. (a) Show that $\alpha^3 + \beta^3 = (\alpha + \beta)(\alpha^2 - \alpha\beta + \beta^2)$.
(b) Hence find $\alpha^3 + \beta^3$, where α and β are the zeroes of the function $y = 3x^2 - 5x - 4$.

15. The line $y = 6x + 9$ crosses the parabola $y = x^2 + 1$ at A and B.
(a) By solving the line and parabola simultaneously, show that the x-coordinates of A and B satisfy the equation $x^2 - 6x - 8 = 0$.
(b) Without solving the equation, find the sum of the roots of the quadratic.
(c) Hence find the x-coordinate of the midpoint M of AB.
(d) Substitute into the equation of the line to find the y-coordinate of M.

16. The line $y = 2x + 3$ crosses the circle $x^2 + y^2 = 25$ at P and Q.
(a) Show that the x-coordinates of P and Q satisfy the equation $5x^2 + 12x - 16 = 0$.
(b) Without solving the equation, find the sum of the roots of the equation.
(c) Hence find the coordinates of the midpoint M of PQ.

10 I Quadratic Identities

This final section presents a useful theorem about the coefficients of quadratics. The theorem says that if we know just three points on the graph of a quadratic function, then we can calculate the whole function. This can be applied to proving quadratic identites and to finding the coefficients of a quadratic.

Theorem — Three Values Determine a Quadratic: The main theorem states that if two quadratic functions are equal for at least three distinct values of x, then they are equal for *all* values of x and their coefficients are equal.

23 THEOREM: Suppose that the two quadratic functions

$$f(x) = ax^2 + bx + c \qquad \text{and} \qquad g(x) = a_1x^2 + b_1x + c_1$$

take the same value for at least three distinct values of x. Then:
- the coefficients a_1, b_1 and c_1 are respectively equal to the coefficients a, b and c,
- $f(x)$ and $g(x)$ are equal for all values of x.

The proof is rather elusive and is given in the appendix to this chapter.

A Notation for Identically Equal Quadratics: If two quadratic functions $f(x)$ and $g(x)$ are equal for all values of x, they are called *identically equal*. The notation for this is $f(x) \equiv g(x)$.

24 IDENTICALLY EQUAL QUADRATIC FUNCTIONS: The notation

$$f(x) \equiv g(x)$$

means that $f(x) = g(x)$ for all values of x.

For example, the well-known difference of squares identity can now be written as

$$(x - a)(x + a) \equiv x^2 - a^2.$$

Application of the Theorem to Identities: The theorem at the start of this section means that an identity involving quadratics can be proven by showing that it is true for just three values of x. These three values can be conveniently chosen to simplify the calculations.

WORKED EXERCISE:

Given that a, b and c are distinct constants, prove that

$$(x - a)(x - b) + (x - b)(x - c) \equiv (x - b)(2x - a - c).$$

SOLUTION:

It will be sufficient to prove the result for $x = a$, $x = b$ and $x = c$.

When $x = a$, LHS $= 0 + (a - b)(a - c)$ and RHS $= (a - b)(a - c)$,

when $x = b$, LHS $= 0 + 0$ and RHS $= 0$,

when $x = c$, LHS $= (c - a)(c - b)$ and RHS $= (c - b)(c - a)$.

Hence, being true for three distinct values of x, the identity holds for all values of x.

Application of the Theorem to Finding Coefficients: If two quadratic expressions are identically equal, there are two ways of generating equations for finding unknown coefficients.

25 TWO METHODS OF GENERATING EQUATIONS FOR FINDING COEFFICIENTS:

- Equate coefficients of like terms.
- Substitute carefully chosen values of x.

WORKED EXERCISE:

Express n^2 in the form $a(n - 3)^2 + b(n - 3) + c$.

SOLUTION:

Let	$n^2 \equiv a(n - 3)^2 + b(n - 3) + c.$	
Equating coefficients of n^2,	$1 = a.$	(1)
Put $n = 3$, then since $n - 3 = 0$,	$9 = 0 + 0 + c.$	(2)
Put $n = 0$, then	$0 = 9a - 3b + c,$	
and since $a = 1$ and $c = 9$,	$0 = 9 - 3b + 9$	
	$b = 6.$	(3)
Hence	$n^2 \equiv (n - 3)^2 + 6(n - 3) + 9.$	

Geometrical Implications of the Theorem: Here are some of the geometrical versions of the theorem, given in the language of coordinate geometry.

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GEOMETRICAL IMPLICATIONS:
- The graph of a quadratic function is completely determined by any three points on the curve.
- The graphs of two distinct quadratic functions cannot intersect in more than two points.
- A line cannot intersect a parabola in more than two points.

- Algebraically, a quadratic $ax^2 + bx + c$ is said to have *degree 2* because it has only a term in x^2, a term in x and a constant term.
- Geometrically, a parabola is said to have *degree 2* because some lines intersect it in two points, but no line intersects it in more than two points.

The third statement above shows that these two contrasting ideas of degree coincide. Hence the theorem at the start of this section links the algebra of the quadratic function with the geometry of the parabola.

Exercise 10I

1. Given that $a(x+2) + b(x-1) = 5x + 1$ for all values of x, expand the LHS and show that $a + b = 5$ and $2a - b = 1$. Solve the equations simultaneously to find a and b.

2. Given that $a(x+3) + b(x-2) = 3x - 1$ for all values of x, expand the LHS and show that $a + b = 3$ and $3a - 2b = -1$. Solve the equations simultaneously to find a and b.

3. Suppose that $2x^2 - 5x + 3 \equiv a(x-2)^2 + b(x-2) + c$.
   (a) Substitute $x = 2$ into the identity to find the value of c.
   (b) Equate the coefficients of x^2 to find the value of a.
   (c) Substitute $x = 0$ into the identity to find the value of b.

4. Suppose that $x^2 + x + 1 = a(x-1)^2 + b(x-1) + c$ for all values of x.
   (a) Substitute $x = 1$ into the identity to find the value of c.
   (b) Equate the coefficients of x^2 to find the value of a.
   (c) Substitute $x = 0$ into the identity to find the value of b.

5. (a) Find a, b and c if $n^2 - n \equiv a(n-4)^2 + b(n-4) + c$.
   (b) Express $2x^2 + 3x - 6$ in the form $a(x+1)^2 + b(x+1) + c$.
   (c) Find a, b and c if $2x^2 + 4x + 5 \equiv a(x-3)^2 + b(x-3) + c$.

DEVELOPMENT

6. (a) Show that the quadratic identity $x(x-b) + x(x-a) \equiv x(2x-a-b)$ holds when $x = a$, $x = b$ and $x = 0$, where a and b are distinct and non-zero.
   (b) Explain why it follows that the identity is true for all values of x.

7. Prove the identity $\dfrac{x(x-a)}{b-a} + \dfrac{x(x-b)}{a-b} = x$, where a and b are distinct and non-zero. [HINT: Substitute $x = a$, $x = b$ and $x = 0$.]

8. Prove the following identities, where p, q and r are distinct.

(a) $(p+q+x)(pq+px+qx) - pqx \equiv (p+q)(p+x)(q+x)$
[Let $x = 0$, $x = -p$ and $x = -q$.]

(b) $\dfrac{p(x-q)(x-r)}{(p-q)(p-r)} + \dfrac{q(x-p)(x-r)}{(q-r)(q-p)} + \dfrac{r(x-p)(x-q)}{(r-p)(r-q)} \equiv x$ [Let $x = p$, $x = q$ and $x = r$.]

(c) $\dfrac{(p-x)(q-x)}{(p-r)(q-r)} + \dfrac{(q-x)(r-x)}{(q-p)(r-p)} + \dfrac{(r-x)(p-x)}{(r-q)(p-q)} \equiv 1$ [Let $x = p$, $x = q$ and $x = r$.]

9. (a) Express x^2 in the form $a(x+1)^2 + b(x+1) + c$.
(b) Express n^2 in the form $a(n-4)^2 + b(n-4) + c$.
(c) Express x^2 in the form $a(x+2)^2 + b(x+2) + c$.

CHALLENGE

10. Find the values of a, b and c for which $m^2 \equiv a(m-1)^2 + b(m-2)^2 + c(m-3)^2$.

11. (a) Show that $m^2 - (m-1)^2 = 2m - 1$.
(b) Hence find the sum of the series $1 + 3 + 5 + \cdots + 61$.
(c) Check your answer, using the formula for the sum of an arithmetic series.
(d) Find the sum of the series $1 + 3 + 5 + \cdots$ to n terms, using each of these two methods.

10J Chapter Review Exercise

1. Use factoring to solve each quadratic equation.
(a) $x^2 - 5x + 6 = 0$ (b) $x^2 - 4x - 21 = 0$ (c) $2x^2 - 7x - 15 = 0$

2. Use the quadratic formula to solve each quadratic equation.
(a) $x^2 - 2x - 1 = 0$ (b) $x^2 - 7x + 11 = 0$ (c) $2x^2 - 4x + 1 = 0$

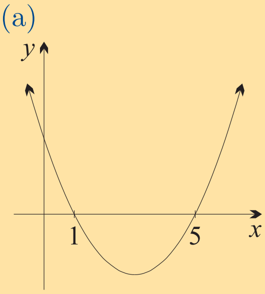
3. Complete the square in each quadratic.
(a) $y = x^2 + 6x + 7$ (b) $y = x^2 - 8x + 3$ (c) $y = -x^2 - 4x + 5$

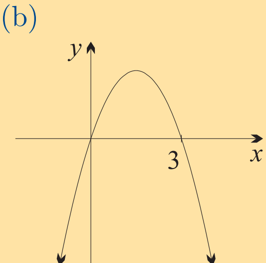
4. Explain how to shift the graph of $y = x^2$ in order to graph each function.
(a) $y = x^2 + 5$ (b) $y = x^2 - 1$ (c) $y = (x-3)^2$ (d) $y = (x+4)^2 + 7$

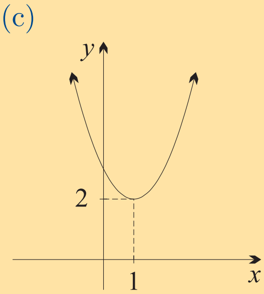
5. State the equation of the monic quadratic function with zeroes:
(a) 2 and 5 (b) 3 and -1 (c) -4 and 7 (d) -5 and -2

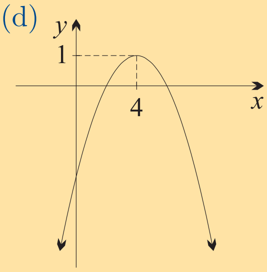
6. Write down the equation of the monic quadratic with vertex:
(a) $(1, 0)$ (b) $(0, -2)$ (c) $(-1, 5)$ (d) $(4, -9)$

7. State the equation of each quadratic function sketched below, given that the coefficient of x^2 in each function is either 1 or -1.

(a)


(b)


(c)


(d)


8. Sketch the graph of each function, indicating the vertex and all intercepts with the axes.
(a) $y = x^2 + 2x - 15$ (b) $y = 8 + 2x - x^2$ (c) $y = x^2 - 6x + 7$

9. Sketch a graph to solve each quadratic inequation.
(a) $x^2 - 4 > 0$ (b) $x^2 + x - 12 \leq 0$ (c) $x^2 - 3x - 10 \geq 0$

10. Solve each pair of equations simultaneously. Hence state how many times the parabola and the straight line intersect.
(a) $y = x^2$ and $y = 3x + 4$, (b) $y = x^2 + 7x - 6$ and $y = 3x + 2$.

11. By using the given substitution, reduce each equation to a quadratic equation and solve it.
(a) $x^4 - 10x^2 + 9 = 0$ [HINT: Let $u = x^2$.]
(b) $x^6 - 7x^3 - 8 = 0$ [HINT: Let $u = x^3$.]
(c) $3^{2x} - 10 \times 3^x + 9 = 0$ [HINT: Let $u = 3^x$.]
(d) $(x^2 - x)^2 - 14(x^2 - x) + 24 = 0$ [HINT: Let $u = x^2 - x$.]

12. Find the minimum or maximum value of each function.
(a) $y = x^2 - 2x + 5$ (c) $y = 2x^2 - 8x + 11$
(b) $y = -x^2 + 4x - 3$ (d) $y = -x^2 - 3x + 1$

13. Two circles have radii r_1 and r_2, where $r_1 + r_2 = 10$.
(a) Show that the sum of the areas of the two circles is given by $A = 2\pi({r_1}^2 - 10r_1 + 50)$.
(b) Hence find the radius of each circle so that the sum of the areas is a minimum.

14. Find the discriminant Δ of each equation. Hence state how many roots each equation has, and whether or not they are rational.
(a) $x^2 - 7x + 6 = 0$ (b) $4x^2 - 4x + 1 = 0$ (c) $x^2 + 3x - 5 = 0$

15. Find the values of k for which the quadratic equation $x^2 + 6x + k = 0$ has:
(a) real roots, (b) equal roots, (c) unreal roots.

16. Find the values of m for which the quadratic equation $x^2 - 3mx + 9 = 0$ has:
(a) real roots, (b) equal roots, (c) unreal roots.

17. Evaluate the discriminant and look carefully at the coefficient of x^2 to determine whether each quadratic is positive definite, negative definite or indefinite.
(a) $y = x^2 - 5x + 7$ (b) $y = -2x^2 + x - 3$ (c) $y = 3x^2 - 4x - 2$

18. Find the value of g for which the function $y = 3x^2 - 4x + 2g$ is positive definite.

19. Find the values of ℓ that will make the quadratic $y = (\ell + 6)x^2 - 2\ell x + 3$ a perfect square.

20. Let α and β be the roots of $x^2 - 3x - 2 = 0$. Without solving the equation, find:
(a) $\alpha + \beta$ (b) $\alpha\beta$ (c) $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$ (d) $\alpha^2 + \beta^2$

21. Let α and β be the roots of $3x^2 - 4x - 1 = 0$. Without solving the equation, find:
(a) $\alpha + \beta$ (b) $\alpha\beta$ (c) $\alpha^2\beta^3 + \alpha^3\beta^2$ (d) $\alpha^2 + \beta^2$

22. Form a quadratic equation with roots $2 + \sqrt{3}$ and $2 - \sqrt{3}$.

23. Find the values of a, b and c, given that $3x^2 - 5x + 7 \equiv a(x - 1)^2 + b(x - 1) + c$.

24. Prove the quadratic identity

$$\frac{a^2(x - b)(x - c)}{(a - b)(a - c)} + \frac{b^2(x - c)(x - a)}{(b - c)(b - a)} + \frac{c^2(x - a)(x - b)}{(c - a)(c - b)} \equiv x^2$$

where a, b and c are distinct constants. [HINT: Let $x = a$, $x = b$ and $x = c$.]

Appendix — Identically Equal Quadratics

Here is a proof of the main theorem of Section 8I.

Theorem — Three Values Determine a Quadratic: Suppose that the two expressions

$$f(x) = ax^2 + bx + c \qquad \text{and} \qquad g(x) = a_1x^2 + b_1x + c_1$$

take the same value for at least three distinct values of x. Then:

- the coefficients a_1, b_1 and c_1 are respectively equal to the coefficients a, b and c,
- $f(x)$ and $g(x)$ are equal for all values of x.

PROOF:

A. First we prove the following particular case:

'Suppose that the expression $Ax^2 + Bx + C$ is zero for the three distinct values $x = \alpha$, $x = \beta$ and $x = \gamma$. Then $A = B = C = 0$.'

Substituting,
$$A\alpha^2 + B\alpha + C = 0 \qquad (1)$$
$$A\beta^2 + B\beta + C = 0 \qquad (2)$$
$$A\gamma^2 + B\gamma + C = 0. \qquad (3)$$

Subtracting (2) from (1), $A(\alpha^2 - \beta^2) + B(\alpha - \beta) = 0$,

and since $\alpha \neq \beta$, we can divide through by $\alpha - \beta$ so that
$$A(\alpha + \beta) + B = 0. \qquad (4)$$

Similarly from (2) and (3), $A(\beta + \gamma) + B = 0. \qquad (5)$

Now subtracting (5) from (4), $A(\alpha - \gamma) = 0$,

and since $\alpha \neq \gamma$, $A = 0$.

Then from (4), $B = 0$ and from (1), $C = 0$.

B. To prove the main theorem, let
$$h(x) = f(x) - g(x) = (a - a_1)x^2 + (b - b_1)x + (c - c_1).$$

Then since $f(x)$ and $g(x)$ agree at three distinct values of x,
it follows that $h(x)$ is zero for these three values of x,
so, by the result in part A, $a - a_1 = b - b_1 = c - c_1 = 0$, as required.

CHAPTER **ELEVEN**

Locus and the Parabola

This short chapter deals with the use of algebraic methods to investigate curves that have been defined geometrically. Such a curve is usually called a *locus*.

In particular, the parabola has been regarded until now simply as the graph of a quadratic function. This curve will now be given a geometric definition in terms of a fixed line called the *directrix* and a fixed point called the *focus*.

11 A A Locus and its Equation

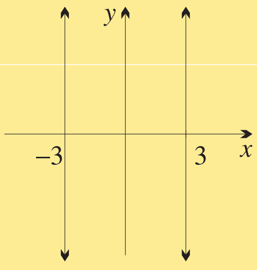
A *locus* is a set of points. The word usually implies that the set has been described geometrically. Usually a locus is some sort of curve, and it is often thought of as the path traced out by a moving point.

This section has two purposes:
- to find the algebraic equation of a locus that has been described geometrically,
- to describe geometrically a locus that has been defined algebraically.

Simple Loci — Sketch and Write Down the Equation: Some simple loci require no more than a sketch, from which the equation can easily be written down.

1 SIMPLE LOCI: Sketch the locus, then write down its equation.

WORKED EXERCISE:
Sketch the locus of a point whose distance from the y-axis is 3 units, then write down its equation.



SOLUTION:
From the sketch, the equation of the locus is

$$x = 3 \text{ or } x = -3.$$

Notice that this locus is not a function. It can also be written down algebraically as a single equation, $x^2 = 9$, or as $|x| = 3$.

Finding the Locus Algebraically: Generally, however, algebraic methods are required to find the equation of a locus. Draw a sketch with a general point $P(x, y)$ placed on the coordinate plane. Then the formal algebraic work should begin as follows:

2 HARDER LOCI: 'Let $P(x, y)$ be any point in the plane.
The condition for P to lie on the locus is . . . '.
NOTE: When the distance formula is used, it is best to square the geometric condition first, to avoid the complication of square roots.

We start with *any point* in the plane, because we seek an algebraic equation such that the point lies on the locus if and only if its coordinates satisfy the equation.

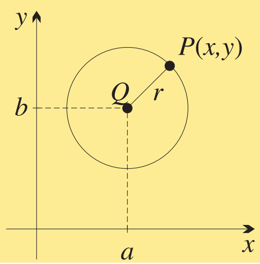
Example — Finding the Equation of a Circle: The precise definition of *circle* defines a circle geometrically as a locus in terms of its centre and radius.

3 THE GEOMETRIC DEFINITION OF A CRICLE:
A *circle* is the locus of all points in the plane that are a fixed distance (called the *radius*) from a given point (called the *centre*).

The first worked exercise below establishes the well-known equation for a circle from this definition of the circle as a locus.

WORKED EXERCISE:
Use the geometric definition of a circle to find the equation of the circle with centre $Q(a, b)$ and radius r.

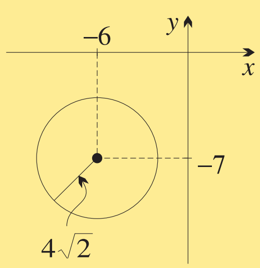
SOLUTION:
Let $P(x, y)$ be any point in the plane.
The condition for P to lie on the locus is
$$PQ = r.$$
Squaring both sides, $PQ^2 = r^2$.
Using the distance formula for the distance from $P(x, y)$ to $Q(a, b)$,
$$(x - a)^2 + (y - b)^2 = r^2.$$



WORKED EXERCISE:
(a) Find the equation of the locus of a point that moves so that its distance from the point $A(2, 1)$ is twice its distance from the point $B(-4, -5)$.
(b) Complete the square and describe the locus geometrically.

SOLUTION:
(a) Let $P(x, y)$ be any point in the plane.
The condition for P to lie on the locus is
$$PA = 2 \times PB \quad \text{(This was stated in the question.)}$$
[square] $$PA^2 = 4 \times PB^2$$
$$(x - 2)^2 + (y - 1)^2 = 4(x + 4)^2 + 4(y + 5)^2$$
$$x^2 - 4x + 4 + y^2 - 2y + 1 = 4x^2 + 32x + 64 + 4y^2 + 40y + 100$$
$$0 = 3x^2 + 36x + 3y^2 + 42y + 159$$
[$\div 3$] $$x^2 + 12x + y^2 + 14y = -53.$$

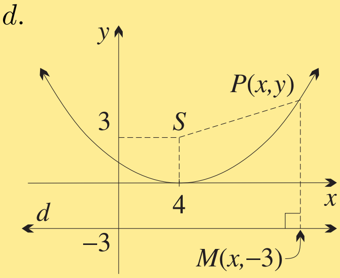
(b) Completing the squares in x and in y,
[$+36 + 49$] $$x^2 + 12x + 36 + y^2 + 14y + 49 = -53 + 36 + 49$$
$$(x + 6)^2 + (y + 7)^2 = 32.$$
$$(x + 6)^2 + (y + 7)^2 = (4\sqrt{2})^2.$$
Hence, using the standard result from the previous worked exercise, the locus is a circle with centre $(-6, -7)$ and radius $4\sqrt{2}$.



Example — Finding the Equation of a Parabola: The locus in the worked example below is a parabola, whose geometric definition is the subject of the next section.

WORKED EXERCISE:
Find the equation of the locus of all points that are equidistant from the point $S(4,3)$ and the line $d : y = -3$.

SOLUTION:
Let $P(x,y)$ be any point in the plane,
and let $M(x,-3)$ be the foot of the perpendicular from P to d.
The condition for P to lie on the locus is



$$PS = PM$$

square
$$PS^2 = PM^2$$
$$(x-4)^2 + (y-3)^2 = (y+3)^2$$
$$(x-4)^2 + y^2 - 6y + 9 = y^2 + 6y + 9$$
$$(x-4)^2 = 12y.$$

Exercise 11A

NOTE: Solve question 1 by sketching the points $P(x,y)$ on the number plane. Solve all questions from question 5 onwards by starting with the phrase:

'The condition for $P(x,y)$ to lie on the locus is ...'

1. Sketch the locus of the point $P(x,y)$ that is subject to each of the following conditions. Then use the graph to write down its equation.
(a) P is 2 units below the x-axis.
(b) P is 1 unit to the left of the y-axis.
(c) P is equidistant from the lines $y = -1$ and $y = 5$.
(d) P is 3 units from the origin.
(e) P is 3 units from the point $(-3,1)$.
(f) P is equidistant from the lines with equations $y = x + 3$ and $y = x + 7$.
(g) The distance of P from the x-axis is three times its distance from the y-axis.

2. Write down the equation of the circle with:
(a) centre $(0,0)$, radius 2
(b) centre $(2,3)$, radius 1
(c) centre $(0,-4)$, radius 3
(d) centre $(-5,-2)$, radius $\frac{1}{2}$

3. Write down the centre and radius of each circle.
(a) $x^2 + y^2 = 1$
(b) $(x+1)^2 + y^2 = 4$
(c) $(x-2)^2 + (y+3)^2 = 5$
(d) $x^2 + (y-4)^2 = 64$

4. Complete the squares to find the centre and radius of each circle.
(a) $x^2 + y^2 - 2y = 3$
(b) $x^2 + 6x + y^2 + 8 = 0$
(c) $x^2 - 4x + y^2 + 6y - 3 = 0$
(d) $x^2 + y^2 - 8x + 14y = 35$

5. (a) Using the distance formula, find an expression for the square of the distance between the points $P(x,y)$ and $A(3,1)$.
(b) Hence derive the equation of the locus of the point $P(x,y)$ that moves so that it is always a distance of 4 units from A.

6. (a) Use the distance formula to find the equation of the locus of the point $P(x, y)$ that moves so that it is equidistant from the points $R(-2, 4)$ and $S(1, 2)$.

(b) (i) Find the midpoint M of RS, and the gradient of RS.

(ii) Find the equation of the line through M perpendicular to RS.

(iii) How does this compare with the answer to part (a)?

7. (a) Given the points $A(4, 0)$, $B(-2, 0)$ and $P(x, y)$, find the gradients of AP and BP.

(b) Hence show that the locus of the point $P(x, y)$ that moves so that $\angle APB$ is a right angle has equation $x^2 + y^2 - 2x - 8 = 0$.

(c) Complete the square in x, and hence describe the locus geometrically.

DEVELOPMENT

8. Given the points $A(1, 4)$ and $B(-3, 2)$, find the equation of each locus of the point $P(x, y)$, and describe each locus geometrically.

(a) P is equidistant from A and B.

(b) $\angle APB$ is a right angle.

(c) P is equidistant from A and the x-axis.

9. (a) Find the locus of the point $P(x, y)$ whose distance from the point $A(4, 0)$ is always twice its distance from the point $B(1, 0)$.

(b) Find the locus of the point $P(x, y)$ whose distance from the point $A(2, 5)$ is always twice its distance from the point $B(4, -1)$.

10. (a) Find an expression for the square of the distance from the point $P(x, y)$ to the point $K(-1, 3)$.

(b) Find an expression for the square of the distance of $P(x, y)$ from the line $y = 0$.

(c) The point $P(x, y)$ moves so that it is equidistant from K and the line $y = 0$. Show that the locus of P is a parabola, and find its vertex.

11. (a) Use the distance formula to find the square of the distance between the point $P(x, y)$ and each of the points $A(0, 4)$, $B(0, -4)$ and $C(6, 3)$.

(b) A point $P(x, y)$ moves so that the sum of the squares of its distances from the points A, B and C is 77. Show that the locus is a circle and find its centre and radius.

12. A point $P(x, y)$ moves so that its distance from the point $K(2, 5)$ is twice its distance from the line $x = -1$. Draw a diagram, and find the equation of the locus of P.

CHALLENGE

13. (a) By using the perpendicular distance formula, find the locus of a point $P(x, y)$ that is equidistant from the lines $3x + 4y = 36$ and $4x + 3y = 24$.

(b) Show that the locus consists of two perpendicular lines, and sketch all four lines on the same number plane.

14. (a) Sketch the locus of all points that are 1 unit from the circle with centre the origin and radius 2 units.

(b) Sketch the locus of all points that are 1 unit from the square with centre the origin and vertices $A(1, 1)$, $B(-1, 1)$, $C(-1, -1)$ and $D(1, -1)$.

15. Find the locus of all points equidistant from the three points $A(3, 0)$, $B(0, 9)$ and $C(7, 8)$.

11 B The Geometric Definition of the Parabola

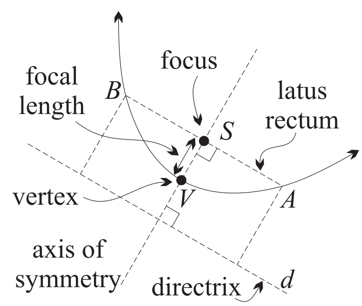
Until now, the word 'parabola' has meant any curve whose equation is a quadratic. The parabola, however, is a geometric object, and needs a proper geometric definition. This locus definition is similar to the locus definition of a circle, and the general algebraic equation can then be derived from this geometric definition.

4

> GEOMETRIC DEFINITION OF A PARABOLA:
> A *parabola* is the locus of all points equidistant from
> - a given point (called the *focus*), and
> - a given line (called the *directrix*),
>
> where the directrix does not pass through the focus.

The Vertex, the Axis of Symmetry and the Latus Rectum: The diagram below begins to explain the definition and to define some special points and lines associated with a parabola. Let S be the focus and let the line d be the directrix.

- The *vertex* V is the point on the parabola midway between the focus and the directrix.
- The *axis of symmetry* is the line through the focus, perpendicular to the directrix. The whole diagram is symmetric in this line.
- Draw a line passing through the focus S, parallel to the directrix d. Let this line meet the parabola at A and B. The interval AB is called the *latus rectum*, which means *the line at right angles* (to the axis of symmetry).
- Because the points A and B lie on the parabola, they are equidistant from the focus and the directrix. This results in *two squares*, one on each side of the axis.



The Focal Length: The *focal length* is the distance between the vertex and the focus. It is usually given the pronumeral a, and it is positive, because it is a distance.

Other important distances can be expressed in terms of the focal length, using the two squares in the diagram above.

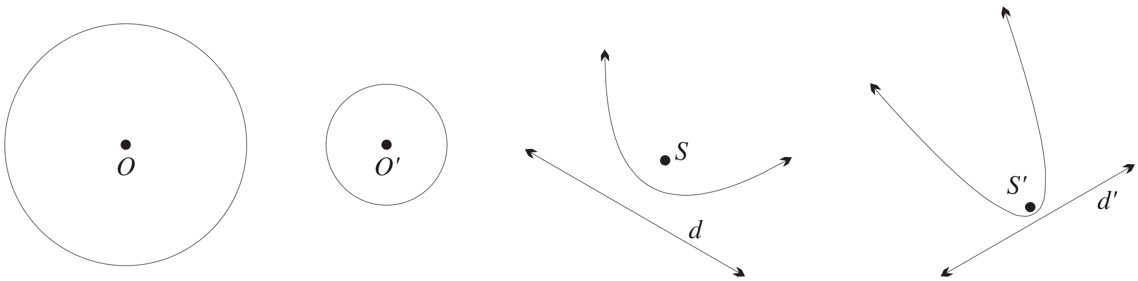
5

> THE FOCAL LENGTH OF A PARABOLA:
> - Let distance from focus to vertex $= a$ (the focal length).
> - Then distance from focus to directrix $= 2a$ (twice the focal length),
> - and length of latus rectum $= 4a$ (four times the focal length).

Chords, Focal Chords and the Latus Rectum: An interval joining two points on the parabola is called a *chord*, just like a chord of a circle. A chord that passes through the focus is called a *focal chord*.

The *latus rectum* is different from all other chords because it is the focal chord that is parallel to the directrix.

The focus of a parabola is like the centre of a circle. Focal chords, which must pass through the focus, are like the diameters of a circle, which must pass through its centre. The focal length gives a measure of how opened out the arms of the parabola are, just as the radius of a circle is the measure of a circle's size.



Any two circles of the same radius are *congruent*, and any two circles are *similar*.

In the same way, any two parabolas with the same focal length are *congruent* — to see this, translate the second focus onto the first, then rotate the second directrix until it coincides with the first. Hence any two parabolas must be *similar*, because an enlargement can be used to change the focal length.

SIMILARITY AND CONGRUENCE OF CIRCLES AND PARABOLAS:

6

- Any two circles with the same radius are congruent.
  Any two parabolas with the same focal length are congruent.
- Any two circles are similar.
  Any two parabolas are similar.

Using the Definition of a Parabola to Find its Equation: The following worked exercise uses the definition of a parabola to find its equation. The methods are the locus methods of the last section.

WORKED EXERCISE: [The locus method]

(a) Use the definition of the parabola to find the equation of the parabola with focus $S(0, 2)$ and directrix $d : y = -2$.

(b) What are the vertex, focal length, and length of the latus rectum?

SOLUTION:

(a) Let $P(x, y)$ be any point in the plane,
and let $M(x, -2)$ be the foot of the perpendicular from P to d.
The condition for P to lie on the parabola is

$$PS = PM$$

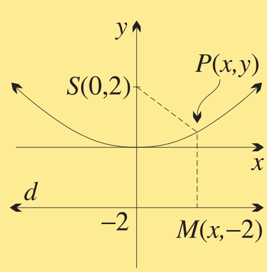
square

$$PS^2 = PM^2$$

$$x^2 + (y - 2)^2 = (y + 2)^2$$

$$x^2 + y^2 - 4y + 4 = y^2 + 4y + 4$$

$$x^2 = 8y.$$



(b) The diagram makes it clear that the vertex is $(0, 0)$ and the focal length is 2. Hence the length of the latus rectum is 8. (See Box 5 above.)

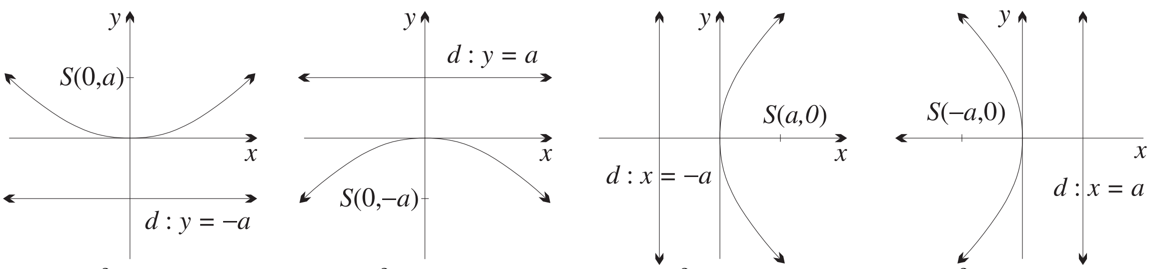
The Four Standard Positions of the Parabola: Although a parabola can be placed anywhere on the plane, in any orientation, its equation will be simpler if the directrix is parallel to one of the axes. The equation will be even simpler if the vertex is at the origin.

This gives *four standard positions* for a parabola with focal length a:

facing up, facing down, facing right, facing left.

The four diagrams below show these four positions.

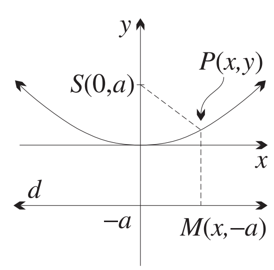
7 THE FOUR STANDARD POSITIONS OF THE PARABOLA:
The equation of every parabola whose vertex is at the origin and whose axis is vertical or horizontal can be put into exactly one of the four forms below:



$x^2 = 4ay$ $x^2 = -4ay$ $y^2 = 4ax$ $y^2 = -4ax$

The first position — vertex at the origin and facing upwards — is the most usual. We will prove that its equation is $x^2 = 4ay$. The other three equations then follow, using reflections in the x-axis and in the line $y = x$.

PROOF: The parabola with vertex at the origin, with focal length a and facing upwards will have focus $S(0, a)$ and directrix $d : y = -a$.
Let $P(x, y)$ be any point in the plane,
and let $M(x, -a)$ be the foot of the perpendicular from P to d.
The condition for P to lie on the parabola is



$$PS = PM$$

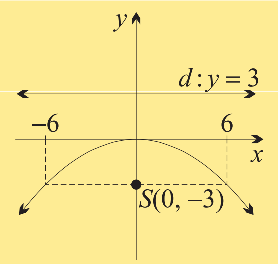
square
$$PS^2 = PM^2$$
$$x^2 + (y - a)^2 = (y + a)^2$$
$$x^2 + y^2 - 2ay + a^2 = y^2 + 2ay + a^2$$
$$x^2 = 4ay.$$

Using the Four Standard Equations of a Parabola: Most of the time, there is no need to go back to the definition of a parabola. One can simply use the standard equations of the parabola established above.

- You need to be able to describe the parabola geometrically, given its equation.
- Conversely, you need to be able to write down the equation of a parabola that has been described geometrically.

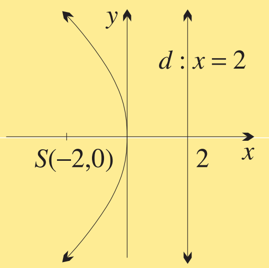
8 FIND THE VALUE OF a FIRST:
- First establish the orientation of the parabola.
- Then find the values of $4a$ and of a.

WORKED EXERCISE: [Geometric description of an equation]
Sketch the parabola $x^2 = -12y$, showing the focus, the directrix, and the endpoints of the latus rectum.



SOLUTION:
The parabola faces down, with $4a = 12$ (by Box 7 above)
$a = 3$.
Hence the focus is $S(0, -3)$, the directrix is $y = 3$,
and the latus rectum has endpoints $(6, -3)$ and $(-6, -3)$.

WORKED EXERCISE: [Writing down the equation]
Write down the equation of the parabola with vertex at the origin and directrix $x = 2$.



SOLUTION:
The parabola is facing left, with $a = 2$ and $4a = 8$.
Hence (by Box 7) its equation is $y^2 = -8x$.

Exercise 11B

1. For each parabola, find the focal length a. Then find: (i) the coordinates of the vertex, (ii) the coordinates of the focus, (iii) the equation of the directrix. Then sketch a graph of each parabola, showing these features.

(a) $x^2 = 4y$	(g) $x^2 = -4y$	(m) $y^2 = 4x$	(s) $y^2 = -4x$
(b) $x^2 = 8y$	(h) $x^2 = -8y$	(n) $y^2 = 12x$	(t) $y^2 = -8x$
(c) $x^2 = 20y$	(i) $x^2 = -12y$	(o) $y^2 = 16x$	(u) $y^2 = -12x$
(d) $x^2 = y$	(j) $x^2 = -y$	(p) $y^2 = x$	(v) $y^2 = -x$
(e) $x^2 = 2y$	(k) $x^2 = -2y$	(q) $y^2 = 6x$	(w) $y^2 = -2x$
(f) $x^2 = 6y$	(l) $x^2 = -0{\cdot}4y$	(r) $y^2 = 10x$	(x) $y^2 = -1{\cdot}2x$

2. Rearrange each equation into the form $x^2 = 4ay$, $x^2 = -4ay$, $y^2 = 4ax$ or $y^2 = -4ax$. Hence sketch a graph of each parabola, indicating the vertex, focus and directrix.

(a) $x^2 - 16y = 0$
(b) $4y + x^2 = 0$
(c) $2y^2 = 4x$
(d) $y^2 + 10x = 0$

DEVELOPMENT

3. First draw a sketch. Then use the four standard forms to find the equation of the parabola with vertex at the origin, axis vertical and:

(a) focus $(0, 5)$,
(b) focus $(0, -3)$,
(c) directrix $y = -2$,
(d) directrix $y = \frac{1}{2}$,
(e) latus rectum having equation $y = 1$,
(f) latus rectum having equation $y = -\frac{1}{4}$.

4. First draw a sketch. Then use the four standard forms to find the equation of the parabola with vertex at the origin, axis horizontal and:

(a) focus $(\frac{1}{2}, 0)$,
(b) focus $(-1, 0)$,
(c) directrix $x = -4$,
(d) directrix $x = 2$,
(e) latus rectum having equation $x = 3$,
(f) latus rectum having equation $x = -\frac{3}{2}$.

5. First draw a sketch. Then use the four standard forms to find the equation of the parabola with vertex at the origin and:
(a) axis vertical, passing through $(4, 1)$,
(b) axis vertical, passing through $(-2, 8)$,
(c) axis horizontal, passing through $(2, -2)$,
(d) axis horizontal, passing through $(-1, 1)$.

6. Find the equation of the parabola with vertex at the origin, in which:
(a) the axis is vertical, and the latus rectum is 8 units in length (2 parabolas),
(b) the focal length is 3 units, and the axis is horizontal or vertical (4 parabolas),
(c) the parabola passes through $(1, 1)$, and the axis is horizontal or vertical (2 parabolas),
(d) the axis is horizontal, and the focal length is $\frac{1}{2}$ (2 parabolas).

CHALLENGE

NOTE: When a question asks 'Use the definition of a parabola to find its equation', the solution should begin 'Let $P(x, y)$ be any point in the plane. The condition for P to lie on the parabola is ....' Otherwise the four standard forms may be used.

7. The variable point $P(x, y)$ moves so that it is equidistant from the point $S(0, 3)$ and the line $y + 3 = 0$. Draw a diagram, and let L be the point $(x, -3)$.
(a) Show that $PS^2 = x^2 + (y - 3)^2$ and $PL^2 = (y + 3)^2$.
(b) By setting $PS^2 = PL^2$, derive the equation of the locus of P.

8. Applying the method outlined in the previous question, use the definition of a parabola to derive the equations of the following parabolas:
(a) focus $(0, 5)$, directrix $y + 5 = 0$,
(b) focus $(2, 0)$, directrix $x + 2 = 0$,

9. Derive the equation of the locus of the point $P(x, y)$ that moves so that:
(a) it is equidistant from the point $S(0, -a)$ and the line $y - a = 0$,
(b) it is equidistant from the point $S(a, 0)$ and the line $x + a = 0$.

11 C Translations of the Parabola

When the vertex of a parabola is not at the origin, the normal rules for shifting curves around the plane apply. To move the vertex from $(0, 0)$ to (h, k), replace x by $x - h$ and y by $y - k$.

Again, there are two tasks that you need to be able to perform:
- First, write down the equation of a parabola, given its geometric description.
- Conversely, describe a parabola geometrically, given its equation.

THE FOUR SHIFTED STANDARD FORMS OF THE PARABOLA:

9

Every parabola whose axis is vertical or horizontal has an equation that can be put into exactly one of the four forms

$$(x - h)^2 = 4a(y - k) \qquad (x - h)^2 = -4a(y - k)$$
$$(y - k)^2 = 4a(x - h) \qquad (y - k)^2 = -4a(x - h)$$

where $a > 0$ is the focal length, and (h, k) is the vertex.

Writing Down the Equation of a Given Parabola: A sketch must be drawn before anything else is done. Writing down the equation requires finding the focal length a, the vertex (h, k), and the orientation of the parabola.

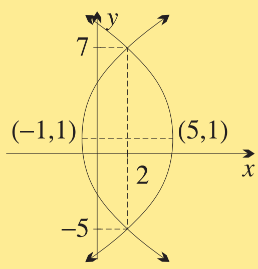
WORKED EXERCISE:

(a) Write down the equations of the two parabolas that have focal length 3, focus $(2, 1)$ and axis parallel to the x-axis.

(b) Sketch them, and find and describe their points of intersection.

SOLUTION:

(a) The parabola facing right has vertex $(-1, 1)$,
so its equation is $(y-k)^2 = 4a(x-h)$
$$(y-1)^2 = 12(x+1).$$
The parabola facing left has vertex $(5, 1)$,
so its equation is $(y-k)^2 = -4a(x-h)$
$$(y-1)^2 = -12(x-5).$$



(b) The two parabolas meet at $(2, 7)$ and $(2, -5)$, which are the endpoints of their common latus rectum.

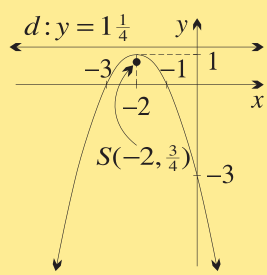
Describing a Parabola Given its Equation: If the equation of a parabola is given, the parabola should be forced into the appropriate standard form by completing the square. As always, find the focal length a. A sketch is then essential.

WORKED EXERCISE:

Find the focus, directrix, focal length and endpoints of the latus rectum of the parabola $y = -3 - 4x - x^2$.

SOLUTION:

Completing the square, $\quad x^2 + 4x = -y - 3$
$$x^2 + 4x + 4 = -y - 3 + 4$$
$$(x+2)^2 = -(y-1).$$
Hence (by Box 9) $4a = 1$ and $a = \frac{1}{4}$, the vertex is $(-2, 1)$,
and the parabola is concave down.
Thus the focus is $(-2, \frac{3}{4})$ and the directrix is $y = 1\frac{1}{4}$,
and the endpoints of the latus rectum are $(-2\frac{1}{2}, \frac{3}{4})$ and $(-1\frac{1}{2}, \frac{3}{4})$.



Exercise 11C

1. Sketch each parabola, clearly indicating the coordinates of the vertex and focus, and the equation of the directrix:

(a) (i) $x^2 = 4(y+1)$ (ii) $(x+2)^2 = 4y$ (iii) $(x-3)^2 = 8(y+5)$ (iv) $(x+1)^2 = 12(y-2)$ (v) $x^2 = 2(y-4)$ (vi) $(x-1)^2 = 6(y+3)$

(b) (i) $(x-4)^2 = -8y$ (ii) $x^2 = -12(y-1)$ (iii) $(x+5)^2 = -4(y-3)$ (iv) $(x-1)^2 = -8(y+2)$ (v) $x^2 = -2(y+3)$ (vi) $(x+3)^2 = -10(y+1)$

(c) (i) $y^2 = 4(x+1)$ (ii) $(y-2)^2 = 4x$ (iii) $y^2 = 6(x+2)$ (iv) $(y-1)^2 = 16x$ (v) $(y+7)^2 = 12(x-5)$ (vi) $(y-3)^2 = 8(x+1)$

(d) (i) $y^2 = -4(x-3)$ (ii) $(y-1)^2 = -8x$ (iii) $y^2 = -10(x+6)$ (iv) $(y-3)^2 = -2x$ (v) $(y+8)^2 = -4(x-3)$ (vi) $(y-3)^2 = -12(x+1)$

DEVELOPMENT

2. Using the four standard forms, find the equation of the parabola with the given focus S and vertex V.

(a) $S(-2,6)$, $V(-2,4)$
(b) $S(5,1)$, $V(1,1)$
(c) $S(2,-1)$, $V(2,2)$
(d) $S(0,0)$, $V(1,0)$
(e) $S(-5,4)$, $V(-5,2)$
(f) $S(-3,-2)$, $V(-7,-2)$
(g) $S(8,-10)$, $V(8,-7)$
(h) $S(-3,-3)$, $V(-1,-3)$
(i) $S(6,0)$, $V(6,-3)$

3. Use the four standard forms to find the equation of the parabola with the given vertex V and directrix d.

(a) $V(2,-1)$, d: $y=-3$
(b) $V(1,0)$, d: $x=0$
(c) $V(-3,4)$, d: $y-6=0$
(d) $V(2,5)$, d: $x=5$
(e) $V(3,1)$, d: $y=-1$
(f) $V(-4,2)$, d: $x=-7$
(g) $V(0,-\frac{3}{2})$, d: $y=\frac{1}{2}$
(h) $V(-1,-4)$, d: $x=2$
(i) $V(-7,-5)$, d: $y=-5\frac{1}{2}$

4. Use the four standard forms to find the equation of the parabola with the given focus S and directrix d.

(a) $S(0,4)$, d: $y=0$
(b) $S(6,0)$, d: $x=0$
(c) $S(0,-2)$, d: $y=0$
(d) $S(-4,0)$, d: $x=0$
(e) $S(1,7)$, d: $y=3$
(f) $S(3,-2)$, d: $x=1$
(g) $S(-1,4)$, d: $y=5$
(h) $S(3,\frac{1}{2})$, d: $x=5$
(i) $S(5,-4)$, d: $y=-9$

5. Express the equation of each parabola in the form $(x-h)^2=4a(y-k)$ or in the form $(x-h)^2=-4a(y-k)$. Sketch a graph, clearly indicating the focus, vertex and directrix.

(a) $y=x^2+6x+5$
(b) $x^2=1-y$
(c) $6y=x^2-12x$
(d) $x^2=2(1+2y)$
(e) $y=(x+8)(x-2)$
(f) $(x+3)(x+5)=8y-25$
(g) $x^2-6x+2y+12=0$
(h) $x^2-8x+12y+4=0$

6. Express the equation of each parabola in the form $(y-k)^2=4a(x-h)$ or in the form $(y-k)^2=-4a(x-h)$. Sketch a graph, clearly indicating the focus, vertex and directrix.

(a) $y^2-4x=0$
(b) $y^2=6-2x$
(c) $6x=y^2+18$
(d) $y^2-2y=4x-5$
(e) $y(y-4)=8x$
(f) $y^2-6y-2x+7=0$
(g) $y^2+4y+6x-26=0$
(h) $(y-4)(y-6)=12x+11$

CHALLENGE

7. The variable point $P(x,y)$ moves so that it is equidistant from the point $S(3,3)$ and the line $y+1=0$. Let L be the point $(x,-1)$.

(a) Show that $PS^2=(x-3)^2+(y-3)^2$ and $PL^2=(y+1)^2$.
(b) By setting $PS^2=PL^2$, derive the equation of the locus of P.

8. Applying the method outlined in the previous question, use the definition of a parabola to derive the equation of:

(a) the parabola with focus $(-7,-2)$ and directrix $y+8=0$,
(b) the parabola with focus $(0,2)$ and directrix $x+2=0$.

9. Find the equation of the parabola that has:

(a) vertex at $(1,4)$, axis parallel to the y-axis, and passes through $(3,5)$;
(b) vertex at $(-3,-2)$, axis parallel to the x-axis, and passes through $(-1,0)$.

10. Find all possible equations of the parabolas described below, assuming that the axis is parallel to one of the coordinate axes.
(a) The vertex is $(3, -1)$ and the focal length is 2 units (4 parabolas).
(b) The latus rectum has endpoints $(1, 3)$ and $(1, -5)$ (2 parabolas).
(c) The focus is $(-2, 4)$ and one endpoint of the latus rectum is $(0, 4)$ (2 parabolas).
(d) The axis is $y - 2 = 0$, the vertex is $(3, 2)$ and the latus rectum has length 6 units (2 parabolas).
(e) The focus is $(6, -3)$ and the vertex is on the line $y = x - 4$ (2 parabolas).

11D Chapter Review Exercise

1. Sketch the locus of the point $P(x, y)$ that is subject to each set of conditions. Hence write down its equation.
(a) P is 3 units above the x-axis.
(b) P is 1 unit to the right of the y-axis.
(c) P is 4 units from the origin.
(d) P is 2 units from the point $(1, -2)$

2. Write down the centre and radius of each circle.
(a) $x^2 + y^2 = 9$ (b) $(x - 1)^2 + (y + 3)^2 = 4$ (c) $x^2 + y^2 - 4x + 8y - 5 = 0$

3. Given the points $A(2, 3)$ and $B(-1, 7)$, find the equation of each locus of the point $P(x, y)$.
(a) P is equidistant from A and B.
(b) $\angle APB$ is a right angle.

4. Sketch each parabola, indicating the focus, vertex and directrix.
(a) $x^2 = 4y$ (c) $x^2 = -8y$ (e) $y^2 = 16x$ (g) $y^2 = -12x$
(b) $x^2 = 2y$ (d) $x^2 = -y$ (f) $y^2 = x$ (h) $y^2 = -6x$

5. Write down the equation of the parabola with vertex at the origin, axis vertical and:
(a) focus $(0, 4)$, (b) directrix $y = 2$, (c) directrix $y = -1$.

6. Write down the equation of the parabola with vertex at the origin, axis horizontal and:
(a) focus $(5, 0)$, (b) focus $(-2, 0)$, (c) directrix $x = 3$.

7. Sketch each parabola, indicating the focus, vertex and directrix.
(a) $x^2 = 4(y - 2)$ (c) $(x + 1)^2 = -2(y - 1)$ (e) $(y - 1)^2 = 16(x + 3)$
(b) $(x - 2)^2 = 8(y + 3)$ (d) $(y - 3)^2 = 12x$ (f) $(y + 2)^2 = -6(x - 1)$

8. Find the equation of the parabola with:
(a) vertex $(1, 4)$ and focus $(1, 6)$,
(b) vertex $(-2, 3)$ and directrix $x = -3$,
(c) focus $(0, -1)$ and directrix $y = 5$.

9. Express each equation in the appropriate standard form. Then sketch a graph of the parabola, clearly indicating the focus, vertex and directrix.
(a) $x^2 - 2x - 4y - 7 = 0$ (c) $y^2 + 6y - 8x + 25 = 0$
(b) $x^2 - 6x + 2y + 11 = 0$ (d) $y^2 + 6x + 12 = 0$

CHAPTER **TWELVE**

The Geometry of the Derivative

Now that we have the derivative, the systematic approach to sketching curves, begun in Chapters Three and Four, can be extended by two further questions:

1. Where is the curve sloping upwards, where is it sloping downwards, and where does it have any maximum or minimum values?
2. Where is the curve concave up, where is it concave down, and where does it change from one concavity to the other?

These will become standard procedures for investigating unfamiliar curves. In particular, the algorithm for finding the maximum and minimum values of a function can be applied to all sorts of practical and theoretical questions.

Curve-sketching software is useful when studying this chapter, because it can easily show the effect on the graph of changing the equation of the curve.

12 A Increasing, Decreasing and Stationary at a Point

At a point where a curve is sloping upwards, the tangent has positive gradient, and y is increasing as x increases. At a point where it is sloping downwards, the tangent has negative gradient, and y is decreasing as x increases.

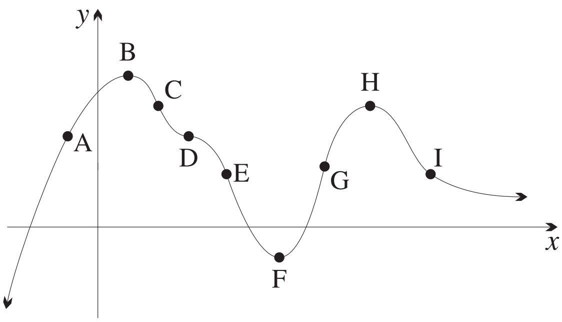
Let $f(x)$ be a function defined at $x = a$. Then:

1 INCREASING, DECREASING AND STATIONARY AT A POINT:

If $f'(a) > 0$, then $f(x)$ is called *increasing* at $x = a$.
If $f'(a) < 0$, then $f(x)$ is called *decreasing* at $x = a$.
If $f'(a) = 0$, then $f(x)$ is called *stationary* at $x = a$.

For example, the curve in the diagram to the right is:

- increasing at A and G,
- decreasing at C, E and I,
- stationary at B, D, F and H.



WORKED EXERCISE:

Differentiate $f(x) = x^3 - 12x$. Hence find whether the curve $y = f(x)$ is increasing, decreasing or stationary at the point where:

(a) $x = 5$ (b) $x = 2$ (c) $x = 0$

SOLUTION:

Differentiating, $f'(x) = 3x^2 - 12$.

(a) Hence $f'(5) = 75 - 12 > 0$, so the curve is increasing at $x = 5$,

(b) and $f'(2) = 12 - 12 = 0$, so the curve is stationary at $x = 2$,

(c) and $f'(0) = -12 < 0$, so the curve is decreasing at $x = 0$.

WORKED EXERCISE:

For what value(s) of x is the curve $y = x^4 - 4x$ stationary?

SOLUTION:

Differentiating, $y' = 4x^3 - 4$
$= 4(x^3 - 1)$.

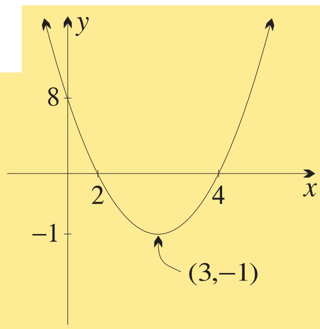
Put $y' = 0$ to find where the curve is stationary.

Then $x^3 = 1$
$x = 1$.

WORKED EXERCISE:

(a) Differentiate $y = (x - 2)(x - 4)$.

(b) Hence find the values of x where the curve is stationary, and where it is decreasing.



SOLUTION:

(a) Expanding, $y = x^2 - 6x + 8$,
and differentiating, $y' = 2x - 6$
$= 2(x - 3)$.

(b) Since $y' = 0$ when $x = 3$, the curve is stationary at $x = 3$.
Since $y' < 0$ when $x < 3$, the curve is decreasing for $x < 3$.

WORKED EXERCISE:

(a) Show that $f(x) = x^3 + x - 1$ is always increasing.

(b) Find $f(0)$ and $f(1)$, and hence explain why the curve has exactly one x-intercept.

SOLUTION:

(a) Differentiating, $f'(x) = 3x^2 + 1$.
Since squares can never be negative, $f'(x)$ can never be less than 1,
so the function is increasing for every value of x.

(b) Substituting, $f(0) = -1$ and $f(1) = 1$.
Since $f(0)$ is negative and $f(1)$ is positive, and the curve is continuous,
the curve must cross the x-axis somewhere between 0 and 1.
Because the function is increasing for every value of x,
it can never go back and cross the x-axis at a second point.

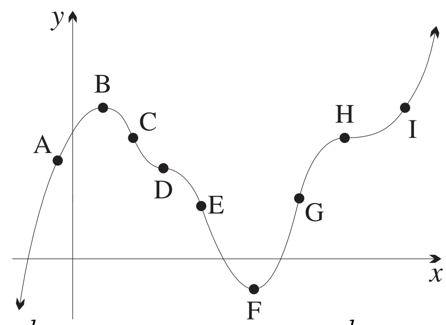
Exercise 12A

1. In the diagram to the right, name the points where:

(a) $f'(x) > 0$

(b) $f'(x) < 0$

(c) $f'(x) = 0$



2. Find the derivative of each function. By substituting $x = 1$ into the derivative, determine whether the function is increasing, decreasing or stationary at $x = 1$.

[A function is increasing when $\dfrac{dy}{dx} > 0$, decreasing when $\dfrac{dy}{dx} < 0$, stationary when $\dfrac{dy}{dx} = 0$.]

(a) $y = x^2$ (c) $y = 3x^2 - 8x$ (e) $y = x^3 + 5x$

(b) $y = x^2 - 2x$ (d) $y = x^2 - 3x + 7$ (f) $y = 4x^3 - 3x^4$

3. (a) Find the derivative $f'(x)$ of $f(x) = x^2 - 6x + 11$.

(b) Hence find whether the curve $y = f(x)$ is increasing, decreasing or stationary at:

(i) $x = 0$ (ii) $x = 1$ (iii) $x = 3$ (iv) $x = 4$ (v) $x = -1$

4. (a) Find the derivative $f'(x)$ of $f(x) = x^3 - 6x^2 + 9x$.

(b) Hence find whether the curve $y = f(x)$ is increasing, decreasing or stationary at:

(i) $x = 0$ (ii) $x = 1$ (iii) $x = 2$ (iv) $x = 3$ (v) $x = -1$

5. By finding where the derivative equals zero, find the x-coordinates of the points where each function is stationary.

(a) $y = x^2 - 2x$ (c) $y = x^2 + 6x + 9$ (e) $y = x^3 - 3x^2$

(b) $y = x^2 - 4x + 3$ (d) $y = 2x^2 - 16x$ (f) $y = x^3 - 12x$

6. (a) Explain why $y = -5x + 2$ is decreasing for all x.

(b) Explain why $y = x + 7$ is increasing for all x.

(c) Explain why $f(x) = x^3$ is increasing for all values of x, apart from $x = 0$, where it is stationary.

(d) Explain why $f(x) = x^2$ is increasing for $x > 0$ and decreasing for $x < 0$. What happens at $x = 0$?

DEVELOPMENT

7. Differentiate each function using the chain rule. Then evaluate $f'(0)$ to establish whether the curve is increasing, decreasing or stationary at $x = 0$.

(a) $f(x) = (x - 1)^3$ (b) $f(x) = (2x - 1)^4$ (c) $f(x) = (x^2 + 3)^2$

8. Differentiate each function using the product rule. Then evaluate $f'(1)$ to establish whether the curve is increasing, decreasing or stationary at $x = 1$.

(a) $f(x) = (x - 5)(x + 3)$ (b) $f(x) = (x - 2)(x^2 + 5)$ (c) $f(x) = (x^4 + 2)(1 - x^3)$

9. Differentiate each function using the quotient rule. Then evaluate $f'(2)$ to establish whether the curve is increasing, decreasing or stationary at $x = 2$.

(a) $f(x) = \dfrac{x}{x + 1}$ (b) $f(x) = \dfrac{x + 1}{x - 1}$ (c) $f(x) = \dfrac{x^2}{x + 2}$

10. Differentiate each function by first writing it in index form. Then evaluate $f'(1)$ to establish whether the curve is increasing, decreasing or stationary at $x = 1$.

(a) $f(x) = \sqrt{x}$ (b) $f(x) = \dfrac{1}{x}$ (c) $f(x) = -\dfrac{1}{x^2}$

11. (a) Find $f'(x)$ for the function $f(x) = 4x - x^2$.

(b) For what values of x is: (i) $f'(x) > 0$, (ii) $f'(x) < 0$, (iii) $f'(x) = 0$?

(c) Find $f(2)$. Then, by interpreting these results geometrically, sketch a graph of $f(x)$.

12. (a) Find $f'(x)$ for the function $f(x) = x^2 - 4x + 3$.

(b) For what values of x is: (i) $f'(x) > 0$, (ii) $f'(x) < 0$, (iii) $f'(x) = 0$?

(c) Evaluate $f(2)$. Then, by interpreting these results geometrically, sketch $y = f(x)$.

13. (a) Let $f(x) = x^3 - 3x^2 - 9x - 2$. Show that $f'(x) = 3(x - 3)(x + 1)$.

(b) By sketching a graph of $y = f'(x)$, show that $f(x)$ is increasing when $x > 3$ or $x < -1$.

14. (a) Find the derivative $f'(x)$ of $f(x) = x^3 + 2x^2 + x + 7$.

(b) Use factoring or the quadratic formula to find the zeroes of $f'(x)$.

(c) Sketch the graph of $y = f'(x)$.

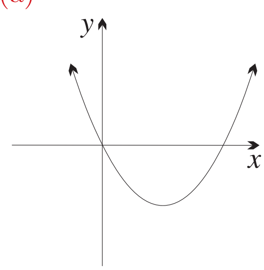
(d) Hence find the values of x for which $f(x)$ is decreasing.

15. Find the derivative of each function. By solving $y' > 0$, find the values of x for which the function is increasing.

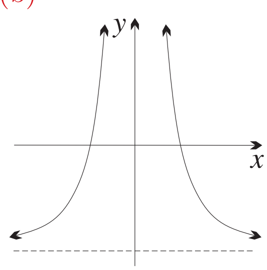
(a) $y = x^2 - 4x + 1$ (b) $y = 7 - 6x - x^2$ (c) $y = 2x^3 - 6x$ (d) $y = x^3 - 3x^2 + 7$

16. The graphs of four functions (a), (b), (c) and (d) are shown below. The graphs of the derivatives of these functions, in scrambled order, are shown in I, II, III and IV. Match the graph of each function with the graph of its derivative.

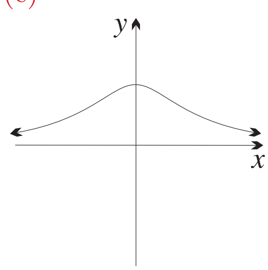
(a)



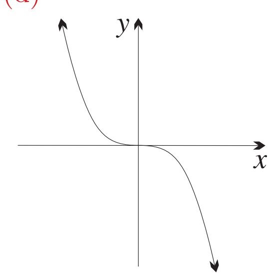
(b)



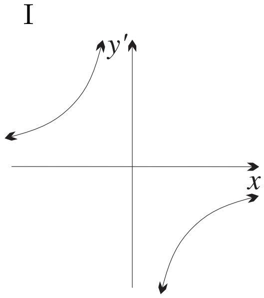
(c)



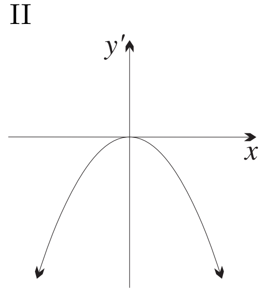
(d)



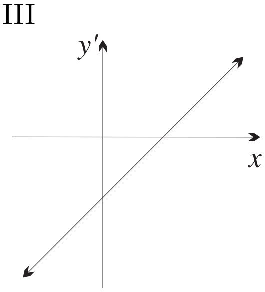
I



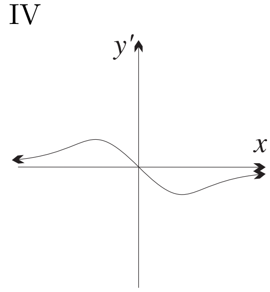
II



III



IV



17. (a) Differentiate $f(x) = -\dfrac{1}{x}$, and hence prove that $f(x)$ increases for all x in its domain.

(b) Sketch a graph of $f(x) = -\dfrac{1}{x}$, and explain why $f(-1) > f(2)$ despite this fact.

18. (a) Use the quotient rule to find the derivative $f'(x)$ of $f(x) = \dfrac{2x}{x-3}$.

(b) Explain why $f(x)$ is decreasing for all $x \neq 3$.

19. (a) Use the quotient rule to find the derivative $f'(x)$ of $f(x) = \dfrac{x^3}{x^2+1}$.

(b) Explain why $f(x)$ is increasing for all x, apart from $x = 0$ where it is stationary.

20. (a) Find $f'(x)$ for the function $f(x) = \frac{1}{3}x^3 + x^2 + 5x + 7$.

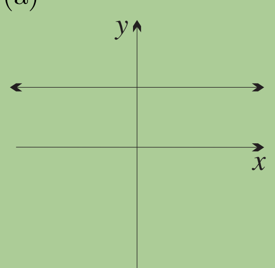
(b) By completing the square, show that $f'(x) = (x+1)^2 + 4$, and hence explain why $f(x)$ is increasing for all x.

(c) Evaluate $f(-3)$ and $f(0)$ and hence explain why the curve $y = f(x)$ has exactly one x-intercept.

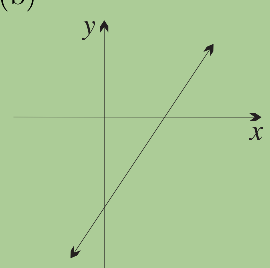
CHALLENGE

21. Look carefully at each function drawn below to establish where it is increasing, decreasing and stationary. Hence draw a graph of the derivative of the function.

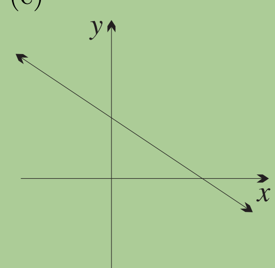
(a)



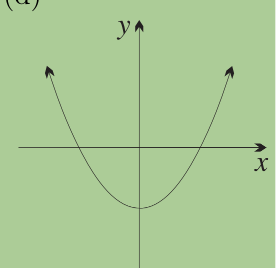
(b)



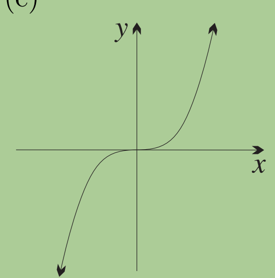
(c)



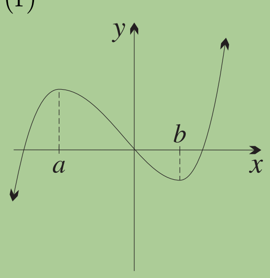
(d)



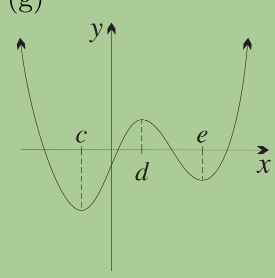
(e)



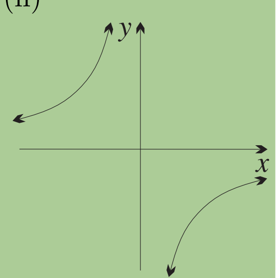
(f)



(g)



(h)



22. (a) If $f(x) = -x^3 + 2x^2 - 5x + 3$, find $f'(x)$.

(b) By evaluating the discriminant Δ, show that $f'(x) < 0$ for all values of x.

(c) Hence deduce the number of solutions to the equation $3 - 5x + 2x^2 - x^3 = 0$.

23. Sketch graphs of continuous curves that have the properties below.

(a) $f(1) = f(-3) = 0$,
$f'(-1) = 0$,
$f'(x) > 0$ when $x < -1$,
$f'(x) < 0$ when $x > -1$.

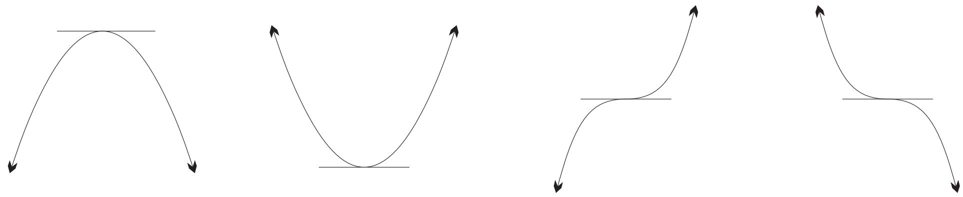
(b) $f(2) = f'(2) = 0$,
$f'(x) > 0$ for all $x \neq 2$.

(c) $f(x)$ is odd,
$f(3) = 0$ and $f'(1) = 0$,
$f'(x) > 0$ for $x > 1$,
$f'(x) < 0$ for $0 \le x < 1$.

(d) $f(x) > 0$ for all x,
$f'(0) = 0$,
$f'(x) < 0$ for $x < 0$,
$f'(x) > 0$ for $x > 0$.

12 B Stationary Points and Turning Points

Stationary points can be classified into four different types:



Maximum turning point | Minimum turning point | Stationary point of inflexion | Stationary point of inflexion

Turning Points: The first stationary point is called a *maximum turning point* — the curve turns smoothly from increasing to decreasing, with a maximum value at the point.

The second stationary point is called a *minimum turning point* — the curve turns smoothly from decreasing to increasing, with a minimum value at the point.

2 TURNING POINTS: A stationary point is called a *turning point* if the derivative changes sign around the point.
- At a *maximum turning point*, the curve changes from increasing to decreasing.
- At a *minimum turning point*, the curve changes from decreasing to increasing.

Stationary Points of Inflexion: In the third and fourth diagrams above, there is no turning point. In the third diagram, the curve is increasing on both sides of the stationary point, and in the fourth, the curve is decreasing on both sides.

Instead, the curve *flexes* around the stationary point, changing *concavity* from downwards to upwards, or from upwards to downwards. The surprising effect is that *the tangent at the stationary point actually crosses the curve.*

3 POINTS OF INFLEXION: A *point of inflexion* is a point on the curve where the tangent crosses the curve. This means that the concavity changes from upwards to downwards, or from downwards to upwards, around the point.

STATIONARY POINTS OF INFLEXION: A *stationary point of inflexion* is a point of inflexion where the tangent is horizontal. This means that it is both a point of inflexion and a stationary point.

Local or Relative Maximum and Minimum: A *local* or *relative maximum* is a point where the curve reaches a maximum in its immediate neighbourhood. (Sometimes there is no tangent at the point — look at points C and I in the following diagram.)

Let $A\big(a, f(a)\big)$ be a point on a curve $y = f(x)$.

4 LOCAL OR RELATIVE MAXIMUM: The point A is called a *local* or *relative maximum* if

$$f(x) \le f(a), \quad \text{for all } x \text{ in some small interval around } a.$$

LOCAL OR RELATIVE MINIMUM: Similarly, A is called a *local* or *relative minimum* if

$$f(x) \ge f(a), \quad \text{for all } x \text{ in some small interval around } a.$$

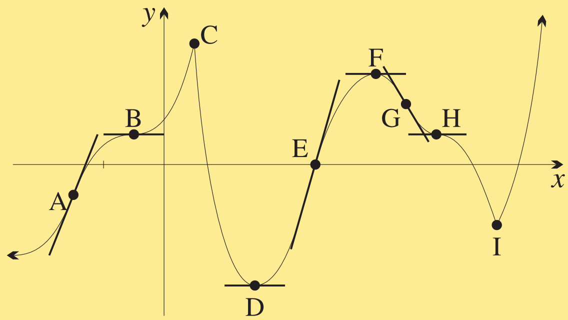
WORKED EXERCISE:
Classify the points labelled A–I in the diagram below.

SOLUTION:
C and F are local maxima,
but only F is a maximum turning point.

D and I are local minima,
but only D is a minimum turning point.

B and H are stationary points of inflexion.

A, E and G are also points of inflexion,
but are not stationary points.



Analysing Stationary Points with a Table of Slopes: Chapter Four explained how a function can only change sign at a zero or a discontinuity. Similarly, the derivative $f'(x)$ can only change sign at a stationary point or at a discontinuity of $f'(x)$.

This gives a straightforward method for analysing the stationary points. The method also gives an overall picture of the shape of the function.

5 USING THE DERIVATIVE $f'(x)$ TO ANALYSE STATIONARY POINTS AND SLOPE:
1. Find the zeroes and discontinuities of the derivative $f'(x)$.
2. Draw up a table of test points of the derivative $f'(x)$ around its zeroes and discontinuities, followed by a table of slopes, to see where its sign changes.

The table of slopes shows not only the nature of each stationary point, but also where the function is increasing and decreasing across its whole domain. This gives an outline of the shape of the curve, in preparation for a proper sketch.

WORKED EXERCISE:
Find the stationary points of the cubic $y = x^3 - 6x^2 + 9x - 4$, determine their nature, and sketch the curve.

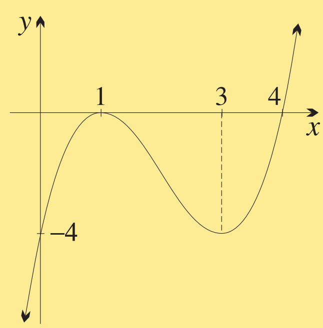
SOLUTION:

$$\frac{dy}{dx} = 3x^2 - 12x + 9$$
$$= 3(x^2 - 4x + 3)$$
$$= 3(x - 1)(x - 3),$$

so y' has zeroes at $x = 1$ and 3, and no discontinuities.

x	0	1	2	3	4
y'	9	0	-3	0	9
Slope	/	—	\	—	/

When $x = 1$, $y = 1 - 6 + 9 - 4$
$= 0$,
and when $x = 3$, $y = 27 - 54 + 27 - 4$
$= -4$.



Hence $(1, 0)$ is a maximum turning point, and $(3, -4)$ is a minimum turning point.

NOTE: Only the signs of y' are relevant, but if the actual values of y' are not calculated, some other argument should be given as to how the signs were obtained.

WORKED EXERCISE:

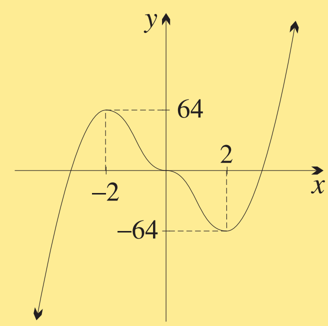
Find the stationary points of the quintic $f(x) = 3x^5 - 20x^3$, determine their nature, and sketch the curve.

SOLUTION:

$$f'(x) = 15x^4 - 60x^2$$
$$= 15x^2(x^2 - 4)$$
$$= 15x^2(x - 2)(x + 2),$$

so $f'(x)$ has zeroes at $x = -2$, $x = 0$ and $x = 2$,
and has no discontinuities:

x	-3	-2	-1	0	1	2	3
$f'(x)$	675	0	-45	0	-45	0	675
Slope	/	—	\	—	\	—	/



When $x = 0$, $y = 0 - 0 = 0$,
when $x = 2$, $y = 96 - 160 = -64$,
and when $x = -2$, $y = -96 + 160 = 64$.
Hence $(-2, 64)$ is a maximum turning point, $(2, -64)$ is a minimum turning point, and $(0, 0)$ is a stationary point of inflexion.

NOTE: This function $f(x) = 3x^5 - 20x^3$ is odd, and it has as its derivative $f'(x) = 15x^4 - 60x^2$, which is even. In general, the derivative of an even function is odd, and the derivative of an odd function is even. This provides a useful check.

Finding Pronumerals in a Function: In this worked exercise, the pronumerals in a function are found using information about a stationary point of the curve.

WORKED EXERCISE:

The graph of the cubic $f(x) = x^3 + ax^2 + bx$ has a stationary point at $A(2, 2)$. Find a and b.

SOLUTION:

To find the two unknown constants, we need two independent equations.

Since $f(2) = 2$, $\quad 2 = 8 + 4a + 2b$
$$2a + b = -3. \qquad (1)$$
Differentiating, $\quad f'(x) = 3x^2 + 2ax + b$
and since $f'(2) = 0$, $\quad 0 = 12 + 4a + b$
$$4a + b = -12. \qquad (2)$$
Subtracting (1) from (2), $\quad 2a = -9$
$$a = -4\tfrac{1}{2},$$
and substituting into (1), $\quad -9 + b = -3$
$$b = 6.$$

Exercise 12B

1. By finding where the derivative equals zero, determine the x-coordinates of any stationary points of each function.
   (a) $y = x^2 - 6x + 8$ (b) $y = x^2 + 4x + 3$ (c) $y = x^3 - 3x$

2. By finding where the derivative equals zero, determine the coordinates of any stationary points of each function. [HINT: Remember that you find the y-coordinate by substituting the x-coordinate into the original function.]

(a) $y = x^2 - 4x + 7$
(b) $y = x^2 - 8x + 16$
(c) $y = 3x^2 - 6x + 1$
(d) $y = -x^2 + 2x - 1$
(e) $y = x^3 - 3x^2$
(f) $y = x^4 - 4x + 1$

3. Find the derivative of each function and complete the given table to determine the nature of the stationary point. Sketch each graph, indicating all important features.

(a) $y = x^2 - 4x + 3$:

x	1	2	3
y'			

(b) $y = 12 + 4x - x^2$:

x	1	2	3
y'			

(c) $y = x^2 + 6x + 8$:

x	-4	-3	-2
y'			

(d) $y = 15 - 2x - x^2$:

x	-2	-1	0
y'			

4. Differentiate each function and show that it has a stationary point at $x = 1$. Then use a table of values of $f'(x)$ to determine the nature of that stationary point.

(a) $f(x) = x^2 - 2x - 3$
(b) $f(x) = 15 + 2x - x^2$
(c) $f(x) = x^3 + 3x^2 - 9x + 2$
(d) $f(x) = x^3 - 3x^2 + 3x + 1$

5. Find the stationary point of each function and use a table of values of $\dfrac{dy}{dx}$ to determine its nature. Sketch each graph, indicating all intercepts with the axes.

(a) $y = x^2 + 4x - 12$
(b) $y = 5 - 4x - x^2$

DEVELOPMENT

6. (a) Show that the derivative of $y = x^3 - 3x^2$ is $\dfrac{dy}{dx} = 3x(x - 2)$.

(b) Use a table of values of $\dfrac{dy}{dx}$ to show that there is a maximum turning point at $(0, 0)$ and a minimum turning point at $(2, -4)$.

(c) Sketch a graph of the function, showing all important features.

7. (a) Show that the derivative of $y = 12x - x^3$ is $y' = 3(2 - x)(2 + x)$.

(b) Use a table of values of y' to show that there is a maximum turning point at $(2, 16)$ and a minimum turning point at $(-2, -16)$.

(c) Sketch a graph of the function, showing all important features.

8. Find the stationary points of each function, then determine their nature using a table of values of $\dfrac{dy}{dx}$. Sketch each graph. (You need not find the x-intercepts.)

(a) $y = 2x^3 + 3x^2 - 36x + 15$
(b) $y = x^3 + 4x^2 + 4x$
(c) $y = 16 + 4x^3 - x^4$
(d) $y = 3x^4 - 16x^3 + 24x^2 + 11$

9. (a) Use the product rule to show that if $y = x(x - 2)^3$, then $y' = 2(2x - 1)(x - 2)^2$.

(b) Find any stationary points and use a table of slopes to analyse them.

(c) Sketch a graph of the function, indicating all important features.

10. (a) Use the product rule to show that if $y = x^2(x - 4)^2$, then $\dfrac{dy}{dx} = 4x(x - 4)(x - 2)$.

(b) Find any stationary points and use a table of slopes to analyse them.

(c) Sketch a graph of the function, indicating all important features.

11. (a) Use the product rule to show that if $y = (x-5)^2(2x+1)$, then $y' = 2(x-5)(3x-4)$.
(b) Find any stationary points and use a table of slopes to analyse them.
(c) Sketch a graph of the function, indicating all important features.

12. (a) The tangent to the curve $y = x^2 + ax - 15$ is horizontal at the point where $x = 4$. Find the value of a.
(b) The curve $y = x^2 + ax + 7$ has a turning point at $x = -1$. Find the value of a.

13. (a) The curve $f(x) = ax^2 + 4x + c$ has a turning point at $(-1, 1)$. Find a and c.
(b) Find b and c if $y = x^3 + bx^2 + cx + 5$ has stationary points at $x = -2$ and $x = 4$.

14. The curve $y = ax^2 + bx + c$ passes through the points $(1, 4)$ and $(-1, 6)$, and reaches its maximum value when $x = -\frac{1}{2}$.
(a) Show that $a + b + c = 4$, $a - b + c = 6$ and $-a + b = 0$.
(b) Hence find the values of a, b and c.

15. The curve $y = ax^2 + bx + c$ touches $y = 2x$ at the origin and has a maximum when $x = 1$.
(a) Explain why $c = 0$.
(b) Explain why $\frac{dy}{dx} = 2$ when $x = 0$ and use this fact to deduce that $b = 2$.
(c) Show that $2a + b = 0$ and hence find the value of a.

CHALLENGE

16. (a) If $f(x) = \dfrac{3x}{x^2+1}$, show that $f'(x) = \dfrac{3(1-x)(1+x)}{(x^2+1)^2}$.
(b) Hence find any stationary points and analyse them.
(c) Sketch a graph of $y = f(x)$, indicating all important features.
(d) Hence state how many roots the equation $\dfrac{3x}{x^2+1} = c$ has for:
(i) $c > \frac{3}{2}$ (ii) $c = \frac{3}{2}$ (iii) $0 < c < \frac{3}{2}$ (iv) $c = 0$
[HINT: Sketch the horizontal line $y = c$ on the same number plane and see how many times the graphs intersect.]

17. The function $y = ax^3 + bx^2 + cx + d$ has a relative maximum at $(-2, 27)$ and a relative minimum at $(1, 0)$. Find the values of a, b, c and d using the following steps.
(a) Find $\frac{dy}{dx}$ and show that $3a + 2b + c = 0$ and $12a - 4b + c = 0$.
(b) Using the fact that $(1, 0)$ and $(-2, 27)$ lie on the curve, show that $a + b + c + d = 0$ and $-8a + 4b - 2c + d = 27$. By subtracting, eliminate d from these two equations.
(c) Solve the simultaneous equations $3a+2b+c = 0$, $12a-4b+c = 0$ and $9a-3b+3c = -27$.
(d) Find the value of d.

12 C Second and Higher Derivatives

The derivative of the derivative of a function is called the *second derivative* of the function. As for the derivative, there is a variety of notations, including

$$\frac{d^2y}{dx^2} \quad \text{and} \quad f''(x) \quad \text{and} \quad f^{(2)}(x) \quad \text{and} \quad y'' \quad \text{and} \quad y^{(2)}.$$

This section is concerned with the algebraic manipulation of the second derivative — the geometric implications are left until the next section.

WORKED EXERCISE:

Find the successive derivatives of $y = x^4 + x^3 + x^2 + x + 1$.

SOLUTION:

$y = x^4 + x^3 + x^2 + x + 1$ $\qquad \dfrac{d^2y}{dx^2} = 12x^2 + 6x + 2$ $\qquad \dfrac{d^4y}{dx^4} = 24$

$\dfrac{dy}{dx} = 4x^3 + 3x^2 + 2x + 1$ $\qquad \dfrac{d^3y}{dx^3} = 24x + 6$ $\qquad \dfrac{d^5y}{dx^5} = 0$

Since the fifth derivative is zero, all the higher derivatives are also zero.

WORKED EXERCISE:

Fnd the first four derivatives of $f(x) = x^{-1}$, giving each answer as a fraction.

SOLUTION:

$f'(x) = -x^{-2} = -\dfrac{1}{x^2}$ $\qquad f''(x) = 2x^{-3} = \dfrac{2}{x^3}$ $\qquad f^{(3)}(x) = -6x^{-4} = -\dfrac{6}{x^4}$ $\qquad f^{(4)}(x) = 24x^{-5} = \dfrac{24}{x^5}$

Exercise 12C

1. Find the first, second and third derivatives of each function.
(a) y $=x^3$ (c) $y = x^7$ (e) $y = 2x^4$ (g) $y = 4 - 3x$ (i) $y = 4x^3 - x^2$
(b) $y = x^{10}$ (d) $y = x^2$ (f) $y = 3x^5$ (h) $y = x^2 - 3x$ (j) $y = 4x^5 + 2x^3$

2. Expand each product, then find the first and second derivatives.
(a) $y = x(x+3)$ (c) $y = (x-2)(x+1)$ (e) $y = 3x^2(2x^3 - 3x^2)$
(b) $y = x^2(x-4)$ (d) $y = (3x+2)(x-5)$ (f) $y = 4x^3(x^5 + 2x^2)$

DEVELOPMENT

3. Find the first, second and third derivatives of each function.
(a) $y = x^{0{\cdot}3}$ (b) $y = x^{-1}$ (c) $y = x^{-2}$ (d) $y = 5x^{-3}$ (e) $y = x^2 + x^{-1}$

4. By writing each function with a negative index, find its first and second derivatives.
(a) $f(x) = \dfrac{1}{x^3}$ (b) $f(x) = \dfrac{1}{x^4}$ (c) $f(x) = \dfrac{3}{x^2}$ (d) $f(x) = \dfrac{2}{x^3}$

5. Use the chain rule to find the first and second derivatives of each function.
(a) y $=(x+1)^2$ (b) $y = (3x-5)^3$ (c) $y = (1-4x)^2$ (d) $y = (8-x)^{11}$

6. By writing each function with a negative index, find its first and second derivatives.
(a) $y = \dfrac{1}{x+2}$ (b) $y = \dfrac{1}{(3-x)^2}$ (c) $y = \dfrac{1}{(5x+4)^3}$ (d) $y = \dfrac{2}{(4-3x)^2}$

7. By writing each function with fractional indices, find its first and second derivatives.
(a) $f(x) = \sqrt{x}$ (c) $f(x) = x\sqrt{x}$ (e) $f(x) = \sqrt{x+2}$
(b) $f(x) = \sqrt[3]{x}$ (d) $f(x) = \dfrac{1}{\sqrt{x}}$ (f) $f(x) = \sqrt{1-4x}$

8. (a) Find $f'(x)$ and $f''(x)$ for the function $f(x) = x^3 + 3x^2 + 5x - 6$.
(b) Hence evaluate: (i) $f'(0)$ (ii) $f'(1)$ (iii) $f''(0)$ (iv) $f''(1)$

9. (a) If $f(x) = 3x + x^3$, find: (i) $f'(2)$ (ii) $f''(2)$ (iii) $f'''(2)$ (iv) $f''''(2)$

(b) If $f(x) = (2x - 3)^4$, find: (i) $f'(1)$ (ii) $f''(1)$ (iii) $f'''(1)$ (iv) $f''''(1)$

10. Use the quotient rule to find the first derivative of each function. Then use the chain rule to find the second derivative.

(a) $y = \dfrac{x}{x+1}$ (b) $y = \dfrac{x-1}{2x+5}$

11. If $f(x) = x(x-1)^4$, use the product rule to find $f'(x)$ and $f''(x)$.

12. Find the values of x for which $y'' = 0$ if:

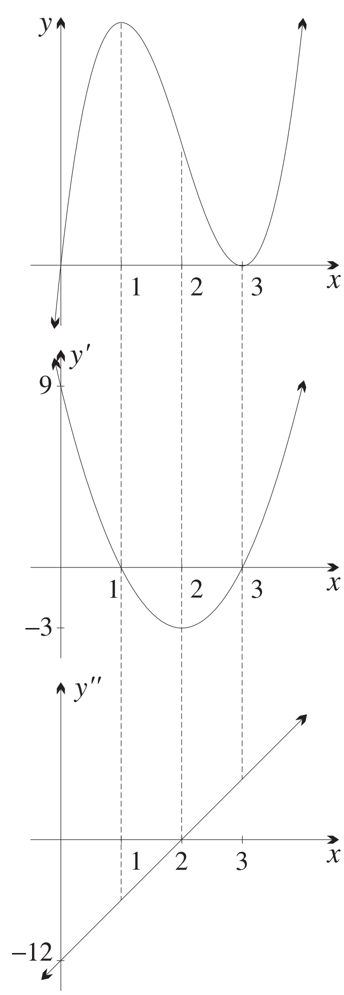
(a) $y = x^4 - 6x^2 + 11$ (b) $y = x^3 + x^2 - 5x + 7$

CHALLENGE

13. (a) Find the first, second and third derivatives of x^n.

(b) Find the nth and $(n+1)$th derivatives of x^n.

12 D Concavity and Points of Inflexion



Sketched to the right are a cubic function and its first and second derivatives. These sketches will show how the *concavity* of the original graph can be determined from the sign of the second derivative.

$$\begin{aligned} y &= x^3 - 6x^2 + 9x = x(x-3)^2 \\ y' &= 3x^2 - 12x + 9 = 3(x-1)(x-3) \\ y'' &= 6x - 12 \qquad = 6(x-2) \end{aligned}$$

The sign of each derivative tells whether the function above it is increasing or decreasing. Thus the second graph describes the gradient of the first, and the third graph describes the gradient of the second.

To the right of $x = 2$, the top graph is concave up. This means that as one moves along the curve to the right from $x = 2$, the tangent gets steeper, with its gradient steadily increasing. Thus for $x > 2$, the gradient function y' is increasing as x increases, as can be seen seen in the middle graph. The bottom graph is the gradient of the middle graph, and accordingly y'' is positive for $x > 2$.

Similarly, to the left of $x = 2$ the top graph is concave down. This means that its gradient function y' is steadily decreasing as x increases. The bottom graph is the derivative of the middle graph, so y'' is negative for $x < 2$.

This example demonstrates that the concavity of a graph $y = f(x)$ at any value $x = a$ is determined by the sign of its second derivative at $x = a$.

6 CONCAVITY AND THE SECOND DERIVATIVE:

- If $f''(a)$ is negative, the curve is concave down at $x = a$.
- If $f''(a)$ is positive, the curve is concave up at $x = a$.

Points of Inflexion: A *point of inflexion* is a point where the tangent crosses the curve, according to the definition in Section 12B. This means that the curve must curl away from the tangent on opposite sides of the tangent, so the concavity must change around the point.

The three diagrams above show how the point of inflexion at $x = 2$ results in a minimum turning point at $x = 2$ in the middle graph of y'. Hence the bottom graph of y'' has a zero at $x = 2$, and changes sign around $x = 2$.

This discussion gives the full method for analysing concavity and finding points of inflexion. Once again, the method uses the fact that y'' can only change sign at a zero or a discontinuity of y''.

7

USING $f''(x)$ TO ANALYSE CONCAVITY AND FIND POINTS OF INFLEXION:
1. Find the zeroes and discontinuities of the second derivative $f''(x)$.
2. Use a table of test points of the second derivative $f''(x)$ around its zeroes and discontinuities, followed by a table of concavities, to see where its sign changes.

The table will show not only any points of inflexion, but also the concavity of the graph across its whole domain.

Before drawing the sketch, it is often useful to find the gradient of the tangent at each point of inflexion. Such tangents are called *inflexional tangents*.

WORKED EXERCISE:

Find any points of inflexion of $f(x) = x^5 - 5x^4$ and the gradients of the inflexional tangents, and describe the concavity. Find any turning points, and sketch.

SOLUTION:

Here, $f(x) = x^5 - 5x^4 \quad = x^4(x-5)$
$f'(x) = 5x^4 - 20x^3 \quad = 5x^3(x-4)$
$f''(x) = 20x^3 - 60x^2 = 20x^2(x-3)$.

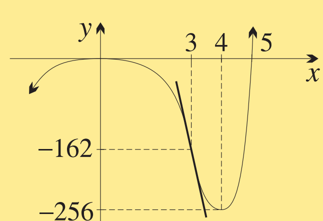
First, $f'(x)$ has zeroes at $x = 0$ and $x = 4$, and no discontinuities:

x	-1	0	1	4	5
$f'(x)$	25	0	-15	0	625
Slope	/	—	\	_	/

so $(0, 0)$ is a maximum turning point, and $(4, -256)$ is a minimum turning point.

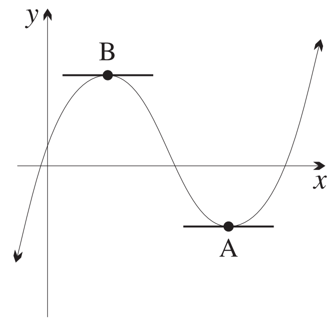
Secondly, $f''(x)$ has zeroes at $x = 0$ and $x = 3$, and no discontinuities:

x	-1	0	1	3	4
$f''(x)$	-80	0	-40	0	320
Concavity	⌢	·	⌢	·	⌣



so $(3, -162)$ is a point of inflexion, but $(0, 0)$ is not.
Since $f'(3) = -135$, the inflexional tangent has gradient -135.
The graph is concave down for $x < 0$ and $0 < x < 3$, and concave up for $x > 3$.

NOTE: The example given above is intended to show that $f''(x) = 0$ is NOT a sufficient condition for a point of inflexion — *the sign of $f''(x)$ must also change around the point.*



Using the Second Derivative to Test Stationary Points: If a curve is concave up at a stationary point, then the point must be a minimum turning point, as in the point A on the diagram to the right.

Similarly, the curve is concave down at B, which must therefore be a maximum turning point. This gives an alternative test of the nature of a stationary point.

8

USING THE SECOND DERIVATIVE TO TEST A STATIONARY POINT:
Suppose that the curve $y = f(x)$ has a stationary point at $x = a$.
- If $f''(a) > 0$, the curve is concave up at $x = a$, and there is a minimum turning point there.
- If $f''(a) < 0$, the curve is concave down at $x = a$, and there is a maximum turning point there.
- If $f''(a) = 0$, *more work is needed. Go back to the table of values of* $f'(x)$.

The third point is most important — all four cases shown on page 318 are possible for the shape of the curve at $x = a$ when the second derivative is zero there.

The previous example of the point $(0, 0)$ on $y = x^5 - 5x^4$ shows that such a point can be a turning point. The following worked exercise is an example where such a point turns out to be a point of inflexion.

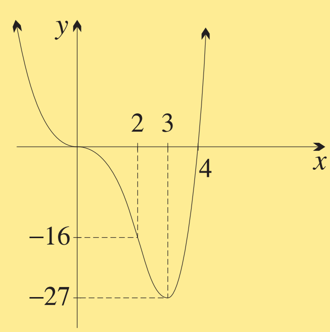
WORKED EXERCISE:
Use the second derivative, if possible, to determine the nature of the stationary points of the graph of $f(x) = x^4 - 4x^3$. Find also any points of inflexion, examine the concavity over the whole domain, and sketch the curve.

SOLUTION:
Here, $f(x) = x^4 - 4x^3 \quad = x^3(x - 4)$
$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$
$f''(x) = 12x^2 - 24x = 12x(x - 2)$,
so $f'(x)$ has zeroes at $x = 0$ and $x = 3$, and no discontinuities.
Since $f''(3) = 36$ is positive, $(3, -27)$ is a minimum turning point,
but $f''(0) = 0$, *so no conclusion can be drawn about* $x = 0$.

x	-1	0	1	3	4
$f'(x)$	-16	0	-8	0	64
Slope	\	—	\	—	/

so $(0, 0)$ is a stationary point of inflexion.
$f''(x)$ has zeroes at $x = 0$ and $x = 2$, and no discontinuities:

x	-1	0	1	2	3
$f''(x)$	36	0	-12	0	36
Concavity	⌣	·	⌢	·	⌣



so, besides the horizontal inflexion at $(0, 0)$, there is a non-stationary inflexion at $(2, -16)$, and the inflexional tangent at $(2, -16)$ has gradient -16.
The graph is concave down for $0 < x < 2$, and concave up for $x < 0$ and for $x > 2$.

Finding Pronumerals in a Function: In this worked exercise, a pronumeral in a function is found using information about the concavity of the graph.

WORKED EXERCISE:

For what values of b is $y = x^4 - bx^3 + 5x^2 + 6x - 8$ concave down when $x = 2$?

SOLUTION:

Differentiating, $y' = 4x^3 - 3bx^2 + 10x + 6$

and differentiating again, $y'' = 12x^2 - 6bx + 10$,

so when $x = 2$, $y'' = 48 - 12b + 10$

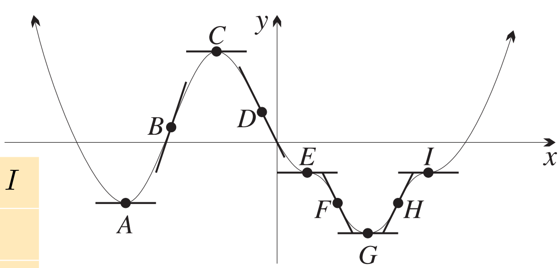
$= 58 - 12b$.

In order for the curve is concave down at $x = 2$,

$$58 - 12b < 0$$
$$12b > 58$$
$$b > 4\tfrac{5}{6}.$$

Exercise 12D

1. Complete the table below for the function to the right. At each point, state whether the first and second derivatives are positive, negative or zero.



Point	A	B	C	D	E	F	G	H	I
y'									
y''									

2. Find $f''(x)$ for each function. By evaluating $f''(0)$, state whether the curve is concave up ($f''(x) > 0$) or concave down ($f''(x) < 0$) at $x = 0$.

(a) $f(x) = x^3 - 3x^2$

(b) $f(x) = x^3 + 4x^2 - 5x + 7$

(c) $f(x) = x^4 + 2x^2 - 3$

(d) $f(x) = 6x - 7x^2 - 8x^4$

3. By showing that $f'(2) = 0$, prove that each curve has a stationary point at $x = 2$. Evaluate $f''(2)$ to determine the nature of the stationary point.

(a) $f(x) = x^2 - 4x + 4$

(b) $f(x) = 5 + 4x - x^2$

(c) $f(x) = x^3 - 12x$

(d) $f(x) = 2x^3 - 3x^2 - 12x + 5$

4. A curve is concave up when $\dfrac{d^2y}{dx^2} > 0$ and concave down when $\dfrac{d^2y}{dx^2} < 0$.

(a) Explain why $y = x^2 - 3x + 7$ is concave up for all values of x.

(b) Explain why $y = -3x^2 + 2x - 4$ is concave down for all values of x.

5. (a) Find the second derivative $\dfrac{d^2y}{dx^2}$ of $y = x^3 - 3x^2 - 5x + 2$.

(b) Hence find the values of x for which the curve is: (i) concave up, (ii) concave down.

6. (a) Find the second derivative $\dfrac{d^2y}{dx^2}$ of $y = x^3 - x^2 - 5x + 1$.

(b) Hence find the values of x for which the curve is: (i) concave up, (ii) concave down.

DEVELOPMENT

7. (a) If $f(x) = x^3 - 3x$, show that $f'(x) = 3(x-1)(x+1)$ and $f''(x) = 6x$.
(b) By solving $f'(x) = 0$, find the coordinates of any stationary points.
(c) Examine the sign of $f''(1)$ and $f''(-1)$ to determine their nature.
(d) Find the coordinates of the point of inflexion. Remember that you must show that the sign of $f''(x)$ changes about this point.
(e) Sketch a graph of the function, indicating all important features.

8. (a) If $f(x) = x^3 - 6x^2 - 15x + 1$, show that $f'(x) = 3(x-5)(x+1)$ and $f''(x) = 6(x-2)$.
(b) Find any stationary points and use the sign of $f''(x)$ to determine their nature.
(c) Find the coordinates of any points of inflexion.
(d) Sketch a graph of the function, indicating all important features.

9. (a) If $y = x^3 - 3x^2 - 9x + 11$, show that $y' = 3(x-3)(x+1)$ and $y'' = 6(x-1)$.
(b) Find any stationary points and use the sign of y'' to determine their nature.
(c) Find the coordinates of any points of inflexion.
(d) Sketch a graph of the function, indicating all important features.

10. (a) If $y = 3 + 4x^3 - x^4$, show that $y' = 4x^2(3-x)$ and $y'' = 12x(2-x)$.
(b) Find any stationary points and use a table of values of y' to determine their nature.
(c) Find the coordinates of any points of inflexion.
(d) Sketch a graph of the function, indicating all important features.

11. Find the range of values of x for which the curve $y = 2x^3 - 3x^2 - 12x + 8$ is:
(a) increasing, that is $y' > 0$,
(b) decreasing, that is $y' < 0$,
(c) concave up, that is $y'' > 0$,
(d) concave down, that is $y'' < 0$.

12. (a) If $y = x^3 + 3x^2 - 72x + 14$, find y' and y''.
(b) Show that the curve has a point of inflexion at $(-1, 88)$.
(c) Show that the gradient of the tangent at the point of inflexion is -75.
(d) Hence find the equation of the tangent at the point of inflexion.

13. (a) If $f(x) = x^3$ and $g(x) = x^4$, find $f'(x)$, $f''(x)$, $g'(x)$ and $g''(x)$.
(b) Both $f(x)$ and $g(x)$ have a stationary point at $(0, 0)$. Evaluate $f''(x)$ and $g''(x)$ when $x = 0$. Can you determine the nature of the stationary points from this calculation?
(c) Use tables of values of $f'(x)$ and $g'(x)$ to determine the nature of the stationary points.

14. (a) Find a if the curve $y = x^3 - ax^2 + 3x - 4$ has an inflexion at the point where $x = 2$.
(b) For what values of a is $y = x^3 + 2ax^2 + 3x - 4$ concave up at the point where $x = -1$?
(c) Find a and b if the curve $y = x^4 + ax^3 + bx^2$ has an inflexion at $(2, 0)$.
(d) For what values of a is $y = x^4 + ax^3 - x^2$ concave up and increasing when $x = 1$?

CHALLENGE

15. (a) If $f(x) = 7 + 5x - x^2 - x^3$, show that $f'(x) = (5+3x)(1-x)$ and $f''(x) = -2(1+3x)$.
(b) Find any stationary points and use the sign of $f''(x)$ to distinguish between them.
(c) Find the coordinates of any points of inflexion.
(d) Sketch a graph, showing all important features.
(e) Evaluate $f'(-\frac{1}{3})$ and hence find the gradient of the curve at the point of inflexion.
(f) Hence show that the tangent at the point of inflexion has equation $144x - 27y + 190 = 0$.

16. Sketch a small section of the graph of the continuous function $f(x)$ about $x = a$ if:
(a) $f'(a) > 0$ and $f''(a) > 0$,
(b) $f'(a) > 0$ and $f''(a) < 0$,
(c) $f'(a) < 0$ and $f''(a) > 0$,
(d) $f'(a) < 0$ and $f''(a) < 0$.

17. A curve has equation $y = ax^3 + bx^2 + cx + d$ and crosses the x-axis at $x = -1$. It has a turning point at $(0, 5)$ and a point of inflexion at $x = \frac{1}{2}$. Find the values of a, b, c and d.

12 E A Review of Curve Sketching

The main purpose of this chapter so far has been to use calculus to examine the gradient and curvature of curves and to find their turning points and inflexions. This section is principally intended as a review of those methods.

Chapters Three and Four presented various non-calculus approaches to curve-sketching. The table below summarises all the methods presented so far. The first four steps were presented in Chapters Three and Four, and the last two were developed in this chapter, using calculus.

Readers should be assured that few curves in this course would require consideration of all the points in the summary. Examination questions almost always give a guide as to which methods to use for any particular function.

9

A SUMMARY OF CURVE-SKETCHING METHODS:

1. DOMAIN: Find the domain of $f(x)$. (*Always* do this first.)
2. SYMMETRY: Find whether the function is even or odd, or neither.
3. A. INTERCEPTS: Find the y-intercept and all x-intercepts (zeroes).
   B. SIGN: Use a table of test points of $f(x)$ to find where the function is positive, and where it is negative.
4. A. VERTICAL ASYMPTOTES: Examine any discontinuities to see whether there are vertical asymptotes there.
   B. HORIZONTAL ASYMPTOTES: Examine the behaviour of $f(x)$ as $x \to \infty$ and as $x \to -\infty$, noting any horizontal asymptotes.
5. THE FIRST DERIVATIVE:
   A. Find the zeroes and discontinuities of $f'(x)$.
   B. Use a table of test points of $f'(x)$ to determine the nature of the stationary points, and the slope of the function throughout its domain.
6. THE SECOND DERIVATIVE:
   A. Find the zeroes and discontinuities of $f''(x)$.
   B. Use a table of test points of $f''(x)$ to find any points of inflexion, and the concavity of the function throughout its domain.
7. ANY OTHER FEATURES

The final Step 7 is a routine warning that many important features of functions will not be picked up using this menu. For example, every parabola has an axis of symmetry, but most parabolas are not even functions. Also, the trigonometric functions repeat periodically, and tests for periodicity are not mentioned.

An Example of a Curve with Turning Points and Asymptotes: The curve in the worked exercise below has three asymptotes and a turning point. Such curves are never easy to analyse, but it is worth having one such example that combines the calculus approaches of the present chapter with the previous non-calculus approaches.

WORKED EXERCISE:

Consider the curve $y = \dfrac{1}{x(x-4)}$.

(a) Write down the domain of the function.

(b) Use a table of test points to analyse the sign of the function.

(c) Find any vertical and horizontal asymptotes.

(d) Show that the derivative is $y' = \dfrac{2(2-x)}{x^2(x-4)^2}$.

(e) Find all the zeroes and discontinuities of $f'(x)$. Then use a table of test points of $f'(x)$ to analyse stationary points and find where the function is increasing and decreasing.

(f) Sketch the curve and hence write down the range of the function.

SOLUTION:

(a) The domain of the function is $x \neq 0$ and $x \neq 4$.

(b) The function is never zero, and it has discontinuites at $x = 0$ and $x = 4$.

x	-1	0	2	4	5
y	$\frac{1}{5}$	$*$	$-\frac{1}{4}$	$*$	$\frac{1}{5}$

Hence y is positive for $x < 0$ or $x > 4$,
and y is negative for $0 < x < 4$.

(c) The lines $x = 0$ and $x = 4$ are vertical asymptotes.
Also, $y \to 0$ as $x \to \infty$ and as $x \to -\infty$,
so the x-axis is a horizontal asymptote.

(d) Differentiating using the chain rule,

$$y' = \frac{-1}{x^2(x-4)^2} \times (2x-4)$$
$$= \frac{2(2-x)}{x^2(x-4)^2}.$$

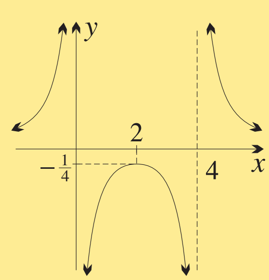
Let $u = x^2 - 4x$.
Then $y = \dfrac{1}{u}$.
Hence $\dfrac{du}{dx} = 2x - 4$
and $\dfrac{dy}{du} = -\dfrac{1}{u^2}$.

(e) Hence y' has a zero at $x = 2$
and discontinuities at $x = 0$ and $x = 4$.

x	-1	0	1	2	3	4	5
y'	$\frac{6}{25}$	$*$	$\frac{2}{9}$	0	$-\frac{2}{9}$	$*$	$-\frac{6}{25}$
Slope	/	$*$	/	—	\	$*$	\

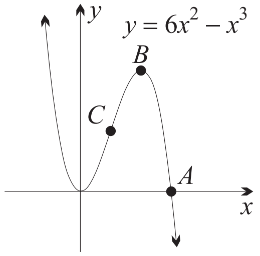
Thus there is a maximum turning point at $\left(2, -\frac{1}{4}\right)$,
the curve is increasing for $x < 2$ (except at $x = 0$),
and it is decreasing for $x > 2$ (except at $x = 4$).

(f) The graph is sketched to the right.
From the graph, the range is $y > 0$ or $y \le -\frac{1}{4}$.



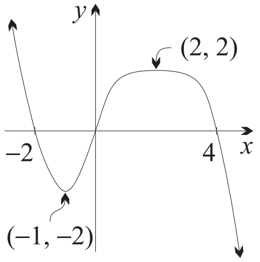
Exercise 12E

1. The diagram to the right shows a sketch of $y = 6x^2 - x^3$. The curve cuts the x-axis at A, and it has a maximum turning point at B and a point of inflexion at C.



(a) Find the coordinates of A.
(b) Find the coordinates of B.
(c) Find the coordinates of C.

2. The diagram to the right shows a curve $y = f(x)$. From the sketch, find the values of x for which:



(a) $f'(x) = 0$,
(b) $f''(x) = 0$,
(c) $f(x)$ is increasing,
(d) $f''(x) > 0$.

3. (a) Find the x-intercepts of the function $y = x^2 - 5x - 14$. (You may use either factoring or the quadratic formula.)
(b) By putting $x = 0$, find the y-intercept.
(c) Solve $\dfrac{dy}{dx} = 0$ and hence find the coordinates of the stationary point.
(d) By examining the sign of $\dfrac{d^2y}{dx^2}$, establish the nature of the stationary point.
(e) Sketch a graph of the function, indicating all important features.

4. Using the steps outlined in the previous question, sketch graphs of:
(a) $y = x^2 - 8x$ (b) $y = 6 - x - x^2$

DEVELOPMENT

5. (a) Show that $y = 27x - x^3$ is an odd function. What symmetry does its graph display?
(b) Show that $y' = 3(9 - x^2)$ and $y'' = -6x$.
(c) Find the coordinates of the stationary points. Then determine their nature, either by examining the sign of $f''(3)$ and $f''(-3)$, or by means of a table of values of y'.
(d) Show, using a table of values of y'', that $x = 0$ is a point of inflexion.
(e) By substituting into the gradient function y', find the gradient at the inflexion.
(f) Sketch a graph of the function, indicating all important features.

6. (a) If $f(x) = 2x^3 - 3x^2 + 5$, show that $f'(x) = 6x(x - 1)$ and $f''(x) = 6(2x - 1)$.
(b) Find the coordinates of the stationary points. Then determine their nature, either by examining the sign of $f''(0)$ and $f''(1)$, or by means of a table of values of y'.
(c) Explain why there is a point of inflexion at $x = \frac{1}{2}$, and find the gradient there.
(d) Sketch a graph of the function, indicating all important features.

7. Find the first and second derivatives of each function below. Hence find the coordinates of any stationary points and determine their nature. Then find any points of inflexion. Sketch a graph of each function. You do not need to find the x-intercepts in part (b).
(a) $y = x(x - 6)^2$ (b) $y = x^3 - 3x^2 - 24x + 5$

8. (a) If $y = 12x^3 - 3x^4 + 11$, show that $y' = 12x^2(3 - x)$ and $y'' = 36x(2 - x)$.
(b) By solving $y' = 0$, find the coordinates of any stationary points.
(c) By examining the sign of y'', establish the nature of the stationary point at $x = 3$. Why does this method fail for the stationary point at $x = 0$?
(d) Use a table of values of y' to show that there is a stationary point of inflexion at $x = 0$.
(e) Show that there is a change in concavity at $x = 2$.
(f) Sketch a graph of the function, showing all important features.

9. Using the method outlined in the previous question, sketch $y = x^4 - 16x^3 + 72x^2 + 10$.

CHALLENGE

NOTE: These three questions involve functions that may have both turning points and asymptotes. As a consequence, the analysis of each function is quite long and complicated.

10. (a) Show that the derivative of $f(x) = \dfrac{1}{x^2 - 4}$ is $f'(x) = -\dfrac{2x}{(x^2 - 4)^2}$.
(b) Show that $y = f(x)$ has a stationary point at $x = 0$. Then determine its nature, using a table of values of $f'(x)$.
(c) Show that the function is even. What sort of symmetry does its graph have?
(d) State the domain of the function and the equations of any vertical asymptotes.
(e) What value does $f(x)$ approach as $x \to \infty$ and as $x \to -\infty$? Hence write down the equation of the horizontal asymptote.
(f) Sketch the graph of $y = f(x)$, showing all important features.
(g) Use the graph to state the range of the function.

11. (a) Show that the derivative of $f(x) = \dfrac{x}{x^2 - 4}$ is $f'(x) = -\dfrac{x^2 + 4}{(x^2 - 4)^2}$.
(b) Explain why the curve $y = f(x)$ has no stationary points, and why the curve is always decreasing.
(c) Given that $f''(x) = \dfrac{2x^3 + 24x}{(x^2 - 4)^3}$, show that $(0, 0)$ is a point of inflexion. Then find the gradient of the tangent at this point.
(d) State the domain of the function and the equations of any vertical asymptotes.
(e) What value does $f(x)$ approach as x becomes large? Hence write down the equation of the horizontal asymptote.
(f) Show that the function is odd. What symmetry does its graph have?
(g) Use a table of test points of y to analyse the sign of the function.
(h) Sketch the graph of $y = f(x)$, showing all important features.
(i) Use the graph to state the range of the function.

12. (a) Show that the derivative of $y = x + \dfrac{1}{x}$ is $y' = \dfrac{x^2 - 1}{x^2}$.
(b) Find the stationary points and determine their nature.
(c) Show that the function is odd. What symmetry does its graph have?
(d) State the domain of the function and the equation of the vertical asymptote.
(e) Use a table of values of y to analyse the sign of the function.
(f) Sketch the graph of the function. [You may assume that the diagonal line $y = x$ is an asymptote to the curve. This is because for large x, the term $\dfrac{1}{x}$ becomes very small.]
(g) Write down the range of the function.

12 F Global Maximum and Minimum

Australia has many high mountain peaks, each of which is a *local* or *relative maximum*, because each is the highest point relative to other peaks in its immediate locality. Mount Kosciuszko is the highest of these, but it is still not a *global* or *absolute maximum*, because there are higher peaks on other continents of the globe. Mount Everest in Asia is the global maximum over the whole world.

Let $A\big(a, f(a)\big)$ be a point on a curve $y = f(x)$.

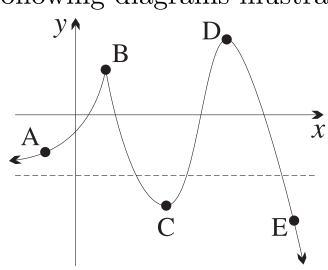
10 GLOBAL OR ABSOLUTE MAXIMUM: The point A is a *global* or *absolute maximum* if

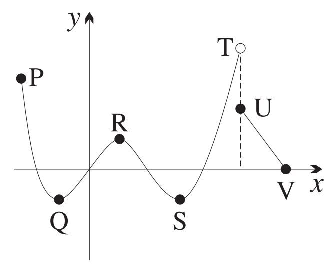
$$f(x) \le f(a), \text{ for all } x \text{ in the domain.}$$

GLOBAL OR ABSOLUTE MINIMUM: Similarly, A is a *global* or *absolute minimum* if

$$f(x) \ge f(a), \text{ for all } x \text{ in the domain.}$$

The following diagrams illustrate what has to be considered in the general case.





The domain here is all real numbers.

1. There are local maxima at the point B, where $f'(x)$ is undefined, and at the turning point D. This point D is also the global maximum.
2. There is a local minimum at the turning point C, which is lower than all points on the curve to the left past A. There is no global minimum, however, because the curve goes infinitely far downwards to the right of E.

The domain of $f(x)$ is the closed interval on the x-axis from P to V.

1. There are local maxima at the turning point R and at the endpoint P. There is no global maximum, however, because the point T has been omitted from the curve.
2. There are local minima at the two turning points Q and S, and at the endpoint V. These points Q and S have equal heights and are thus both global minima.

Testing for Global Maximum and Minimum: These examples show that there are three types of points that must be considered and compared when finding the global maximum and minimum of a function $f(x)$ defined on some domain.

11 TESTING FOR GLOBAL MAXIMUM AND MINIMUM: Examine and compare:

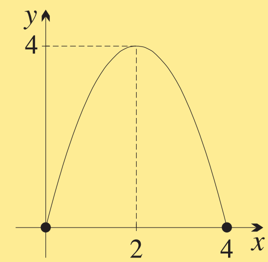
1. Turning points
2. Boundaries of the domain (or the behaviour for large x)
3. Discontinuities of $f'(x)$ (to pick up sharp corners or discontinuities)

More simply, examine and compare turning points and boundary points — curves that occur in this course are unlikely to have a maximum or minimum at a point where $f'(x)$ is undefined.

WORKED EXERCISE:

Find the absolute maximum and minimum of $f(x) = 4x - x^2$ over the domain $0 \le x \le 4$. NOTE: Calculus is not needed, because the function is a quadratic.

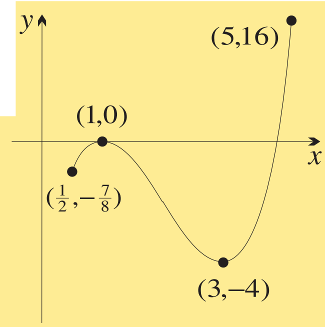
SOLUTION:



The graph is a concave-down quadratic.
Factoring, $f(x) = x(4 - x)$,
so the x-intercepts are $x = 0$ and $x = 4$.
Taking their average, the axis of symmetry is $x = 2$,
and substituting, the vertex is $(2, 4)$.
Hence, from the sketch, the absolute maximum is 4 at $x = 2$,
and the absolute minimum is 0 at the endpoints where $x = 0$ or 4.

WORKED EXERCISE:

Find the global maximum and minimum of the function $f(x) = x^3 - 6x^2 + 9x - 4$, where $\frac{1}{2} \le x \le 5$.



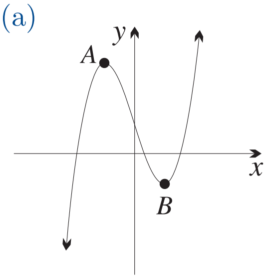
SOLUTION:

The unrestricted curve was sketched in Section 12B,
and substituting the boundaries, $f(\frac{1}{2}) = -\frac{7}{8}$ and $f(5) = 16$.
Hence the global maximum is 16, at $x = 5$,
and the global minimum is -4, at $x = 3$.

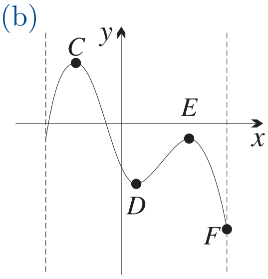
Exercise 12F

1. In each diagram below, name the points that are: (i) absolute maxima, (ii) absolute minima, (iii) relative maxima, (iv) relative minima, (v) stationary points of inflexion.

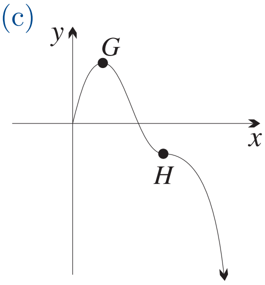
(a)



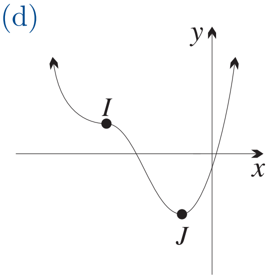
(b)



(c)



(d)



2. Sketch each function and state its global minimum and maximum in the specified domain:

(a) $y = x^2$, $-2 \le x \le 2$

(b) $y = 5 - x$, $0 \le x \le 3$

(c) $y = \sqrt{16 - x^2}$, $-4 \le x \le 4$

(d) $y = |x|$, $-5 \le x \le 1$

(e) $y = \sqrt{x}$, $0 \le x \le 8$

(f) $y = 1/x$, $-4 \le x \le -1$

(g) $y = \begin{cases} -1, & \text{for } x < -2, \\ x + 1, & \text{for } -2 \le x < 1, \\ 2, & \text{for } x \ge 1. \end{cases}$

DEVELOPMENT

3. Sketch the graph of each function, clearly indicating any stationary points. Determine the absolute minimum and maximum of the function in the specified domain.

(a) $y = x^2 - 4x + 3$, $0 \le x \le 5$

(b) $y = x^3 - 3x^2 + 5$, $-3 \le x \le 2$

(c) $y = 3x^3 - x + 2$, $-1 \le x \le 1$

(d) $y = x^3 - 6x^2 + 12x$, $0 \le x \le 3$

CHALLENGE

4. Find (i) the local maxima or minima, and (ii) the global maximum and minimum of the function $y = x^4 - 8x^2 + 11$ on each of the following domains:
(a) $1 \le x \le 3$ (b) $-4 \le x \le 1$ (c) $-1 \le x \le 0$

12 G Applications of Maximisation and Minimisation

Here are some of many practical applications of maximisation and minimisation.

- Maximise the volume of a box built from a rectangular sheet of cardboard.
- Minimise the fuel used in a flight.
- Maximise the profits from manufacturing and selling an article.
- Minimise the amount of metal used in a can of soft drink.

Such problems can be solved using calculus, provided that a clear functional relationship can first be established.

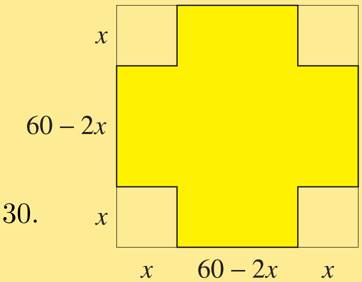
12

MAXIMISATION AND MINIMISATION PROBLEMS: After drawing a diagram:
1. Introduce the two variables from which the function is to be formed.
   'Let y (or whatever) be the quantity that is to be maximised,
   and let x (or whatever) be the quantity that can be varied.'
2. Form an equation in the two variables, noting any restrictions.
3. Find the global maximum or minimum.
4. Write a careful conclusion.

NOTE: A claim that a stationary point is a maximum or minimum must be justified by a proper analysis of the nature of the stationary point.

WORKED EXERCISE:
An open rectangular box is to be made by cutting square corners out of a square piece of cardboard measuring $60\,\text{cm} \times 60\,\text{cm}$, and folding up the sides. What is the maximum volume of the box, and what are its dimensions then?

SOLUTION:



Let V be the volume of the box,
and let x be the side lengths of the cut-out squares.
Then the box is $x\,\text{cm}$ high,
with base a square of side length $60 - 2x$,

so $V = x(60 - 2x)^2$,
$= 3600x - 240x^2 + 4x^3$, where $0 \le x \le 30$.

Differentiating, $V' = 3600 - 480x + 12x^2$
$= 12(x - 30)(x - 10)$,

so V' has zeroes at $x = 10$ and $x = 30$, and no discontinuities.

Furthermore, $V'' = -480 + 24x$
so $V''(10) = -240 < 0$ and $V''(30) = 240 > 0$.

Hence $(10, 16\,000)$ is the global maximum in the domain $0 \le x \le 30$,
and the maximum volume is $16\,000\,\text{cm}^3$ when the box is $10\,\text{cm} \times 40\,\text{cm} \times 40\,\text{cm}$.

WORKED EXERCISE:

A certain cylindrical soft drink can is required to have a volume of $250\,\text{cm}^3$.

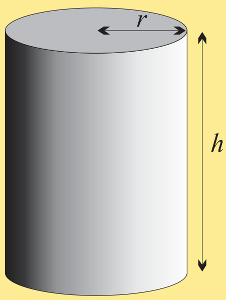
(a) Show that the height of the can must be $\dfrac{250}{\pi r^2}$, where r is the base radius.

(b) Show that the total surface area is $S = 2\pi r^2 + \dfrac{500}{r}$.

(c) Show that $r = \dfrac{5}{\pi^{\frac{1}{3}}}$ gives a global minimum of S in the domain $r > 0$.

(d) Show that to minimise the surface area of the can, the diameter of its base should equal its height.

SOLUTION:



(a) Let the height of the can be h cm.

$$\text{Then volume} = \pi r^2 h$$
$$250 = \pi r^2 h$$
$$h = \frac{250}{\pi r^2}.$$

(b) Each end has area πr^2 and the curved side has area $2\pi r h$, so

$$\begin{aligned} S &= 2\pi r^2 + 2\pi r h \\ &= 2\pi r^2 + 2\pi r \times \frac{250}{\pi r^2} \\ &= 2\pi r^2 + \frac{500}{r}, \quad \text{where } r > 0. \end{aligned}$$

(c) Differentiating,
$$\begin{aligned} \frac{dS}{dr} &= 4\pi r - \frac{500}{r^2} \\ &= \frac{4\pi r^3 - 500}{r^2}. \end{aligned}$$

To find stationary points, put $\dfrac{dS}{dr} = 0$

$$4\pi r^3 = 500$$
$$r^3 = \frac{125}{\pi}$$
$$r = \frac{5}{\pi^{\frac{1}{3}}}.$$

Differentiating again, $\dfrac{d^2S}{dr^2} = 4\pi + \dfrac{1000}{r^3}$,

which is positive for all $r > 0$.

Hence the stationary point is a global minimum in the domain $r > 0$.

(d) When $r = \dfrac{5}{\pi^{\frac{1}{3}}}$,
$$\begin{aligned} h &= \frac{250}{\pi r^2} \\ &= \frac{250}{\pi} \times \frac{\pi^{\frac{2}{3}}}{25} \\ &= \frac{10}{\pi^{\frac{1}{3}}} \\ &= 2r. \end{aligned}$$

Hence the minimum surface area occurs when the diameter equals the height.

Cost and Time Problems: There is often an optimum speed at which the costs of running a boat or truck are minimised.

- At very slow speeds, wages and fixed costs rise.
- At very high speeds, the costs of fuel and wear rise.

If some formula for these costs can be found, calculus can find the best speed.

WORKED EXERCISE:

The cost C (in dollars per hour) of running a boat depends on the speed v km/h of the boat according to the formula $C = 500 + 40v + 5v^2$.

(a) Show that the total cost for a trip of 100 km is $T = \dfrac{50\,000}{v} + 4000 + 500v$.

(b) What speed will minimise the total cost of the trip?

SOLUTION:

(a) Since time $= \dfrac{\text{distance}}{\text{speed}}$, the time for the trip is $\dfrac{100}{v}$ hours.

Hence the total cost is

$$\begin{aligned} T &= (\text{cost per hour}) \times (\text{time for the trip}) \\ &= (500 + 40v + 5v^2) \times \frac{100}{v} \\ &= \frac{50\,000}{v} + 4000 + 500v, \quad \text{where } v > 0. \end{aligned}$$

(b) Differentiating,

$$\begin{aligned} \frac{dT}{dv} &= -\frac{50\,000}{v^2} + 500 \\ &= \frac{500(-100 + v^2)}{v^2} \\ &= \frac{500(v-10)(v+10)}{v^2}, \end{aligned}$$

so $\dfrac{dT}{dv}$ has a single zero at $v = 10$ in the domain $v > 0$, and no discontinuities.

Differentiating again, $\dfrac{d^2T}{dv^2} = \dfrac{100\,000}{v^3}$, which is positive for all $v > 0$,

so $v = 10$ gives a global minimum in the domain $v > 0$.

Thus a speed of 10 km/h will minimise the cost of the trip.

Exercise 12G

NOTE: You must always prove that any stationary point is a maximum or minimum, either by creating a table of values of the derivative, or by substituting into the second derivative. It is never acceptable to assume this from the wording of a question.

1. At time t seconds, a particle has height $h = 3 + t - 2t^2$ metres.

(a) Find $\dfrac{dh}{dt}$ and show that the maximum height occurs after 0·25 seconds.

(b) Find the maximum height.

2. (a) Given that $2x + y = 11$, express $P = xy$ in terms of x only.

(b) Find $\dfrac{dP}{dx}$ and hence the value of x at which P attains its maximum value.

(c) Calculate the maximum value of P (and prove that it is a maximum).

3. (a) Given that $x + y = 8$, express $P = x^2 + y^2$ in terms of x only.
(b) Find $\dfrac{dP}{dx}$, and hence find the value of x at which P attains its minimum value.
(c) Calculate the minimum value of P (and prove that it is a minimum).

4. (a) A rectangle has a constant perimeter of 20 cm. Let the rectangle have length x cm. Show that the width is $(10 - x)$ cm, and hence show that the area is $A = 10x - x^2$.
(b) Find $\dfrac{dA}{dx}$, and hence find the value of x at which A is maximum.
(c) Hence find the maximum area of the rectangle.

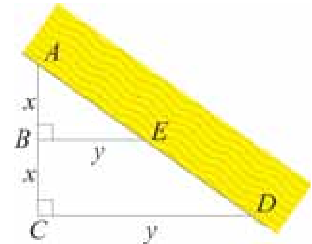
5. A landscaper is constructing a rectangular garden bed. Three of the sides are to be fenced using 40 metres of fencing, while an existing wall will form the fourth side of the rectangle.
(a) Let x be the length of each of the two sides perpendicular to the wall. Show that the side parallel to the wall will have length $(40 - 2x)$ metres.
(b) Show that the area of the garden bed is $A = 40x - 2x^2$.
(c) Find $\dfrac{dA}{dx}$, and hence find the value of x at which A attains its maximum value.
(d) Find the maximum area of the garden bed.

6. The amount V of vitamins present in a patient's bloodstream t hours after taking the vitamin tablets is given by $V = 4t^2 - t^3$, for $0 \le t \le 3$. Find $\dfrac{dV}{dt}$ and the time at which the amount of vitamins in the patients bloodstream is a maximum.

DEVELOPMENT

7. A rectangle has a constant area of $36\,\text{cm}^2$.
(a) Show that the width of the rectangle is $\dfrac{36}{x}$ cm, where x is the length.
(b) Show that the perimeter of the rectangle is $P = 2x + \dfrac{72}{x}$.
(c) Show that $\dfrac{dP}{dx} = 2 - \dfrac{72}{x^2}$ and that the minimum value of P occurs at $x = 6$.
(d) Find the minimum possible perimeter of the rectangle.

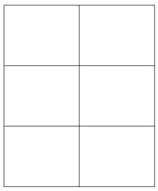
8. A farmer has a field of total area $1200\,\text{m}^2$. To keep his animals separate, he sets up his field with fences at AC, CD and BE, as shown in the diagram. The side AD is beside a river and so no fence is needed there. The point B is the midpoint of AC, and CD is twice the length of BE. Let $AB = x$ metres and $BE = y$ metres.



(a) Show that the total length of fencing is $L = 2x + \dfrac{1800}{x}$.
(b) Hence find the values of x and y that allow the farmer to use the least possible length of fencing.

9. The sum of two positive numbers is 40. [HINT: Let the numbers be x and $40 - x$.]
(a) Find the numbers if their product is a maximum. [HINT: Let $P = x(40 - x)$.]
(b) Find the numbers if the sum of their squares is a minimum.
[HINT: Let $S = x^2 + (40 - x)^2$.]

10. A window frame consisting of 6 equal rectangles is illustrated to the right. Only 12 metres of frame is available for its construction.



(a) Let the entire frame have height h metres and width y metres. Show that $y = \frac{1}{4}(12 - 3h)$.

(b) Show that the area of the window is $A = 3h - \frac{3}{4}h^2$.

(c) Find $\frac{dA}{dh}$, and hence find the dimensions of the frame for which the area of the window is a maximum.

11. A piece of wire of length 10 metres is cut into two pieces and used to form two squares.

(a) Let one piece of wire have length x metres. Find the side length of each square.

(b) Show that the combined area of the squares is $A = \frac{1}{8}(x^2 - 10x + 50)$.

(c) Find $\frac{dA}{dx}$, and hence find the value of x that makes A a minimum.

(d) Find the least possible value of the combined areas.

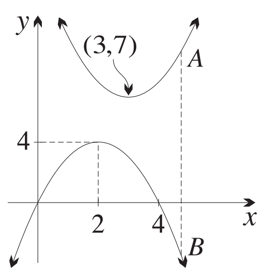
12. The total cost of producing x telescopes per day is given by $C = (\frac{1}{5}x^2 + 15x + 10)$ dollars, and each telescope is sold for a price of $(47 - \frac{1}{3}x)$ dollars.

(a) Find an expression for the revenue R raised from the sale of x telescopes per day.

(b) Find an expression for the daily profit $P = R - C$ made if x telescopes are sold.

(c) How many telescopes should be made daily in order to maximise the profit?

13. A point A lies on the curve $y = (x-3)^2 + 7$. A point B with the same x-coordinate as that of A lies on $y = x(4 - x)$.



(a) Show that the length of AB is $L = 2x^2 - 10x + 16$.

(b) Find $\frac{dL}{dx}$, and hence find the value of x at which the length of AB is a minimum.

(c) Find the minimum length of AB.

14. (a) An open rectangular box is to be formed by cutting squares of side length x cm from the corners of a rectangular sheet of metal that has length 40 cm and width 15 cm.

(b) Show that the volume of the box is given by $V = 600x - 110x^2 + 4x^3$.

(c) Find $\frac{dV}{dx}$, and hence find the value of x that maximises the volume of the box.

15. The sum of the height h of a cylinder and the circumference of its base is 10 metres.

(a) Show that $h = 10 - 2\pi r$, where r is the radius of the cylinder.

(b) Show that the volume of the cylinder is $V = \pi r^2(10 - 2\pi r)$.

(c) Find $\frac{dV}{dr}$, and hence find the value of r at which the volume is a maximum.

(d) Hence find the maximum volume of the cylinder.

16. A closed cylindrical can is to have a surface area of 60π cm^2.

(a) Let the cylinder have height h and radius r. Show that $h = \dfrac{30 - r^2}{r}$.

(b) Show that the volume of the can is $V = \pi r(30 - r^2)$.

(c) Find $\frac{dV}{dr}$, and hence find the maximum possible volume of the can in terms of π.

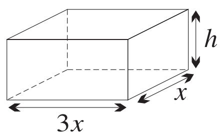
17. A box with volume $32\,\text{cm}^3$ has a square base and no lid. Let the square base have length x cm and the box have height h cm.

(a) Show that the surface area of the box is $S = x^2 + 4xh$.

(b) Show that $h = \dfrac{32}{x^2}$, and hence that $S = x^2 + \dfrac{128}{x}$.

(c) Find S', and hence find the dimensions of the box that minimise its surface area.

18. The steel frame of a rectangular prism, as illustrated in the diagram, is three times as long as it is wide.

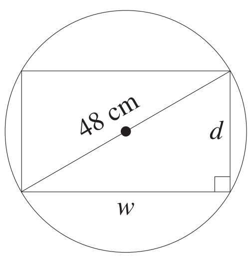


(a) Find an expression in terms of x and y for S, the length of steel required to construct the frame.

(b) The prism has a volume of $4374\,\text{m}^3$. Show that $y = \dfrac{1458}{x^2}$, and hence show that $S = 16x + \dfrac{5832}{x^2}$.

(c) Show that $\dfrac{dS}{dx} = \dfrac{16(x^3 - 729)}{x^3}$, and hence find the dimensions of the frame so that the minimum amount of steel is used.

19. Engineers have determined that the strength s of a rectangular beam varies as the product of the width w and the square of the depth d of the beam, that is, $s = kwd^2$ for some constant k.

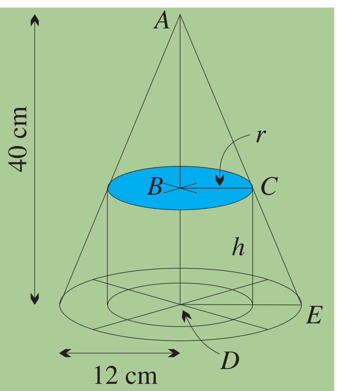


(a) A particular cylindrical log has a diameter of 48 cm. Use Pythagoras' theorem to show that $s = kw(2304 - w^2)$.

(b) Find $\dfrac{ds}{dw}$, and hence find the dimensions of the strongest rectangular beam that can be cut from the log.

CHALLENGE

20. A cylinder of height h cm and radius r cm is enclosed in a cone of height 40 cm and radius 12 cm.



(a) Explain why $\Delta ABC \parallel\!| \Delta ADE$.

(b) By using ratios of corresponding sides, show that $h = 40 - \frac{10}{3}r$.

(c) Show that the volume of the cylinder is given by $V = 40\pi r^2 - \frac{10}{3}\pi r^3$.

(d) Find $\dfrac{dV}{dr}$, and hence find the value of r for which the volume of the cylinder is maximised.

21. A page of a book is to have $80\,\text{cm}^2$ of printed material. There is to be a 2 cm margin at the top and bottom and a 1 cm margin on each side of the page.

(a) Let the page have width x cm and height y cm. Show that $(y-4)(x-2) = 80$ and hence that $y = 4 + \dfrac{80}{x-2}$.

(b) Show that the area of the page is $A = \dfrac{4x^2 + 72x}{x-2}$.

(c) Use the quotient rule to show that $\dfrac{dA}{dx} = \dfrac{4(x^2 - 4x - 36)}{(x-2)^2}$.

(d) What should be the dimensions of the page in order to use the least amount of paper?

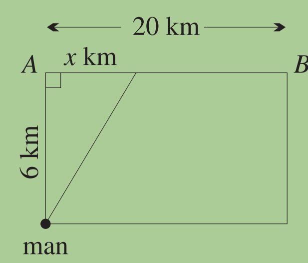
22. A transport company runs a truck from Hobart to Launceston, a distance of 250 km, at a constant speed of v km/h. For a given speed v, the cost per hour is $6400 + v^2$ cents.

(a) Show that the cost of the trip, in cents, is $C = 250\left(\dfrac{6400}{v} + v\right)$.

(b) Find the speed at which the cost of the journey is minimised.

(c) Find the minimum cost of the journey.

23. A man in a rowing boat is presently 6 km from the nearest point A on the shore. He wants to reach, as soon as possible, a point B that is a further 20 km along the shore from A.



(a) He can row at 8 km/h and he can run at 10 km/h. He rows to a point on the shore x km from A, and then he runs to B. Show that the time taken for the journey is $T = \frac{1}{8}\sqrt{36 + x^2} + \frac{1}{10}(20 - x)$.
[HINT: Recall that time = distance/speed.]

(b) Find $\dfrac{dT}{dx}$, and show that the time for the journey is minimised if he lands 8 km from A.

12 H Primitive Functions

This section reverses the process of differentiation and asks, 'What can we say about a function if we know its derivative?'

The results of this section will be needed when integration is introduced in the first chapter of the Year 12 book.

Functions with the Same Derivative: Many different functions can all have the same derivative. For example, all these functions have the same derivative, $2x$:

$$x^2, \qquad x^2 + 3, \qquad x^2 - 2, \qquad x^2 + 4\tfrac{1}{2}, \qquad x^2 - \pi.$$

These functions are all the same apart from a constant term. This is true generally — *any two functions with the same derivative differ only by a constant.*

13 THEOREM:
(a) If a function $f(x)$ has derivative zero in an interval $a < x < b$, then $f(x)$ is a constant function in $a < x < b$.
(b) If $f'(x) = g'(x)$ for all x in an interval $a < x < b$, then $f(x)$ and $g(x)$ differ by a constant in $a < x < b$.

PROOF:

(a) Because the derivative is zero, the gradient of the curve must be zero throughout the interval. The curve must therefore be a horizontal straight line, and $f(x)$ is a constant function.

(b) Take the difference between $f(x)$ and $g(x)$ and apply part (a).

$$\begin{aligned} \text{Let} \quad h(x) &= f(x) - g(x). \\ \text{Then} \quad h'(x) &= f'(x) - g'(x) \\ &= 0, \quad \text{for all } x \text{ in the interval } a < x < b. \end{aligned}$$

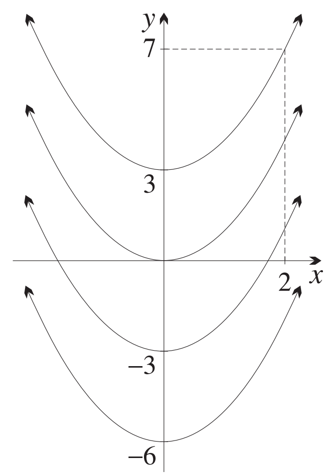
Hence by part (a), $h(x) = C$, where C is a constant,

so $f(x) - g(x) = C$, as required.

The Family of Curves with the Same Derivative: Continuing with the very first example, the various functions whose derivatives are $2x$ must all be of the form

$$f(x) = x^2 + C, \quad \text{where } C \text{ is a constant.}$$

By taking different values of the constant C, these functions form an infinite family of curves, each consisting of the parabola $y = x^2$ translated upwards or downwards.



Boundary Conditions: If we know also that the curve must pass through a particular point, say $(2, 7)$, then we can evaluate the constant C by substituting the point into $f(x) = x^2 + C$:

$$7 = 4 + C.$$

Thus $C = 3$ and hence $f(x) = x^2 + 3$ — in place of the infinite family of functions, there is now a single function.

Such an extra condition is called a *boundary condition*. It is also called an *initial condition* if it involves the value of y when $x = 0$, particularly when x is time.

Primitives: We need a suitable name for the result of this reverse process:

14 A PRIMITIVE OF A FUNCTION:
- A function $F(x)$ is called a *primitive* of $f(x)$ if the derivative of $F(x)$ is $f(x)$:

$$F'(x) = f(x).$$

- In general, *the primitive* of $f(x)$ is then $F(x) + C$, where C is a constant.

For example, these functions are all primitives of $x^2 + 1$:

$$\tfrac{1}{3}x^3 + x, \qquad \tfrac{1}{3}x^3 + x + 7, \qquad \tfrac{1}{3}x^3 + x - 13, \qquad \tfrac{1}{3}x^3 + x + 4\pi,$$

and *the primitive* of $x^2 + 1$ is $\frac{1}{3}x^3 + x + C$, where C is a constant.

A Rule for Finding Primitives: We have seen that a primitive of x is $\frac{1}{2}x^2$, and a primitive of x^2 is $\frac{1}{3}x^3$. Reversing the formula $\dfrac{d}{dx}(x^{n+1}) = (n+1)x^n$ gives the general rule:

15 FINDING PRIMITIVES: Suppose that $n \neq -1$.

$$\text{If } \frac{dy}{dx} = x^n, \text{ then } y = \frac{x^{n+1}}{n+1} + C, \quad \text{for some constant } C.$$

'Increase the index by 1 and divide by the new index.'

WORKED EXERCISE:

Find the primitives of: (a) $x^3 + x^2 + x + 1$ (b) $5x^3 + 7$

SOLUTION:

(a) Let $\dfrac{dy}{dx} = x^3 + x^2 + x + 1$.

Then $y = \frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + C$, where C is a constant.

(b) Let $f'(x) = 5x^3 + 7$.

Then $f(x) = \frac{5}{4}x^4 + 7x + C$, where C is a constant.

WORKED EXERCISE:

Rewrite each function with negative or fractional indices, and find the primitive.

(a) $\dfrac{1}{x^2}$ (b) $\sqrt{x}$

SOLUTION:

(a) Let $f'(x) = \dfrac{1}{x^2}$

$= x^{-2}$.

Then $f(x) = -x^{-1} + C$, where C is a constant,

$= -\dfrac{1}{x} + C$.

(b) Let $\dfrac{dy}{dx} = \sqrt{x}$

$= x^{\frac{1}{2}}$.

Then $y = \frac{2}{3}x^{\frac{3}{2}} + C$, where C is a constant.

Linear Extension: Reversing the formula $\dfrac{d}{dx}(ax+b)^{n+1} = a(n+1)(ax+b)^n$ gives:

16 EXTENSION TO POWERS OF LINEAR FUNCTIONS: Suppose that $n \neq -1$.

If $\dfrac{dy}{dx} = (ax+b)^n$, then $y = \dfrac{(ax+b)^{n+1}}{a(n+1)} + C$, for some constant C.

'Increase the index by 1 and divide by the new index and by the coefficient of x.'

WORKED EXERCISE:

Find the primitives of:

(a) $(3x+1)^4$ (b) $(1-3x)^6$ (c) $\dfrac{1}{(x+1)^2}$ (d) $\sqrt{x+1}$

SOLUTION:

(a) Let $\dfrac{dy}{dx} = (3x+1)^4$.

Then $y = \dfrac{(3x+1)^5}{5 \times 3} + C$,

where C is a constant,

$= \dfrac{(3x+1)^5}{15} + C$.

(b) Let $\dfrac{dy}{dx} = (1-3x)^6$.

Then $y = \dfrac{(1-3x)^7}{7 \times (-3)} + C$,

where C is a constant,

$y = -\dfrac{(1-3x)^7}{21} + C$.

(c) Let $\dfrac{dy}{dx} = \dfrac{1}{(x+1)^2}$

$= (x+1)^{-2}$.

Then $y = \dfrac{(x+1)^{-1}}{-1} + C$,

where C is a constant,

$= -\dfrac{1}{x+1} + C$.

(d) Let $\dfrac{dy}{dx} = \sqrt{x+1}$

$= (x+1)^{\frac{1}{2}}$.

Then $y = \dfrac{(x+1)^{\frac{3}{2}}}{\frac{3}{2}} + C$,

where C is a constant,

$y = \frac{2}{3}(x+1)^{\frac{3}{2}} + C$.

Finding the Primitive, Given a Boundary Condition: If the derivative and a particular value of a function are known, then the original function can be found as follows:

- First find the primitive, taking care to include the constant of integration.
- Then substitute the known value of the function to work out the constant.

WORKED EXERCISE:

Given that $\dfrac{dy}{dx} = 6x^2 + 1$, and $y = 12$ when $x = 2$, find y as a function of x.

SOLUTION:

Since $\dfrac{dy}{dx} = 6x^2 + 1$,

$y = 2x^3 + x + C$, for some constant C.

Substituting $x = 2$ and $y = 12$, $12 = 16 + 2 + C$,

so $C = -6$, and hence $y = 2x^3 + x - 6$.

WORKED EXERCISE:

Given that $f''(x) = 12(x-1)^2$, and $f(0) = f(1) = 0$, find $f(4)$.

SOLUTION:

We know that $f''(x) = 12(x-1)^2$.

Integrating, $f'(x) = 4(x-1)^3 + C$, for some constant C

and integrating again, $f(x) = (x-1)^4 + Cx + D$, for some constant D.

Since $f(0) = 0$, $0 = 1 + 0 + D$

$D = -1$.

Since $f(1) = 0$, $0 = 0 + C - 1$.

Hence $C = 1$, and so $f(4) = 81 + 4 - 1$

$= 84$.

Exercise 12H

1. Find the primitive of each function.

(a) x^6 (b) x^3 (c) x^{10} (d) $3x$ (e) 5 (f) $5x^9$ (g) $21x^6$ (h) 0

2. Find the primitive of each function.

(a) $x^2 + x^4$ (b) $4x^3 - 5x^4$ (c) $2x^2 + 5x^7$ (d) $x^2 - x + 1$ (e) $3 - 4x + 16x^7$ (f) $3x^2 - 4x^3 - 5x^4$

3. Find the primitive of each function, after first expanding the product.

(a) $x(x-3)$ (b) $(x+1)(x-2)$ (c) $(3x-1)(x+4)$ (d) $x^2(5x^3 - 4x)$ (e) $2x^3(4x^4 + 1)$ (f) $(x-3)(1+x^2)$

4. Find y as a function of x if:

(a) $y' = 2x + 3$ and: (i) $y = 3$ when $x = 0$, (ii) $y = 8$ when $x = 1$.

(b) $y' = 9x^2 + 4$ and: (i) $y = 1$ when $x = 0$, (ii) $y = 5$ when $x = 1$.

(c) $y' = 3x^2 - 4x + 7$ and: (i) $y = 0$ when $x = 0$, (ii) $y = -1$ when $x = 1$.

DEVELOPMENT

5. Write each function using a negative power of x. Then find the primitive function, giving the answer in fractional form without negative indices.

(a) $\dfrac{1}{x^2}$ (b) $\dfrac{1}{x^3}$ (c) $-\dfrac{2}{x^3}$ (d) $-\dfrac{3}{x^4}$ (e) $\dfrac{1}{x^2} - \dfrac{1}{x^3}$

6. Write each function using a fractional index, and hence find the primitive:

(a) $\sqrt{x}$ (b) $\dfrac{1}{\sqrt{x}}$ (c) $\sqrt[3]{x}$ (d) $\dfrac{2}{\sqrt{x}}$ (e) $\sqrt[5]{x^3}$

7. Find y as a function of x if $\dfrac{dy}{dx} = \sqrt{x}$ and: (a) $y = 1$ when $x = 0$, (b) $y = 2$ when $x = 9$.

8. Find each family of curves whose gradient function is given below. Then sketch the family, and find the member of the family passing through $A(1, 2)$.

(a) $\dfrac{dy}{dx} = -4x$ (b) $\dfrac{dy}{dx} = 3$ (c) $\dfrac{dy}{dx} = 3x^2$ (d) $\dfrac{dy}{dx} = -\dfrac{1}{x^2}$

9. Recall that if $\dfrac{dy}{dx} = (ax+b)^n$, then $y = \dfrac{(ax+b)^{n+1}}{a(n+1)} + C$, for some constant C. Hence find the primitive of each function.

(a) $(x+1)^3$ (c) $(x+5)^2$ (e) $(3x-4)^6$ (g) $(1-x)^3$ (i) $\dfrac{1}{(x-2)^4}$

(b) $(x-2)^5$ (d) $(2x+3)^4$ (f) $(5x-1)^3$ (h) $(1-7x)^3$ (j) $\dfrac{1}{(1-x)^{10}}$

10. Find the primitive of each function. Use the rule given in the previous question and the fact that $\sqrt{u} = u^{\frac{1}{2}}$.

(a) $\sqrt{x+1}$ (b) $\sqrt{x-5}$ (c) $\sqrt{1-x}$ (d) $\sqrt{2x-7}$ (e) $\sqrt{3x-4}$

11. (a) Find y if $y' = (x-1)^4$, given that $y = 0$ when $x = 1$.
(b) Find y if $y' = (2x+1)^3$, given that $y = -1$ when $x = 0$.
(c) Find y if $y' = \sqrt{2x+1}$, given that $y = \frac{1}{3}$ when $x = 0$.

12. (a) Find the equation of the curve through the origin whose gradient is $\dfrac{dy}{dx} = 3x^4 - x^3 + 1$.
(b) Find the curve passing through $(2, 6)$ with gradient function $\dfrac{dy}{dx} = 2 + 3x^2 - x^3$.
(c) Find the curve through the point $(\frac{1}{5}, 1)$ with gradient function $y' = (2-5x)^3$.

13. Find y if $\dfrac{dy}{dt} = 8t^3 - 6t^2 + 5$, and $y = 4$ when $t = 0$. Hence find y when $t = 2$.

CHALLENGE

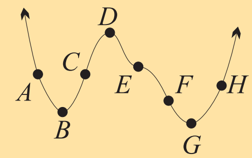
14. Box 15 of the text states the rule that the primitive of x^n is $\dfrac{x^{n+1}}{n+1}$, provided that $n \neq -1$. Why can't this rule be used when $n = -1$?

15. Find y if $y'' = 6x + 4$, given that when $x = 1$, $y' = 2$ and $y = 4$.
[HINT: Find y' and use the condition $y' = 2$ when $x = 1$ to find the constant of integration. Then find y and use the condition $y = 4$ when $x = 1$ to find the second constant.]

16. At any point on a curve, $\dfrac{d^2y}{dx^2} = 2x - 10$. The curve passes through the point $(3, -34)$, and at this point the tangent to the curve has a gradient of 20.
(a) Show that $\dfrac{dy}{dx} = x^2 - 10x + 41$.
(b) Hence find y and show that the curve cuts the y-axis at the point $(0, -121)$.

17. If $y'' = 8 - 6x$, show that $y = 4x^2 - x^3 + Cx + D$, for some constants C and D. Hence, by solving simultaneous equations, find the curve passing through the points $(1, 6)$ and $(-1, 8)$.

12I Chapter Review Exercise



1. In the diagram to the right, name the points where:

(a) $f'(x) > 0$ (c) $f'(x) = 0$ (e) $f''(x) < 0$

(b) $f'(x) < 0$ (d) $f''(x) > 0$ (f) $f''(x) = 0$

2. (a) Find the derivative $f'(x)$ of the function $f(x) = x^3 - x^2 - x - 7$.

(b) Hence find whether $f(x)$ is increasing, decreasing or stationary at:

(i) $x = 0$ (ii) $x = 1$ (iii) $x = -1$ (iv) $x = 3$

3. (a) Find the derivative $f'(x)$ of the function $f(x) = x^2 - 4x + 3$.

(b) Find the values of x for which $f(x)$ is:

(i) increasing, (ii) decreasing, (iii) stationary.

4. Use the appropriate rule to differentiate each function. Then evaluate $f'(1)$ to determine whether the function is increasing, decreasing or stationary at $x = 1$.

(a) $f(x) = x^3$ (c) $f(x) = (x-1)^5$

(b) $f(x) = (x+2)(x-3)$ (d) $f(x) = \dfrac{x+1}{x-3}$

5. Find the first and second derivatives of:

(a) $y = x^7$ (b) $y = x^3 - 4x^2$ (c) $y = (x-2)^5$ (d) $y = \dfrac{1}{x}$

6. Find $f''(x)$ for each function. By evaluating $f''(1)$, state whether the curve is concave up or concave down at $x = 1$.

(a) $f(x) = x^3 - 2x^2 + 4x - 5$ (b) $f(x) = 6 - 2x^3 - x^4$

7. (a) Find the second derivative $f''(x)$ of the function $f(x) = 2x^3 - 3x^2 + 6x - 1$.

(b) Find the values of x for which $f(x)$ is:

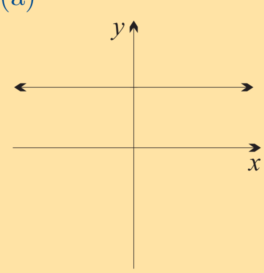
(i) concave up, (ii) concave down.

8. Find the values of x for which the curve $y = x^3 - 6x^2 + 9x - 11$ is:

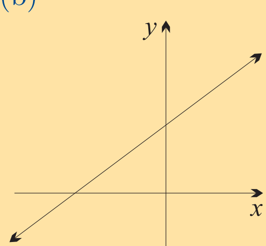
(a) increasing, (b) decreasing, (c) concave up, (d) concave down.

9. Look carefully at each function drawn below to establish where it is increasing, decreasing and stationary. Hence draw a graph of the derivative of each function.

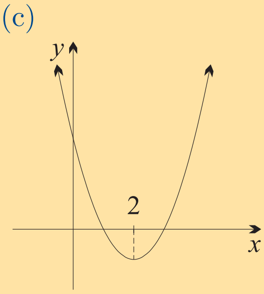
(a)



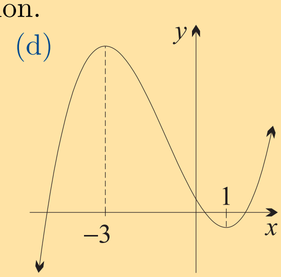
(b)



(c)



(d)

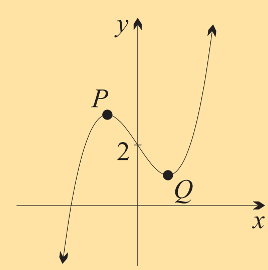


10. The graph of $y = x^3 + x^2 - x + 2$ is sketched to the right. The points P and Q are the turning points.

(a) Find the coordinates of P and Q.

(b) For what values of x is the curve concave up?

(c) For what values of k are there three distinct solutions of the equation $x^2 + x^2 - x + 2 = k$?



11. Sketch a graph of each function, indicating all stationary points and points of inflexion.
(a) $y = x^2 - 6x - 7$ (b) $y = x^3 - 6x^2 + 8$ (c) $y = 2x^3 - 3x^2 - 12x + 1$

12. (a) Sketch a graph of the function $y = x^3 - 3x^2 - 9x + 11$, indicating all stationary points.
(b) Hence determine the absolute maximum and absolute minimum values of the function on the domain $-2 \le x \le 6$.

13. (a) The tangent to $y = x^2 - ax + 9$ is horizontal at $x = -1$. Find the value of a.
(b) The curve $y = ax^2 + bx + 3$ has a turning point at $(-1, 0)$. Find the values of a and b.

14. (a) Show that the curve $y = x^4 - 4x^3 + 7$ has a point of inflexion at $(2, -9)$.
(b) Find the gradient of the curve at this point of inflexion.
(c) Hence show that the equation of the tangent at the point of inflexion is $16x + y - 23 = 0$.

15. The number S of students logged onto a particular website over a five-hour period is given by the formula $S = 175 + 18t^2 - t^4$, for $0 \le t \le 5$.
(a) What is the initial number of students that are logged on?
(b) How many students are logged on at the end of the five hours?
(c) What was the maximum number of students logged onto the website during the five-hour period?

16. A rectangular sheet of cardboard measures 16 cm by 6 cm. Equal squares of side length x cm are cut out of the corners and the sides are turned up to form an open rectangular box.
(a) Show that the volume V of the box is given by $V = 4x^3 - 44x^2 + 96x$.
(b) Find, in exact form, the maximum volume of the box.

17. A coal shute is built in the shape of an upturned cone, in which the sum of the base radius r and the height h is 12 metres.
(a) Show that the volume V of the coal shute is given by the formula $V = 4\pi r^2 - \frac{1}{3}\pi r^3$. (Recall that the volume of a cone is given by $V = \frac{1}{3}\pi r^2 h$.)
(b) Find the radius of the cone that yields the maximum volume.

18. Find the primitive of each function.
(a) x^7 (b) $2x$ (c) 4 (d) $10x^4$ (e) $8x + 3x^2 - 4x^3$

19. Find the primitive of each function after first expanding the brackets.
(a) $3x(x - 2)$ (b) $(x + 1)(x - 5)$ (c) $(2x - 3)^2$

20. Find the primitive of each function.
(a) $(x + 1)^5$ (b) $(x - 4)^7$ (c) $(2x - 1)^3$

21. Find the primitive of each function after writing the function in index form.
(a) $\dfrac{1}{x^2}$ (b) $\sqrt{x}$

22. Find the equation of the curve passing through the point $(2, 5)$ with gradient function $f'(x) = 3x^2 - 4x + 1$.

23. If $f'(x) = 4x - 3$ and $f(2) = 7$, find $f(4)$.

Answers to Exercises

Chapter One

Exercise **1A** (Page 2)

1(a) $8x$ (b) $2x$ (c) $-2x$ (d) $-8x$

2(a) $5a$ (b) $-a$ (c) $-9a$ (d) $-3a$

3(a) $-x$ (b) $2y$ (c) $-4a$ (d) $-3b$ (e) $7x$ (f) $-3ab$ (g) $4pq$ (h) $-3abc$

4(a) $x+3$ (b) $2y-3$ (c) $2a-3$ (d) $8x+4y$ (e) $-10t-5$ (f) $4a-3a^2$ (g) $-5x^2-12x-3$ (h) $9a-3b-5c$

5(a) $-6a$ (b) $12a^2$ (c) a^5 (d) a^6

6(a) $-2a$ (b) 3 (c) a^4 (d) a

7(a) 5 (b) $-7x^2$ (c) $12a$ (d) $-3x^3y^4z$

8(a) $2t^2$ (b) 0 (c) t^4 (d) 1

9(a) $-3x$ (b) $-9x$ (c) $-18x^2$ (d) -2

10(a) -4 (b) -12 (c) 18 (d) 2

11(a) 0 (b) -1 (c) 59 (d) 40

12(a) $2x$ (b) $4x$ (c) $-6a$ (d) $-4b$

13(a) $10a$ (b) $-18x$ (c) $-3a^2$ (d) $6a^3b$ (e) $-8x^5$ (f) $-6p^3q^4$

14(a) -2 (b) $3x$ (c) xy (d) $-a^4$ (e) $-7ab^3$ (f) $5ab^2c^6$

15(a) $6x$ (b) $20a$ (c) $5ab+bc$ (d) $2x^3-5x^2y+2xy^2+3y^3$

16(a) $2x^2-2x+4$ (b) $3a-5b-4c$ (c) $-3a+2b-2c+2d$ (d) $2ab-2bc+2cd$

17(a) $2x^2-2x$ (b) $6x^2y+2y^3$ (c) a^3-c^3-abc (d) $4x^4-5x^3-2x^2-x+2$

18(a) $6a^5b^6$ (b) $-24a^4b^8$ (c) $9a^6$ (d) $-8a^{12}b^3$

19 $-x^3+3x^2+7x-8$

20(a) $3a^2$ (b) $5c^4$ (c) a^2bc^6

21(a) $2x^5$ (b) $9xy^5$ (c) b^4 (d) $2a^3$

22 $-18x^{25}y^{22}$

Exercise **1B** (Page 5)

1(a) $3x-6$ (b) $2x-6$ (c) $-3x+6$ (d) $-2x+6$ (e) $-3x-6$ (f) $-2x-6$ (g) $-x+2$ (h) $-2+x$ (i) $-x-3$

2(a) $3x+3y$ (b) $-2p+2q$ (c) $4a+8b$ (d) x^2-7x (e) $-x^2+3x$ (f) $-a^2-4a$ (g) $5a+15b-10c$ (h) $-6x+9y-15z$ (i) $2x^2y-3xy^2$

3(a) $x+2$ (b) $7a-3$ (c) $2x-4$ (d) $4-3a$ (e) $2-x$ (f) $2c$ (g) $-x-y$ (h) $x+4$ (i) $-7a-2b$ (j) $-s-9t$ (k) x^2+17xy (l) $-8a-3b+5c$

4(a) x^2+5x+6 (b) $y^2+11y+28$ (c) $t^2+3t-18$ (d) x^2-2x-8 (e) t^2-4t+3 (f) $2a^2+13a+15$ (g) $3u^2-10u-8$ (h) $8p^2-2p-15$ (i) $2b^2-13b+21$ (j) $15a^2-a-2$ (k) $-c^2+9c-18$ (l) $2d^2+5d-12$ (m) $2xy-4x+3y-6$ (n) $5ab+4a-10b-8$ (o) $12-8m-9n+6mn$

6(a) $x^2+2xy+y^2$ (b) $x^2-2xy+y^2$ (c) x^2-y^2 (d) a^2+6a+9 (e) $b^2-8b+16$ (f) $c^2+10c+25$ (g) d^2-36 (h) $49-e^2$ (i) $64+16f+f^2$ (j) $81-18g+g^2$ (k) h^2-100 (l) $i^2+22i+121$ (m) $4a^2+4a+1$ (n) $4b^2-12b+9$ (o) $9c^2+12c+4$ (p) $4d^2+12de+9e^2$ (q) $4f^2-9g^2$ (r) $9h^2-4i^2$ (s) $25j^2+40j+16$ (t) $16k^2-40k\ell+25\ell^2$ (u) $16-25m^2$ (v) $25-30n+9n^2$ (w) $49p^2+56pq+16q^2$ (x) $64-48r+9r^2$

7(a) $-a^3+a^2+a$ (b) $-2x^4+4x^3+6x^2-2x$ (c) $6x^3y^2-15x^4y$ (d) $-2a^4b^4+4a^5b^2$

8(a) $11x-3$ (b) $-4b+8c-8$

9(a) $t^2+2+\frac{1}{t^2}$ (b) $t^2-2+\frac{1}{t^2}$ (c) $t^2-\frac{1}{t^2}$

10 $a^2-b^2-c^2+2bc$

11(a) a^2-4b^2 (b) $10-17x-20x^2$ (c) $16x^2+56x+49$ (d) $x^4-x^2y-12y^2$ (e) a^2-ac-b^2+bc (f) $27x^3+1$

12(a) $2ab-b^2$ (b) $2x+3$ (c) $18-6a$ (d) $4pq$ (e) x^2+2x-1 (f) a^2-2a-6

13(a) $10\,404$ (b) $998\,001$ (c) $39\,991$

Exercise **1C** (Page 8)

1(a) $2(x+4)$ **(b)** $3(2a-5)$ **(c)** $a(x-y)$
(d) $5a(4b-3c)$ **(e)** $x(x+3)$ **(f)** $p(p+2q)$
(g) $3a(a-2b)$ **(h)** $6x(2x+3)$ **(i)** $4c(5d-8)$
(j) $ab(a+b)$ **(k)** $2a^2(3+a)$ **(l)** $7x^2y(x-2y)$
2(a) $(p+q)(m+n)$ **(b)** $(x-y)(a+b)$ **(c)** $(x+3)(a+2)$
(d) $(a+b)(a+c)$ **(e)** $(z-1)(z^2+1)$ **(f)** $(a+b)(c-d)$
(g) $(p-q)(u-v)$ **(h)** $(x-3)(x-y)$ **(i)** $(p-q)(5-x)$
(j) $(2a-b)(x-y)$ **(k)** $(b+c)(a-1)$ **(l)** $(x+4)(x^2-3)$
(m) $(a-3)(a^2-2)$ **(n)** $(2t+5)(t^2-5)$
(o) $(x-3)(2x^2-a)$
3(a) $(a-1)(a+1)$ **(b)** $(b-2)(b+2)$ **(c)** $(c-3)(c+3)$
(d) $(d-10)(d+10)$ **(e)** $(5-y)(5+y)$
(f) $(1-n)(1+n)$ **(g)** $(7-x)(7+x)$
(h) $(12-p)(12+p)$ **(i)** $(2c-3)(2c+3)$
(j) $(3u-1)(3u+1)$ **(k)** $(5x-4)(5x+4)$
(l) $(1-7k)(1+7k)$ **(m)** $(x-2y)(x+2y)$
(n) $(3a-b)(3a+b)$ **(o)** $(5m-6n)(5m+6n)$
(p) $(9ab-8)(9ab+8)$
4(a) $(a+1)(a+2)$ **(b)** $(k+2)(k+3)$
(c) $(m+1)(m+6)$ **(d)** $(x+3)(x+5)$
(e) $(y+4)(y+5)$ **(f)** $(t+2)(t+10)$ **(g)** $(x-1)(x-3)$
(h) $(c-2)(c-5)$ **(i)** $(a-3)(a-4)$ **(j)** $(b-2)(b-6)$
(k) $(t+2)(t-1)$ **(l)** $(u-2)(u+1)$ **(m)** $(w-4)(w+2)$
(n) $(a+4)(a-2)$ **(o)** $(p-5)(p+3)$ **(p)** $(y+7)(y-4)$
(q) $(c-3)(c-9)$ **(r)** $(u-6)(u-7)$ **(s)** $(x-10)(x+9)$
(t) $(x+8)(x-5)$ **(u)** $(t-8)(t+4)$ **(v)** $(p+12)(p-3)$
(w) $(u-20)(u+4)$ **(x)** $(t+25)(t-2)$
5(a) $(3x+1)(x+1)$ **(b)** $(2x+1)(x+2)$
(c) $(3x+1)(x+5)$ **(d)** $(3x+2)(x+2)$
(e) $(2x-1)(x-1)$ **(f)** $(5x-3)(x-2)$
(g) $(5x-6)(x-1)$ **(h)** $(3x-1)(2x-3)$
(i) $(2x-3)(x+1)$ **(j)** $(2x+5)(x-1)$
(k) $(3x+5)(x-1)$ **(l)** $(3x-1)(x+5)$
(m) $(2x+3)(x-5)$ **(n)** $(2x-5)(x+3)$
(o) $(6x-1)(x+3)$ **(p)** $(2x-3)(3x+1)$
(q) $(3x-2)(2x+3)$ **(r)** $(5x+3)(x+4)$
(s) $(5x-6)(x+2)$ **(t)** $(5x-4)(x-3)$
(u) $(5x+4)(x-3)$ **(v)** $(5x-2)(x+6)$
(w) $(3x-4)(3x+2)$ **(x)** $(3x-5)(x+6)$
6(a) $(a-5)(a+5)$ **(b)** $b(b-25)$ **(c)** $(c-5)(c-20)$
(d) $(2d+5)(d+10)$ **(e)** $(e+5)(e^2+5)$
(f) $(4-f)(4+f)$ **(g)** $g^2(16-g)$
(h) $(h+8)^2$ **(i)** $(i-18)(i+2)$
(j) $(j+4)(5j-4)$ **(k)** $(2k+1)(2k-9)$
(l) $(k-8)(2k^2-3)$ **(m)** $(2a+b)(a-2)$
(n) $3m^2n^4(2m+3n)$ **(o)** $(7p-11q)(7p+11q)$
(p) $(t-4)(t-10)$ **(q)** $(3t-10)(t+4)$
(r) $(5t+4)(t+10)$ **(s)** $(5t+8)(t+5)$
(t) $5t(t^2+2t+3)$ **(u)** $(u+18)(u-3)$
(v) $(3x-2y)(x^2-5)$ **(w)** $(1-6a)(1+6a)$
(x) $(2a-3)^2$
7(a) $3(a-2)(a+2)$ **(b)** $(x-y)(x+y)(x^2+y^2)$
(c) $x(x-1)(x+1)$ **(d)** $5(x+2)(x-3)$
(e) $y(5-y)(5+y)$ **(f)** $(2-a)(2+a)(4+a^2)$
(g) $2(2x-3)(x+5)$ **(h)** $a(a+1)(a^2+1)$
(i) $(c+1)(c-1)(c+9)$ **(j)** $x(x-1)(x-7)$
(k) $(x-2)(x+2)(x^2+1)$ **(l)** $(x-1)(x+1)(a-2)$

Exercise **1D** (Page 11)

1(a) 1 **(b)** 2 **(c)** $\frac{1}{2}$ **(d)** $\frac{1}{a}$ **(e)** $\frac{x}{3y}$ **(f)** $\frac{3}{a}$
2(a) 1 **(b)** $\frac{1}{2}$ **(c)** $3x$ **(d)** $\frac{b}{2}$ **(e)** $\frac{3}{2x}$ **(f)** $\frac{1}{2a}$ **(g)** $\frac{4}{b}$
(h) 6
3(a) $\frac{7x}{10}$ **(b)** $\frac{a}{6}$ **(c)** $\frac{3x-2y}{24}$ **(d)** $\frac{13a}{6}$ **(e)** $\frac{b}{15}$ **(f)** $-\frac{xy}{20}$
(g) $\frac{3}{2x}$ **(h)** $\frac{25}{12x}$ **(i)** $\frac{b-a}{ab}$ **(j)** $\frac{x^2+1}{x}$ **(k)** $\frac{a^2+b}{a}$ **(l)** $\frac{x-1}{x^2}$
4(a) $\frac{5x+7}{6}$ **(b)** $\frac{18x+11}{20}$ **(c)** $\frac{x+1}{4}$ **(d)** $\frac{x}{6}$ **(e)** $\frac{2x+17}{20}$
(f) $\frac{2x-3}{6}$ **(g)** $\frac{9x+26}{12}$ **(h)** $\frac{12x+3}{5}$ **(i)** $\frac{2x-16}{15x}$
5(a) 2 **(b)** $\frac{3}{2}$ **(c)** $\frac{x}{3}$ **(d)** $\frac{1}{x+y}$ **(e)** $\frac{3}{2b}$ **(f)** $\frac{x}{x-2}$ **(g)** $\frac{a+3}{a+4}$
(h) $\frac{x+1}{x-1}$ **(i)** $\frac{x+5}{x+4}$ **(j)** $\frac{c+d}{a}$ **(k)** $\frac{y-5}{2y+1}$ **(l)** $\frac{3a+2b}{3x+2y}$
6(a) $\frac{2x+1}{x(x+1)}$ **(b)** $\frac{1}{x(x+1)}$ **(c)** $\frac{2x}{(x+1)(x-1)}$
(d) $\frac{5x-13}{(x-2)(x-3)}$ **(e)** $\frac{x-5}{(x+1)(x-1)}$ **(f)** $\frac{10}{(x+3)(x-2)}$
(g) $\frac{x^2+y^2}{x^2-y^2}$ **(h)** $\frac{ax-bx}{(x+a)(x+b)}$ **(i)** $\frac{2x}{x^2-1}$
7(a) $6a^2$ **(b)** $\frac{5c}{2a}$ **(c)** $\frac{x^2}{yz}$ **(d)** $\frac{1}{2a^2}$
8(a) $\frac{3x}{2(x-1)}$ **(b)** a **(c)** $\frac{c+2}{c+4}$ **(d)** x **(e)** $\frac{3x-1}{a+b}$ **(f)** $\frac{x-7}{3(x+3)}$
9(a) $\frac{2}{x^2-1}$ **(b)** $\frac{2x}{(x-2)^2(x+2)}$ **(c)** $\frac{3x}{x^2-y^2}$
(d) $\frac{x+1}{(x-2)(x+3)(x+4)}$ **(e)** $\frac{bx}{a(a-b)(a+b)}$
(f) $\frac{x}{(x-1)(x-2)(x-3)}$
10(a) -1 **(b)** $-u-v$ **(c)** $3-x$ **(d)** $\frac{2}{a-b}$ **(e)** 1
(f) $\frac{-1}{2x+y}$
11(a) $\frac{1}{3}$ **(b)** $\frac{7}{13}$ **(c)** $\frac{3}{11}$ **(d)** $\frac{1}{5}$ **(e)** $\frac{1}{x+2}$ **(f)** $\frac{t^2-1}{t^2+1}$
(g) $\frac{ab}{a+b}$ **(h)** $\frac{x^2+y^2}{x^2-y^2}$ **(i)** $\frac{x^2}{2x+1}$ **(j)** $\frac{x-1}{x-3}$

Exercise **1E** (Page 14)

2(a) $(x+y)(x^2-xy+y^2)$ **(b)** $(a-b)(a^2+ab+b^2)$
(c) $(y+1)(y^2-y+1)$ **(d)** $(k-1)(k^2+k+1)$
(e) $(a+2)(a^2-2a+4)$ **(f)** $(b-2)(b^2+2b+4)$
(g) $(3-t)(9+3t+t^2)$ **(h)** $(3+u)(9-3u+u^2)$
(i) $(x+4)(x^2-4x+16)$ **(j)** $(y-4)(y^2+4y+16)$
(k) $(5+a)(25-5a+a^2)$ **(l)** $(5-b)(25+5b+b^2)$
3(a) $(2p+1)(4p^2-2p+1)$ **(b)** $(2q-1)(4q^2+2q+1)$
(c) $(u-4v)(u^2+4uv+16v^2)$
(d) $(t+4u)(t^2-4tu+16u^2)$
(e) $(3c+2)(9c^2-6c+4)$ **(f)** $(3d-2)(9d^2+6d+4)$
(g) $(4m-5n)(16m^2+20mn+25n^2)$

(h) $(4p+5q)(16p^2-20pq+25q^2)$
(i) $(6e-7f)(36e^2+42ef+49f^2)$
(j) $(6g+7h)(36g^2-42gh+49h^2)$
(k) $(abc+10)(a^2b^2c^2-10abc+100)$
(l) $(9x-11y)(81x^2+99xy+121y^2)$
4(a) $5(x-1)(x^2+x+1)$ **(b)** $2(x+2)(x^2-2x+4)$
(c) $a(a-b)(a^2+ab+b^2)$ **(d)** $3(2t+3)(4t^2-6t+9)$
(e) $y(x-5)(x^2+5x+25)$
(f) $2(5p-6q)(25p^2+30pq+36q^2)$
(g) $x(3x+10y)(9x^2-30xy+100y^2)$
(h) $5(xy-1)(x^2y^2+xy+1)$
(i) $x^3(x+y)(x^2-xy+y^2)$
5(a) $\frac{x^2+x+1}{x+1}$ **(b)** $\frac{a-5}{a^2-2a+4}$ **(c)** $\frac{a^2-a+1}{2a^2}$ **(d)** $\frac{1}{x}$
6(a) $\frac{12a+12}{a^3-8}$ **(b)** $\frac{x^2}{x^3-1}$ **(c)** $\frac{x^2-3x+8}{(x-4)(x+2)(x^2-2x+4)}$
(d) $\frac{3a^2-ab}{a^3+b^3}$

Exercise **1F** (Page 15)

1(a) $a=15$ **(b)** $t=-2$ **(c)** $c=-7$ **(d)** $n=-18$
(e) $x=10$ **(f)** $x=\frac{2}{3}$ **(g)** $a=-5$ **(h)** $x=4$
(i) $x=-1$ **(j)** $y=50$ **(k)** $t=0$ **(l)** $x=-16$
2(a) $x=3$ **(b)** $p=0$ **(c)** $a=8$ **(d)** $w=-1$
(e) $x=9$ **(f)** $x=-5$ **(g)** $x=-4$ **(h)** $x=-7$
(i) $t=-30$ **(j)** $a=-\frac{7}{5}$ **(k)** $y=-\frac{16}{7}$ **(l)** $u=48$
3(a) $n=4$ **(b)** $b=-1$ **(c)** $x=4$ **(d)** $x=-11$
(e) $a=-\frac{1}{2}$ **(f)** $y=2$ **(g)** $x=\frac{7}{9}$ **(h)** $x=-\frac{3}{5}$
(i) $x=\frac{23}{6}$ **(j)** $x=-\frac{2}{5}$ **(k)** There are no solutions.
(l) All real numbers are solutions.
4(a) $x=4$ **(b)** $a=8$ **(c)** $y=16$ **(d)** $x=\frac{1}{3}$
(e) $a=\frac{2}{5}$ **(f)** $y=\frac{3}{2}$ **(g)** $x=-8$ **(h)** $a=\frac{3}{2}$
(i) $x=\frac{1}{4}$ **(j)** $a=-\frac{5}{4}$ **(k)** $t=\frac{3}{5}$ **(l)** $c=\frac{9}{2}$
(m) $a=-1$ **(n)** $x=\frac{1}{5}$ **(o)** $x=\frac{7}{17}$ **(p)** $t=-\frac{26}{27}$
5(a) $x=\frac{19}{6}$ **(b)** $x=\frac{3}{14}$ **(c)** $x=-1$ **(d)** $x=\frac{17}{6}$
6(a) $a=3$ **(b)** $s=16$ **(c)** $v=\frac{2}{3}$ **(d)** $\ell=21$
(e) $C=35$ **(f)** $c=-\frac{14}{5}$
7(a) 6 **(b)** -4 **(c)** 17 **(d)** 65 cents
8(a) $x=15$ **(b)** $y=\frac{2}{3}$ **(c)** $a=-15$ **(d)** $x=\frac{9}{2}$
(e) $x=6$ **(f)** $x=\frac{1}{6}$ **(g)** $x=\frac{1}{2}$ **(h)** $x=20$
(i) $x=-\frac{23}{2}$ **(j)** $x=-\frac{7}{3}$ **(k)** $x=\frac{5}{6}$ **(l)** $a=-11$
9(a) $b=\frac{a+d}{c}$ **(b)** $n=\frac{t-a+d}{d}$ **(c)** $r=\frac{p-qt}{t}$
(d) $v=\frac{3}{u-1}$
10(a) $x=2$ **(b)** $x=0$ **(c)** $x=\frac{34}{57}$ **(d)** $x=-\frac{7}{3}$
(e) $x=-\frac{5}{2}$ **(f)** $x=-\frac{43}{69}$
11(a) 20 **(b)** 80 litres **(c)** 16 **(d)** 30 km/h
12(a) $a=-\frac{2b}{3}$ **(b)** $g=\frac{2fh}{5f-h}$ **(c)** $y=\frac{2x}{1-x}$
(d) $b=\frac{4a+5}{a-1}$ **(e)** $d=\frac{5c-7}{3c+2}$ **(f)** $v=\frac{1+u-w-uw}{1-u}$

Exercise **1G** (Page 18)

1(a) $x=3$ or -3 **(b)** $y=5$ or -5 **(c)** $a=2$ or -2
(d) $c=6$ or -6 **(e)** $t=1$ or -1 **(f)** $x=\frac{3}{2}$ or $-\frac{3}{2}$
(g) $x=\frac{1}{2}$ or $-\frac{1}{2}$ **(h)** $a=2\frac{2}{3}$ or $-2\frac{2}{3}$
(i) $y=\frac{4}{5}$ or $-\frac{4}{5}$
2(a) $x=0$ or 5 **(b)** $y=0$ or -1 **(c)** $c=0$ or -2
(d) $k=0$ or 7 **(e)** $t=0$ or 1 **(f)** $a=0$ or 3
(g) $b=0$ or $\frac{1}{2}$ **(h)** $u=0$ or $-\frac{1}{3}$ **(i)** $x=-\frac{3}{4}$ or 0
(j) $a=0$ or $\frac{5}{2}$ **(k)** $y=0$ or $\frac{2}{3}$ **(l)** $h=0$ or $-\frac{12}{5}$
3(a) $x=-3$ or -1 **(b)** $x=1$ or 2 **(c)** $x=-4$ or -2
(d) $a=2$ or 5 **(e)** $t=-2$ or 6 **(f)** $c=5$
(g) $n=1$ or 8 **(h)** $p=-5$ or 3 **(i)** $a=-2$ or 12
(j) $y=-5$ or 1 **(k)** $p=-2$ or 3 **(l)** $a=-11$ or 12
(m) $c=3$ or 6 **(n)** $t=-2$ or 10 **(o)** $u=-8$ or 7
(p) $k=-4$ or 6 **(q)** $h=-25$ or -2
(r) $\alpha=-22$ or 2
4(a) $x=-\frac{1}{2}$ or -1 **(b)** $a=\frac{1}{3}$ or 2 **(c)** $y=\frac{1}{4}$ or 1
(d) $x=-5$ or $-\frac{1}{2}$ **(e)** $x=-1\frac{1}{2}$ or 1
(f) $n=-1$ or $1\frac{2}{3}$ **(g)** $b=-\frac{2}{3}$ or 2 **(h)** $a=-5$ or $1\frac{1}{2}$
(i) $y=-2\frac{1}{2}$ or 3 **(j)** $y=-4$ or $\frac{2}{3}$ **(k)** $x=\frac{1}{5}$ or 5
(l) $t=\frac{3}{4}$ or 3 **(m)** $t=-\frac{2}{5}$ or 3 **(n)** $u=-\frac{4}{5}$ or $\frac{1}{2}$
(o) $x=\frac{1}{5}$ **(p)** $x=-\frac{2}{3}$ or $-\frac{3}{2}$ **(q)** $b=-\frac{3}{2}$ or $-\frac{1}{6}$
(r) $k=-\frac{8}{3}$ or $\frac{1}{2}$
5(a) $x=\frac{1+\sqrt{5}}{2}$ or $\frac{1-\sqrt{5}}{2}$, $x \doteqdot 1{\cdot}618$ or $-0{\cdot}6180$
(b) $x=\frac{-1+\sqrt{13}}{2}$ or $\frac{-1-\sqrt{13}}{2}$, $x \doteqdot 1{\cdot}303$ or $-2{\cdot}303$
(c) $a=3$ or 4
(d) $u=-1+\sqrt{3}$ or $-1-\sqrt{3}$, $u \doteqdot 0{\cdot}7321$ or $-2{\cdot}732$
(e) $c=3+\sqrt{7}$ or $3-\sqrt{7}$, $c \doteqdot 5{\cdot}646$ or $0{\cdot}3542$
(f) $x=-\frac{1}{2}$
(g) $a=\frac{2+\sqrt{2}}{2}$ or $\frac{2-\sqrt{2}}{2}$, $a \doteqdot 1{\cdot}707$ or $0{\cdot}2929$
(h) $x=-3$ or $\frac{2}{5}$
(i) $b=\frac{-3+\sqrt{17}}{4}$ or $\frac{-3-\sqrt{17}}{4}$, $b \doteqdot 0{\cdot}2808$ or $-1{\cdot}781$
(j) $c=\frac{2+\sqrt{13}}{3}$ or $\frac{2-\sqrt{13}}{3}$, $c \doteqdot 1{\cdot}869$ or $-0{\cdot}5352$
(k) $t=\frac{1+\sqrt{5}}{4}$ or $\frac{1-\sqrt{5}}{4}$, $t \doteqdot 0{\cdot}8090$ or $-0{\cdot}3090$
(l) no solutions
6(a) $x=-1$ or 2 **(b)** $a=2$ or 5 **(c)** $y=\frac{1}{2}$ or 4
(d) $b=-\frac{2}{5}$ or $\frac{2}{3}$ **(e)** $k=-1$ or 3 **(f)** $u=\frac{4}{3}$ or 4
7(a) $x=1+\sqrt{2}$ or $1-\sqrt{2}$ **(b)** $a=2+\sqrt{3}$ or $2-\sqrt{3}$
(c) $a=1+\sqrt{5}$ or $1-\sqrt{5}$ **(d)** $m=\frac{2+\sqrt{14}}{5}$ or $\frac{2-\sqrt{14}}{5}$
(e) $y=1+\sqrt{6}$ or $1-\sqrt{6}$ **(f)** $k=\frac{-5+\sqrt{73}}{4}$ or $\frac{-5-\sqrt{73}}{4}$
8(a) $p=\frac{1}{2}$ or 1 **(b)** $x=-3$ or 5 **(c)** $n=5$
9(a) 7 **(b)** 6 and 9 **(c)** $x=15$
10(a) $a=-\frac{7}{3}$ or 3 **(b)** $k=-4$ or 15
(c) $t=2\sqrt{3}$ or $-\sqrt{3}$ **(d)** $m=\frac{1+\sqrt{2}}{3}$ or $\frac{1-\sqrt{2}}{3}$
11(a) 4 cm **(b)** $\frac{2}{5}$ or $\frac{-9}{-6}$ **(c)** 3 cm
(d) 2 hours, 4 hours **(e)** 55 km/h and 60 km/h

Exercise **1H** (Page 22)

1(a) $x = 3,\ y = 3$ **(b)** $x = 2,\ y = 4$
(c) $x = 2,\ y = 1$ **(d)** $a = -3,\ b = -2$
(e) $p = 3,\ q = -1$ **(f)** $u = 1,\ v = -2$
2(a) $x = 3,\ y = 2$ **(b)** $x = 1,\ y = -2$
(c) $x = 4,\ y = 1$ **(d)** $a = -1,\ b = 3$
(e) $c = 2,\ d = 2$ **(f)** $p = -2,\ q = -3$
3(a) $x = 2,\ y = 4$ **(b)** $x = -1,\ y = 3$
(c) $x = 2,\ y = 2$ **(d)** $x = 9,\ y = 1$
(e) $x = 3,\ y = 4$ **(f)** $x = 4,\ y = -1$
(g) $x = 5,\ y = 3\frac{3}{5}$ **(h)** $x = 13,\ y = 7$
4(a) $x = -1,\ y = 3$ **(b)** $x = 5,\ y = 2$
(c) $x = -4,\ y = 3$ **(d)** $x = 2,\ y = -6$
(e) $x = 1,\ y = 2$ **(f)** $x = 16,\ y = -24$
(g) $x = 1,\ y = 6$ **(h)** $x = 5,\ y = -2$
(i) $x = 5,\ y = 6$ **(j)** $x = 7,\ y = 5$
(k) $x = \frac{1}{2},\ y = \frac{3}{2}$ **(l)** $x = 5,\ y = 8$
5(a) $x = 1$ & $y = 1$ or $x = -2$ & $y = 4$
(b) $x = 2$ & $y = 1$ or $x = 4$ & $y = 5$
(c) $x = 0$ & $y = 0$ or $x = 1$ & $y = 3$
(d) $x = -2$ & $y = -7$ or $x = 3$ & $y = -2$
(e) $x = -3$ & $y = -5$ or $x = 5$ & $y = 3$
(f) $x = 1$ & $y = 6$ or $x = 2$ & $y = 3$
(g) $x = 5$ & $y = 3$ or $x = 5$ & $y = -3$
or $x = -5$ & $y = 3$ or $x = -5$ & $y = -3$
(h) $x = 9$ & $y = 6$ or $x = 9$ & $y = -6$
or $x = -9$ & $y = 6$ or $x = -9$ & $y = -6$
6(a) 53 and 37
(b) The pen cost 60 c, the pencil cost 15 c.
(c) Each apple cost 40 c, each orange cost 60 c.
(d) 44 adults, 22 children
(e) The man is 36, the son is 12.
(f) 189 for, 168 against
7(a) $x = 12$ & $y = 20$ **(b)** $x = 3$ & $y = 2$
8(a) $x = 6$ & $y = 3$ & $z = 1$
(b) $x = 2$ & $y = -1$ & $z = 3$
(c) $a = 3$ & $b = -2$ & $c = 2$
(d) $p = -1$ & $q = 2$ & $r = 5$
(e) $x = 5$ & $y = -3$ & $z = -4$
(f) $u = -2$ & $v = 6$ & $w = 1$
9(a) $x = 5$ & $y = 10$ or $x = 10$ & $y = 5$
(b) $x = -8$ & $y = -11$ or $x = 11$ & $y = 8$
(c) $x = \frac{1}{2}$ & $y = 4$ or $x = 2$ & $y = 1$
(d) $x = 4$ & $y = 5$ or $x = 5$ & $y = 4$
(e) $x = 1$ & $y = 2$ or $x = \frac{3}{2}$ & $y = \frac{7}{4}$
(f) $x = 2$ & $y = 5$ or $x = \frac{10}{3}$ & $y = 3$
10(a) $\frac{2}{15}$ **(b)** 9 \$20 notes, 14 \$10 notes **(c)** 72
(d) 5 km/h, 3 km/h

Exercise **1I** (Page 24)

1(a) 1 **(b)** 9 **(c)** 25 **(d)** 81 **(e)** $\frac{9}{4}$ **(f)** $\frac{1}{4}$ **(g)** $\frac{25}{4}$
(h) $\frac{81}{4}$
2(a) $(x+2)^2$ **(b)** $(y+1)^2$ **(c)** $(p+7)^2$ **(d)** $(m-6)^2$
(e) $(t-8)^2$ **(f)** $(x+10)^2$ **(g)** $(u-20)^2$ **(h)** $(a-12)^2$
3(a) $x^2+6x+9 = (x+3)^2$ **(b)** $y^2+8y+16 = (y+4)^2$
(c) $a^2 - 20a + 100 = (a - 10)^2$
(d) $b^2 - 100b + 2500 = (b - 50)^2$
(e) $u^2+u+\frac{1}{4} = (u+\frac{1}{2})^2$ **(f)** $t^2-7t+\frac{49}{4} = (t-\frac{7}{2})^2$
(g) $m^2 + 50m + 625 = (m + 25)^2$
(h) $c^2 - 13c + \frac{169}{4} = (c - \frac{13}{2})^2$
4(a) $x = -1$ or 3 **(b)** $x = 0$ or 6 **(c)** $a = -4$ or -2
(d) $y = -5$ or 2 **(e)** $b = -2$ or 7 **(f)** $x = -2 + \sqrt{3}$
or $-2 - \sqrt{3}$ **(g)** $x = 5 + \sqrt{5}$ or $5 - \sqrt{5}$
(h) no solution for y **(i)** $a = \frac{-7+\sqrt{21}}{2}$ or $\frac{-7-\sqrt{21}}{2}$
5(a) $x = 2$ or 3 **(b)** $x = \frac{2+\sqrt{6}}{2}$ or $\frac{2-\sqrt{6}}{2}$
(c) no solution for x **(d)** $x = \frac{-4+\sqrt{10}}{2}$ or $\frac{-4-\sqrt{10}}{2}$
(e) $x = -\frac{3}{2}$ or $\frac{1}{2}$ **(f)** $x = \frac{1+\sqrt{5}}{4}$ or $\frac{1-\sqrt{5}}{4}$
(g) $x = -\frac{1}{3}$ or 3 **(h)** $x = -3$ or $\frac{5}{2}$
(i) $x = \frac{5+\sqrt{11}}{2}$ or $\frac{5-\sqrt{11}}{2}$
6(b) $a = 3,\ b = 4$ and $c = 25$
(d) $A = -5,\ B = 6$ and $C = 8$

Review Exercise **1J** (Page 25)

1(a) $-6y$ **(b)** $-10y$ **(c)** $-16y^2$ **(d)** -4
2(a) $-3a^2$ **(b)** $-a^2$ **(c)** $2a^4$ **(d)** 2
3(a) $2t-1$ **(b)** $4p+3q$ **(c)** $x-2y$ **(d)** $5a^2-3a-18$
4(a) $-18k^9$ **(b)** $-2k^3$ **(c)** $36k^{12}$ **(d)** $27k^9$
5(a) $14x-3$ **(b)** $-4a+2b$ **(c)** $-2a$ **(d)** $-6x^3-10x^2$
(e) $2n^2 + 11n - 21$ **(f)** $r^2 + 6r + 9$ **(g)** $y^2 - 25$
(h) $6x^2 - 19x + 15$ **(i)** $t^2 - 16t + 64$ **(j)** $4c^2 - 49$
(k) $16p^2 + 8p + 1$ **(l)** $9u^2 - 12u + 4$
6(a) $18(a + 2)$ **(b)** $4(5b - 9)$ **(c)** $9c(c + 4)$
(d) $(d - 6)(d + 6)$ **(e)** $(e + 4)(e + 9)$ **(f)** $(f - 6)^2$
(g) $(6 - 5g)(6 + 5g)$ **(h)** $(h - 12)(h + 3)$
(i) $(i + 9)(i - 4)$ **(j)** $(2j + 3)(j + 4)$
(k) $(3k + 2)(k - 3)$ **(l)** $(5\ell - 4)(\ell - 2)$
(m) $(2m - 3)(2m + 5)$ **(n)** $(n + 2)(n^2 - 2n + 4)$
(o) $(p - 3)(p^2 + 3p + 9)$ **(p)** $(p + 9)(p^2 + 4)$
(q) $(q - r)(t - 5)$ **(r)** $(u^2 + v)(w - x)$
7(a) $\frac{3x}{4}$ **(b)** $\frac{x}{4}$ **(c)** $\frac{x^2}{8}$ **(d)** 2 **(e)** $\frac{13a}{6b}$ **(f)** $\frac{5a}{6b}$ **(g)** $\frac{a^2}{b^2}$
(h) $\frac{9}{4}$ **(i)** $\frac{x^2+y^2}{xy}$ **(j)** $\frac{x^2-y^2}{xy}$ **(k)** 1 **(l)** $\frac{x^2}{y^2}$
8(a) $\frac{8x-13}{15}$ **(b)** $\frac{8x-13}{(x+4)(x-5)}$ **(c)** $\frac{3x+13}{10}$ **(d)** $\frac{-3x-13}{(x+1)(x-4)}$
(e) $\frac{x-3}{4}$ **(f)** $\frac{-2x+6}{x(x+3)}$
9(a) $\frac{3}{5}$ **(b)** $\frac{2}{x+y}$ **(c)** $\frac{x+3}{x-4}$ **(d)** $\frac{x+1}{x^2+1}$ **(e)** $\frac{1}{a+b}$ **(f)** $\frac{x-7}{3x-2}$
(g) $\frac{x^2+2x+4}{x+2}$ **(h)** $\frac{a-3}{a^2-a+1}$
10(a) $x = 4$ **(b)** $x = \frac{2}{3}$ **(c)** $x = 46$ **(d)** $x = 36$

(e) $a = 3$ **(f)** $a = 10$ **(g)** $a = -17$ **(h)** $a = -42$
11(a) $a = -7$ or 7 **(b)** $b = -7$ or 0
(c) $c = -6$ or -1 **(d)** $d = -7$ or 1 **(e)** $e = 2$ or 3
(f) $f = -\frac{3}{2}$ or 2 **(g)** $g = \frac{1}{2}$ or 6 **(h)** $h = -2$ or $\frac{4}{3}$
12(a) $x = 2 + \sqrt{3}$ or $2 - \sqrt{3}$
(b) $y = \frac{-3+\sqrt{21}}{2}$ or $\frac{-3-\sqrt{21}}{2}$
(c) $y = -3 + \sqrt{5}$ or $-3 - \sqrt{5}$
(d) $y = \frac{1+\sqrt{7}}{3}$ or $\frac{1-\sqrt{7}}{3}$
(e) $y = \frac{-5+\sqrt{65}}{4}$ or $\frac{-5-\sqrt{65}}{4}$
(f) $y = \frac{3+\sqrt{13}}{4}$ or $\frac{3-\sqrt{13}}{4}$
13(a) $x = -2 + \sqrt{10}$ or $-2 - \sqrt{10}$
(b) $x = 3 + \sqrt{6}$ or $3 - \sqrt{6}$
(c) $x = 1 + \sqrt{13}$ or $1 - \sqrt{13}$
(d) $x = -5 + 3\sqrt{2}$ or $-5 - 3\sqrt{2}$

Chapter Two

Exercise **2A** (Page 29)

1(a) $2, 3, 5, 7, 11, 13, 17, 19$ **(b)** $23, 29, 31, 37, 41, 43, 47$
2(a) 2×5 **(b)** 3×7 **(c)** $2^2 \times 3$ **(d)** 2×3^2 **(e)** $2^2 \times 7$
(f) $3^2 \times 5$ **(g)** 3^3 **(h)** $2^3 \times 5$
3(a) 2 **(b)** 3 **(c)** 7 **(d)** 4 **(e)** 9 **(f)** 6 **(g)** 8 **(h)** 12
4(a) 10 **(b)** 6 **(c)** 12 **(d)** 28 **(e)** 24 **(f)** 36 **(g)** 50
(h) 30
5(a) $2 \times 3 \times 5$ **(b)** $2^2 \times 3^2$ **(c)** 3×13 **(d)** $2^4 \times 3$
(e) $2^2 \times 3^3$ **(f)** 2^7 **(g)** $2 \times 7 \times 11$ **(h)** $2^3 \times 17$
6(a) $\frac{1}{3}$ **(b)** $\frac{4}{5}$ **(c)** $\frac{2}{3}$ **(d)** $\frac{3}{4}$ **(e)** $\frac{2}{5}$ **(f)** $\frac{7}{15}$ **(g)** $\frac{4}{7}$
(h) $\frac{5}{6}$ **(i)** $\frac{3}{5}$ **(j)** $\frac{3}{4}$
7(a) $\frac{3}{4}$ **(b)** $\frac{7}{10}$ **(c)** $\frac{5}{6}$ **(d)** $\frac{4}{15}$ **(e)** $\frac{5}{18}$ **(f)** $\frac{1}{24}$ **(g)** $\frac{5}{6}$
(h) $\frac{1}{75}$
8(a) 5 **(b)** 8 **(c)** $\frac{1}{10}$ **(d)** $\frac{1}{7}$ **(e)** $\frac{1}{4}$ **(f)** 6 **(g)** $\frac{1}{4}$
(h) $\frac{2}{3}$ **(i)** 4 **(j)** $\frac{1}{4}$
9(a) $\frac{14}{15}$ **(b)** $\frac{5}{11}$ **(c)** $\frac{1}{2000}$
10(a) $\frac{1}{5}$ **(b)** $\frac{7}{144}$ **(c)** 1
11(a) $2^6 \times 3^2$, 24 **(b)** $5^2 \times 7^2$, 35 **(c)** $2^6 \times 5^6$, 1000

Exercise **2B** (Page 31)

1(a) $\frac{3}{10}$ **(b)** $\frac{4}{5}$ **(c)** $\frac{3}{4}$ **(d)** $\frac{1}{20}$
2(a) $0{\cdot}6$ **(b)** $0{\cdot}27$ **(c)** $0{\cdot}09$ **(d)** $0{\cdot}165$
3(a) 25% **(b)** 40% **(c)** 24% **(d)** 65%
4(a) 32% **(b)** 9% **(c)** $22{\cdot}5\%$ **(d)** 150%
5(a) $0{\cdot}5$ **(b)** $0{\cdot}2$ **(c)** $0{\cdot}6$ **(d)** $0{\cdot}75$ **(e)** $0{\cdot}04$ **(f)** $0{\cdot}35$
(g) $0{\cdot}125$ **(h)** $0{\cdot}625$
6(a) $0{\cdot}\dot{3}$ **(b)** $0{\cdot}\dot{6}$ **(c)** $0{\cdot}\dot{1}$ **(d)** $0{\cdot}\dot{5}$ **(e)** $0{\cdot}\dot{2}\dot{7}$ **(f)** $0{\cdot}\dot{0}\dot{9}$
(g) $0{\cdot}1\dot{6}$ **(h)** $0{\cdot}8\dot{3}$
7(a) $\frac{2}{5}$ **(b)** $\frac{1}{4}$ **(c)** $\frac{3}{20}$ **(d)** $\frac{4}{25}$ **(e)** $\frac{39}{50}$ **(f)** $\frac{1}{200}$ **(g)** $\frac{3}{8}$
(h) $\frac{33}{125}$
8(a) $\frac{2}{9}$ **(b)** $\frac{7}{9}$ **(c)** $\frac{4}{9}$ **(d)** $\frac{65}{99}$ **(e)** $\frac{6}{11}$ **(f)** $\frac{28}{33}$ **(g)** $\frac{2}{33}$
(h) $\frac{5}{37}$ **(i)** $\frac{254}{333}$ **(j)** $\frac{11}{333}$
9(a) $60\,\text{c}$ **(b)** $15\,\text{kg}$ **(c)** $\$7800$ **(d)** $72\,\text{min}$ or $1\frac{1}{5}\,\text{h}$
10(a) $0{\cdot}132$ **(b)** $0{\cdot}025$ **(c)** $0{\cdot}3125$ **(d)** $0{\cdot}3375$
(e) $0{\cdot}58\dot{3}$ **(f)** $1{\cdot}\dot{8}\dot{1}$ **(g)** $0{\cdot}1\dot{3}$ **(h)** $0{\cdot}2\dot{3}\dot{6}$
11(a) $1\frac{2}{3}$ **(b)** $3\frac{7}{33}$ **(c)** $2\frac{47}{111}$ **(d)** $1\frac{2}{27}$ **(e)** $\frac{7}{30}$ **(f)** $\frac{7}{45}$
(g) $\frac{316}{495}$ **(h)** $\frac{19}{55}$
12(a) $\$800$ **(b)** $\$160$ **(c)** $\$120$
13 $\frac{1}{7} = 0{\cdot}\dot{1}4285\dot{7}$, $\frac{2}{7} = 0{\cdot}\dot{2}8571\dot{4}$, etc. The digits of each cycle are in the same order but start at a different place in the cycle.
14(a) $\frac{1}{11} = 0{\cdot}\dot{0}\dot{9}$, $\frac{2}{11} = 0{\cdot}\dot{1}\dot{8}$, $\ldots$, $\frac{5}{11} = 0{\cdot}\dot{4}\dot{5}$, $\frac{6}{11} = 0{\cdot}\dot{5}\dot{4}$, $\ldots$, $\frac{10}{11} = 0{\cdot}\dot{9}\dot{0}$. The first digit runs from 0 to 9, the second runs from 9 to 0.

(b) $\frac{1}{13} = 0{\cdot}\dot{0}7692\dot{3}$, $\frac{2}{13} = 0{\cdot}\dot{1}5384\dot{6}$, $\frac{3}{13} = 0{\cdot}\dot{2}3076\dot{9}$, $\frac{4}{13} = 0{\cdot}\dot{3}0769\dot{2}$, $\frac{5}{13} = 0{\cdot}\dot{3}8461\dot{5}$, $\frac{6}{13} = 0{\cdot}\dot{4}6153\dot{8}$, … $\frac{12}{13} = 0{\cdot}\dot{9}2307\dot{6}$. There are two groups. In six fractions, the digits 076923 cycle, starting at each of the six places. In the other six, the digits 153846 cycle, starting at each of the six places.
16(c) $3{\cdot}000\,300\,03 \neq 3$, showing that some fractions are not stored exactly.

Exercise 2C (Page 34)

1(a) $0{\cdot}3$ **(b)** $5{\cdot}7$ **(c)** $12{\cdot}8$ **(d)** $0{\cdot}1$ **(e)** $3{\cdot}0$ **(f)** $10{\cdot}0$
2(a) $0{\cdot}43$ **(b)** $5{\cdot}4$ **(c)** $5{\cdot}0$ **(d)** $0{\cdot}043$ **(e)** 430 **(f)** 4300
3(a) $3{\cdot}162$ **(b)** $6{\cdot}856$ **(c)** $0{\cdot}563$ **(d)** $0{\cdot}771$ **(e)** $3{\cdot}142$ **(f)** $9{\cdot}870$
4(a) $7{\cdot}62$ **(b)** $5{\cdot}10$ **(c)** 3840 **(d)** $538\,000$ **(e)** $0{\cdot}740$ **(f)** $0{\cdot}00806$
5(a) 1 **(b)** 2 **(c)** 3 **(d)** 2 **(e)** 4 **(f)** either 1, 2 or 3
6(a) rational, $\frac{-3}{1}$ **(b)** rational, $\frac{3}{2}$ **(c)** irrational **(d)** rational, $\frac{2}{1}$ **(e)** rational, $\frac{3}{1}$ **(f)** irrational **(g)** rational, $\frac{2}{3}$ **(h)** rational, $\frac{9}{20}$ **(i)** rational, $\frac{3}{25}$ **(j)** rational, $\frac{333}{1000}$ **(k)** rational, $\frac{1}{3}$ **(l)** rational, $\frac{22}{7}$ **(m)** irrational **(n)** rational, $3\frac{7}{50}$ **(o)** rational, $\frac{0}{1}$
7(a) $45{\cdot}186$ **(b)** $2{\cdot}233$ **(c)** $0{\cdot}054$ **(d)** $0{\cdot}931$ **(e)** $0{\cdot}842$ **(f)** $0{\cdot}111$
8(a) 10, rational **(b)** $\sqrt{41}$, irrational **(c)** 8, rational **(d)** $\sqrt{5}$, irrational **(e)** $\frac{13}{15}$, rational **(f)** 45, rational
9(a) $0{\cdot}3981$ **(b)** $0{\cdot}05263$ **(c)** $1{\cdot}425$ **(d)** $5{\cdot}138$ **(e)** $0{\cdot}1522$ **(f)** $25\,650$ **(g)** $5{\cdot}158$ **(h)** $0{\cdot}7891$ **(i)** $1{\cdot}388 \times 10^{14}$ **(j)** $1{\cdot}134$ **(k)** $0{\cdot}005892$ **(l)** $1{\cdot}173$
10(a) $9{\cdot}46 \times 10^{15}$ m **(b)** $2{\cdot}1 \times 10^{22}$ m **(c)** $4{\cdot}29 \times 10^{17}$ seconds **(d)** $1{\cdot}3 \times 10^{26}$ m
11(a) $1{\cdot}836 \times 10^{3}$ **(b)** 6×10^{26}
12(a) $5{\cdot}2 \times 10^{-46}\ \text{m}^3$ **(b)** $3 \times 10^{18}\ \text{kg/m}^2$ **(c)** 3×10^{15} times denser

Exercise 2D (Page 37)

1(a) 4 **(b)** 6 **(c)** 9 **(d)** 11 **(e)** 12 **(f)** 20 **(g)** 50 **(h)** 100
2(a) $2\sqrt{3}$ **(b)** $3\sqrt{2}$ **(c)** $2\sqrt{5}$ **(d)** $3\sqrt{3}$ **(e)** $2\sqrt{7}$ **(f)** $2\sqrt{10}$ **(g)** $4\sqrt{2}$ **(h)** $3\sqrt{11}$ **(i)** $3\sqrt{6}$ **(j)** $10\sqrt{2}$ **(k)** $2\sqrt{15}$ **(l)** $5\sqrt{3}$ **(m)** $4\sqrt{5}$ **(n)** $7\sqrt{2}$ **(o)** $20\sqrt{2}$ **(p)** $10\sqrt{10}$
3(a) $2\sqrt{3}$ **(b)** $2\sqrt{7}$ **(c)** $\sqrt{5}$ **(d)** $-2\sqrt{2}$ **(e)** $2\sqrt{3}+3\sqrt{2}$ **(f)** $\sqrt{5}-2\sqrt{7}$ **(g)** $3\sqrt{6}-2\sqrt{3}$ **(h)** $-3\sqrt{2}-6\sqrt{5}$ **(i)** $-4\sqrt{10}+2\sqrt{5}$
4(a) $6\sqrt{2}$ **(b)** $10\sqrt{3}$ **(c)** $4\sqrt{6}$ **(d)** $8\sqrt{11}$ **(e)** $9\sqrt{5}$ **(f)** $12\sqrt{13}$ **(g)** $20\sqrt{3}$ **(h)** $8\sqrt{6}$
5(a) $\sqrt{20}$ **(b)** $\sqrt{50}$ **(c)** $\sqrt{128}$ **(d)** $\sqrt{108}$ **(e)** $\sqrt{125}$ **(f)** $\sqrt{112}$ **(g)** $\sqrt{68}$ **(h)** $\sqrt{490}$
6(a) $3\sqrt{2}$ **(b)** $\sqrt{3}$ **(c)** $2\sqrt{2}$ **(d)** $5\sqrt{6}$ **(e)** $\sqrt{5}$ **(f)** $2\sqrt{10}$ **(g)** $4\sqrt{3}$ **(h)** $2\sqrt{5}$ **(i)** $11\sqrt{2}$
7(a) $4\sqrt{6}+10\sqrt{3}$ **(b)** $2\sqrt{2}+6\sqrt{3}$ **(c)** $4\sqrt{7}-10\sqrt{35}$
8(a) 7 **(b)** 20 **(c)** 96

Exercise 2E (Page 39)

1(a) 3 **(b)** $\sqrt{6}$ **(c)** 7 **(d)** $\sqrt{30}$ **(e)** $6\sqrt{2}$ **(f)** $10\sqrt{5}$ **(g)** $6\sqrt{15}$ **(h)** $30\sqrt{14}$ **(i)** 12 **(j)** 63 **(k)** 30 **(l)** 240
2(a) $\sqrt{5}$ **(b)** $\sqrt{7}$ **(c)** $\sqrt{5}$ **(d)** 2 **(e)** $3\sqrt{2}$ **(f)** $\sqrt{3}$ **(g)** $2\sqrt{7}$ **(h)** $5\sqrt{5}$
3(a) $5+\sqrt{5}$ **(b)** $\sqrt{6}-\sqrt{2}$ **(c)** $2\sqrt{3}-3$ **(d)** $2\sqrt{10}-4$ **(e)** $7\sqrt{7}-14$ **(f)** $18-2\sqrt{30}$
4(a) $2\sqrt{3}$ **(b)** $5\sqrt{2}$ **(c)** $3\sqrt{5}$ **(d)** $4\sqrt{11}$ **(e)** 24 **(f)** $12\sqrt{10}$
5(a) $2\sqrt{5}-2$ **(b)** $3\sqrt{6}+3\sqrt{2}$ **(c)** $5\sqrt{3}+4\sqrt{5}$ **(d)** $4\sqrt{3}-2\sqrt{6}$ **(e)** $27\sqrt{3}-9\sqrt{7}$ **(f)** $21\sqrt{2}-42$
6(a) $\sqrt{6}-\sqrt{3}+\sqrt{2}-1$ **(b)** $\sqrt{35}+3\sqrt{5}-2\sqrt{7}-6$ **(c)** $\sqrt{15}+\sqrt{10}+\sqrt{6}+2$ **(d)** $8-3\sqrt{6}$ **(e)** $4+\sqrt{7}$ **(f)** $7\sqrt{3}-4\sqrt{6}$
7(a) 4 **(b)** 2 **(c)** 1 **(d)** 7 **(e)** 15 **(f)** 29
8(a) $4+2\sqrt{3}$ **(b)** $6-2\sqrt{5}$ **(c)** $5+2\sqrt{6}$ **(d)** $12-2\sqrt{35}$ **(e)** $13-4\sqrt{3}$ **(f)** $29+12\sqrt{5}$ **(g)** $33+4\sqrt{35}$ **(h)** $30-12\sqrt{6}$ **(i)** $55+30\sqrt{2}$
9(a) 2 **(b)** $\frac{3}{5}$ **(c)** $2\sqrt{3}$ **(d)** $\frac{5\sqrt{3}}{2}$ **(e)** 5 **(f)** 4
10(a) 3 **(b)** 5 **(c)** 4 **(d)** 6
11(a) $\sqrt{3}$ **(b)** $\frac{6\sqrt{7}}{13}$

Exercise 2F (Page 41)

1(a) $\frac{\sqrt{3}}{3}$ **(b)** $\frac{\sqrt{7}}{7}$ **(c)** $\frac{3\sqrt{5}}{5}$ **(d)** $\frac{5\sqrt{2}}{2}$ **(e)** $\frac{\sqrt{6}}{3}$ **(f)** $\frac{\sqrt{35}}{7}$ **(g)** $\frac{2\sqrt{55}}{5}$ **(h)** $\frac{3\sqrt{14}}{2}$
2(a) $\frac{\sqrt{3}+1}{2}$ **(b)** $\frac{\sqrt{7}-2}{3}$ **(c)** $\frac{3-\sqrt{5}}{4}$ **(d)** $\frac{4+\sqrt{7}}{9}$ **(e)** $\frac{\sqrt{5}+\sqrt{2}}{3}$ **(f)** $\frac{\sqrt{10}-\sqrt{6}}{4}$ **(g)** $\frac{2\sqrt{3}-1}{11}$ **(h)** $\frac{5+3\sqrt{2}}{7}$
3(a) $\sqrt{2}$ **(b)** $\sqrt{5}$ **(c)** $2\sqrt{3}$ **(d)** $3\sqrt{7}$ **(e)** $\frac{\sqrt{6}}{2}$ **(f)** $\frac{\sqrt{15}}{3}$ **(g)** $\frac{4\sqrt{6}}{3}$ **(h)** $\frac{7\sqrt{10}}{5}$
4(a) $\frac{\sqrt{5}}{10}$ **(b)** $\frac{\sqrt{7}}{21}$ **(c)** $\frac{3\sqrt{2}}{10}$ **(d)** $\frac{2\sqrt{3}}{21}$ **(e)** $\frac{5\sqrt{2}}{3}$ **(f)** $\frac{3\sqrt{3}}{4}$ **(g)** $\frac{\sqrt{30}}{20}$ **(h)** $\frac{2\sqrt{77}}{35}$
5(a) $\frac{3\sqrt{5}-3}{4}$ **(b)** $\frac{8\sqrt{2}+4\sqrt{3}}{5}$ **(c)** $\frac{5\sqrt{7}+7}{18}$ **(d)** $\frac{3\sqrt{15}-9}{2}$ **(e)** $\frac{28+10\sqrt{7}}{3}$ **(f)** $\sqrt{2}+1$ **(g)** $2-\sqrt{3}$ **(h)** $\frac{7+2\sqrt{10}}{3}$ **(i)** $8-3\sqrt{7}$ **(j)** $\frac{23+6\sqrt{10}}{13}$ **(k)** $4-\sqrt{15}$ **(l)** $\frac{93+28\sqrt{11}}{5}$
6(a) $\sqrt{3}+1$ **(b)** $4-\sqrt{10}$
7(a) 3 **(b)** 1 **(c)** 7 **(d)** 2
9 $a=-1$, $b=2$

Review Exercise 2G (Page 42)

1(a) composite **(b)** prime **(c)** prime **(d)** composite **(e)** composite **(f)** prime

2(a) 3×5 **(b)** 2^3 **(c)** 2×17 **(d)** $2\times 3\times 7$ **(e)** 3×5^2 **(f)** $2\times 3^2\times 5$

3(a) $\frac{4}{9}$ **(b)** $\frac{3}{4}$ **(c)** $\frac{2}{3}$ **(d)** $\frac{3}{7}$ **(e)** $\frac{8}{15}$ **(f)** $\frac{3}{5}$

4(a) $\frac{8}{15}$ **(b)** $\frac{5}{8}$ **(c)** $\frac{1}{18}$ **(d)** $\frac{4}{15}$ **(e)** $\frac{11}{24}$ **(f)** $\frac{7}{36}$

5(a) $0{\cdot}4$ **(b)** $0{\cdot}25$ **(c)** $0{\cdot}12$ **(d)** $0{\cdot}65$ **(e)** $0{\cdot}375$ **(f)** $0{\cdot}175$

6(a) $0{\cdot}\dot{2}$ **(b)** $0{\cdot}\dot{7}$ **(c)** $0{\cdot}\dot{1}\dot{8}$ **(d)** $0{\cdot}\dot{6}\dot{3}$ **(e)** $0{\cdot}08\dot{3}$ **(f)** $0{\cdot}41\dot{6}$

7(a) $\frac{3}{5}$ **(b)** $\frac{1}{20}$ **(c)** $\frac{2}{25}$ **(d)** $\frac{19}{50}$ **(e)** $\frac{3}{250}$ **(f)** $\frac{27}{40}$

8(a) $\frac{1}{3}$ **(b)** $\frac{8}{9}$ **(c)** $\frac{25}{99}$ **(d)** $\frac{5}{11}$ **(e)** $\frac{31}{111}$ **(f)** $\frac{11}{37}$ **(g)** $\frac{11}{30}$ **(h)** $\frac{26}{45}$ **(i)** $\frac{9}{55}$

9(a) rational, $\frac{7}{1}$ **(b)** rational, $\frac{-9}{4}$ **(c)** rational, $\frac{3}{1}$ **(d)** irrational **(e)** irrational **(f)** rational, $\frac{2}{1}$ **(g)** rational, $\frac{-4}{25}$ **(h)** irrational

10(a) $4{\cdot}12$, $4{\cdot}1$ **(b)** $4{\cdot}67$, $4{\cdot}7$ **(c)** $2{\cdot}83$, $2{\cdot}8$ **(d)** $0{\cdot}77$, $0{\cdot}77$ **(e)** $0{\cdot}02$, $0{\cdot}019$ **(f)** $542{\cdot}41$, 540

11(a) $1{\cdot}67$ **(b)** $70{\cdot}1$ **(c)** $1{\cdot}43$ **(d)** $0{\cdot}200$ **(e)** $0{\cdot}488$ **(f)** $0{\cdot}496$ **(g)** $1{\cdot}27$ **(h)** 1590 **(i)** $0{\cdot}978$

12(a) $2\sqrt{6}$ **(b)** $3\sqrt{5}$ **(c)** $5\sqrt{2}$ **(d)** $10\sqrt{5}$ **(e)** $9\sqrt{2}$ **(f)** $4\sqrt{10}$

13(a) $2\sqrt{5}$ **(b)** 5 **(c)** 28 **(d)** $\sqrt{7}-\sqrt{5}$ **(e)** $\sqrt{7}$ **(f)** $3\sqrt{5}$ **(g)** 4 **(h)** $2\sqrt{5}$ **(i)** $24\sqrt{10}$

14(a) $\sqrt{3}$ **(b)** $7\sqrt{2}$ **(c)** $4\sqrt{2}$ **(d)** $8\sqrt{6}-6\sqrt{5}$

15(a) $3\sqrt{7}-7$ **(b)** $2\sqrt{30}+3\sqrt{10}$ **(c)** $3\sqrt{5}-5\sqrt{15}$ **(d)** $3\sqrt{2}+6$

16(a) $\sqrt{5}+1$ **(b)** $13+7\sqrt{3}$ **(c)** $2\sqrt{35}+4\sqrt{7}-6\sqrt{5}-12$ **(d)** 1 **(e)** 13 **(f)** $11-4\sqrt{7}$ **(g)** $7+2\sqrt{10}$ **(h)** $34-24\sqrt{2}$

17(a) $\frac{\sqrt{5}}{5}$ **(b)** $\frac{3\sqrt{2}}{2}$ **(c)** $\frac{\sqrt{33}}{11}$ **(d)** $\frac{\sqrt{3}}{15}$ **(e)** $\frac{5\sqrt{7}}{14}$ **(f)** $\frac{\sqrt{5}}{15}$

18(a) $\frac{\sqrt{5}-\sqrt{2}}{3}$ **(b)** $\frac{3+\sqrt{7}}{2}$ **(c)** $\frac{2\sqrt{6}+\sqrt{3}}{21}$ **(d)** $\frac{3-\sqrt{3}}{2}$ **(e)** $\frac{\sqrt{11}-\sqrt{5}}{2}$ **(f)** $\frac{6\sqrt{35}+21}{13}$

19(a) $\frac{9-2\sqrt{14}}{5}$ **(b)** $26+15\sqrt{3}$

20 $x=50$

21 $5\sqrt{5}+2$

22 $p=5$, $q=2$

23 $\frac{7}{3}$

Chapter Three

Exercise 3A (Page 47)

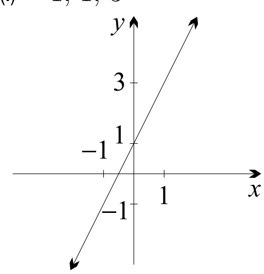
1 a, b, d, e, g, h, j, l

2(a) domain: all real x, range: $y\geq -1$
(b) domain: all real x, range: $y>-1$
(c) domain: all real x, range: all real y
(d) domain: all real x, range: $y=2$
(e) domain: all real x, range: $y<2$
(f) domain: $x\geq -1$, range: all real y
(g) domain: $x\neq 0$, range: $y\neq 0$
(h) domain: all real x, range: all real y
(i) domain: $0\leq x\leq 3$, range: $-3\leq y\leq 3$
(j) domain: $x<4$, range: $y>0$
(k) domain: all real x, range: $y\leq -1, y>1$
(l) domain: all real x, range: $y<1$

3(a) 5, -1, -7 **(b)** 0, 4, 0 **(c)** 16, 8, 0 **(d)** 4, 1, $\frac{1}{4}$

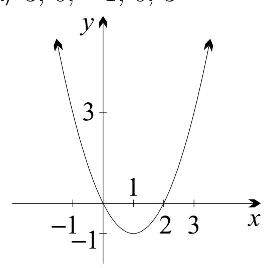
4(a) -4, 3, 12 **(b)** $-\frac{1}{3}$, 2, $\frac{1}{5}$ **(c)** -18, $1\frac{1}{4}$, -10 **(d)** 1, $\frac{3}{\sqrt{2}}$, 3

5(i) -1, 1, 3



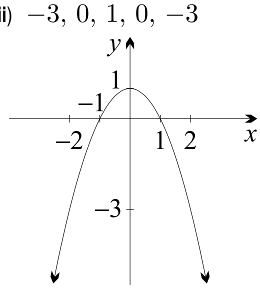
domain: all real x, range: all real y

(ii) 3, 0, -1, 0, 3



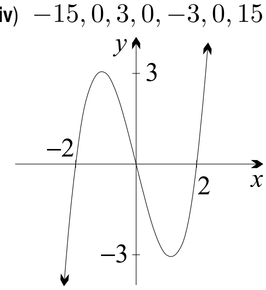
domain: all real x, range: $y\geq -1$

(iii) -3, 0, 1, 0, -3



domain: all real x, range: $y\leq 1$

(iv) $-15, 0, 3, 0, -3, 0, 15$



domain: all real x, range: all real y

6(a) $x\neq 0$ **(b)** $x\neq 3$ **(c)** $x\neq -1$ **(d)** $x\neq -2$

7(a) $x\geq 0$ **(b)** $x\geq 2$ **(c)** $x\geq -3$ **(d)** $x\geq -5$

8(a) $(0,3)$ and $(0,-3)$ **(b)** $(2,0)$ and $(2,1)$ **(c)** $(0,1)$ and $(0,-1)$

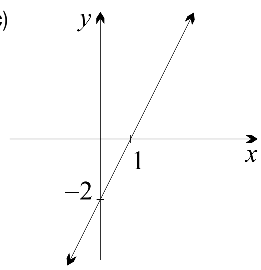
9(a) $2a-4$, $-2a-4$, $2a-2$ **(b)** $2-a$, $2+a$, $1-a$ **(c)** a^2, a^2, a^2+2a+1 **(d)** $\frac{1}{a-1}$, $\frac{1}{-a-1}=-\frac{1}{a+1}$, $\frac{1}{a}$

10(a) $5t^2 + 2,\ 5t,\ 5t - 8$ **(b)** $t,\ \sqrt{t} - 2,\ \sqrt{t-2}$
(c) $t^4 + 2t^2,\ t^2 + 2t - 2,\ t^2 - 2t$
(d) $2 - t^4,\ -t^2,\ -t^2 + 4t - 2$
11(a) all real x **(b)** all real x **(c)** $x \neq 4$ **(d)** $x \neq \frac{1}{2}$
(e) $x \geq -\frac{1}{2}$ **(f)** $x \leq 5$ **(g)** $x > 0$ **(h)** $x > -1$
(i) $x < 1$ **(j)** $x > 1\frac{1}{2}$ **(k)** $x < \frac{3}{4}$ **(l)** $x > -\frac{2}{3}$
12(a) $-2 - 2\sqrt{2}$ **(b)** $3 - 2\sqrt{7}$ **(c)** $2 - 4\sqrt{3}$ **(d)** 0
13(a) $5\frac{1}{2}$ **(b)** $3\frac{5}{6}$
14(a) $-3,\ -2,\ -1,\ 0,\ 1,\ 2,\ -1$ **(b)** $3,\ 0,\ 0,\ 1,\ 4$
15(a) 15 **(b)** 1 **(c)** 27 **(d)** $2a^2 + 1$
16(a) 5 **(b)** $p + q + 5$ **(c)** $2a + 5$
17(a) $x > -2$ **(b)** $x \neq 2$ and $x \neq -2$
(c) $x \neq -1$ and $x \neq 0$ **(d)** $x \neq 2$ and $x \neq 3$
(e) $x \leq -2$ or $x \geq 2$ **(f)** $-1 < x < 1$

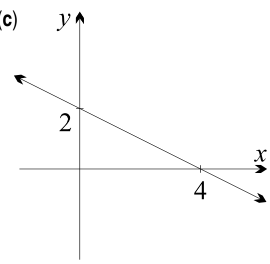
Exercise **3B** (Page 51)

1(a) $x = 1$ **(b)** $y = -2$ **2(a)** $x = 4$ **(b)** $y = 2$

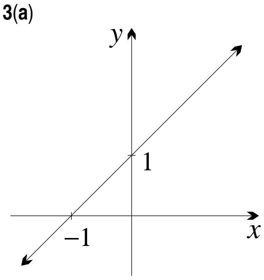
1(c)



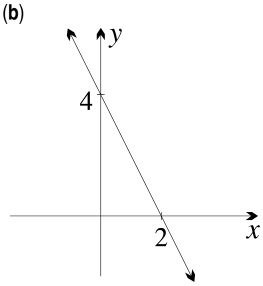
2(c)



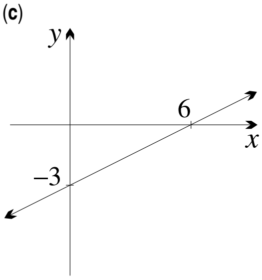
3(a)



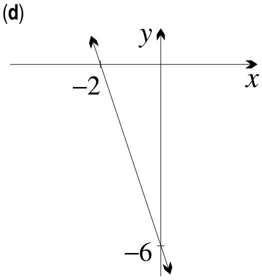
(b)



(c)

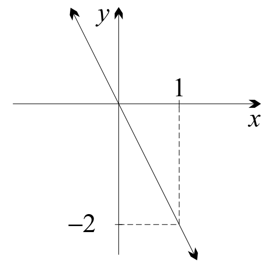


(d)

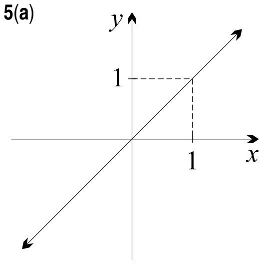


(e) intercepts: $(1, 0)$ and $(0, 1)$
(f) intercepts: $(-1, 0)$ and $(0, 2)$
(g) intercepts: $(3, 0)$ and $(0, -1)$
(h) intercepts: $(4, 0)$ and $(0, -2)$
(i) intercepts: $(6, 0)$ and $(0, -4)$
(j) intercepts: $(-6, 0)$ and $(0, -1\frac{1}{2})$
(k) intercepts: $(2, 0)$ and $(0, 5)$
(l) intercepts: $(3, 0)$ and $(0, -7\frac{1}{2})$

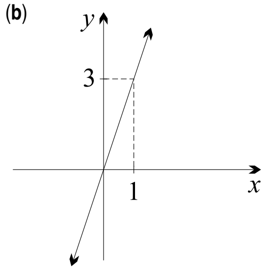
4(b) $(1, -2)$



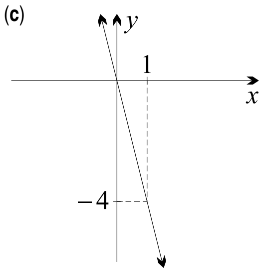
5(a)



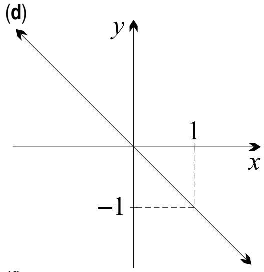
(b)



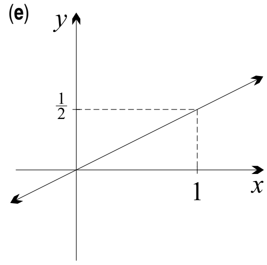
(c)



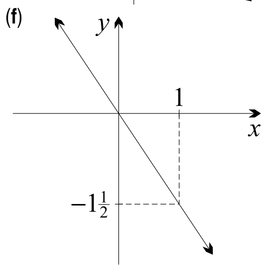
(d)



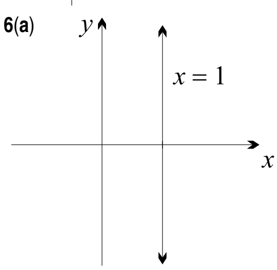
(e)



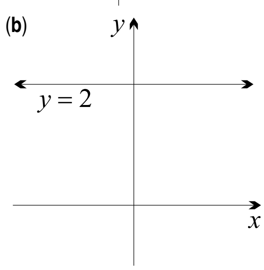
(f)



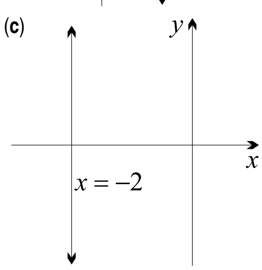
6(a)



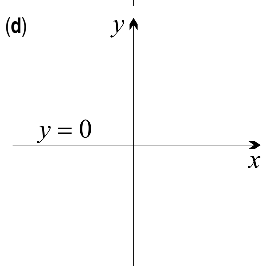
(b)



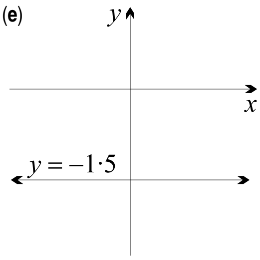
(c)



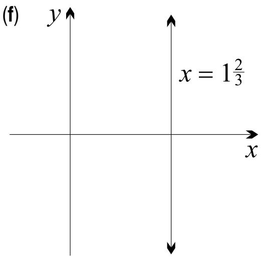
(d)



(e)



(f)

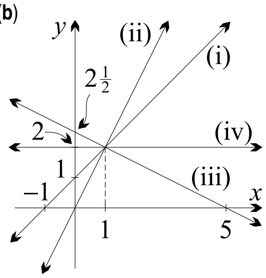


7(a) yes **(b)** no **(c)** yes **(d)** yes **(e)** yes **(f)** no

8(b) $(3, 2)$

9(a) $(-1, 3)$ **(b)** $(1, -2)$ **(c)** $(-2, -1)$

10(b)

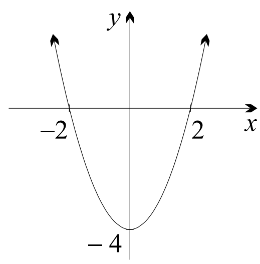


Exercise **3C** (Page 53)

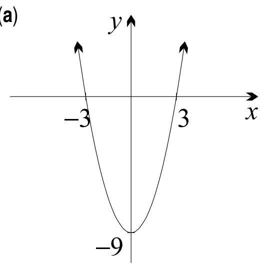
1(a) $y = -4$

(b) $x = 2$ and $x = -2$

(d) domain: all real x, range: $y \geq -4$

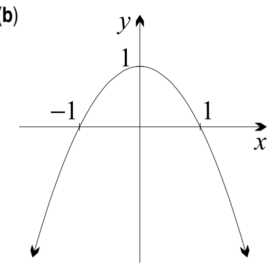


2(a)



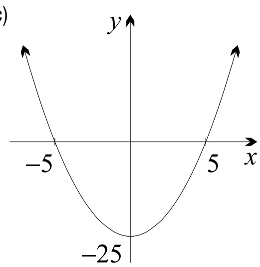
domain: all real x, range: $y \geq -9$

(b)



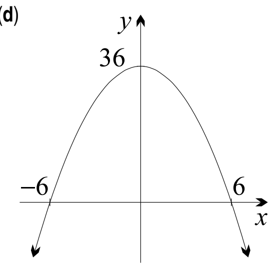
domain: all real x, range: $y \leq 1$

(c)



domain: all real x, range: $y \geq -25$

(d)



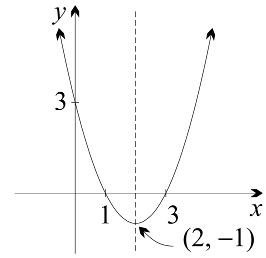
domain: all real x, range: $y \leq 36$

3(a) $x = 1$ and $x = 3$

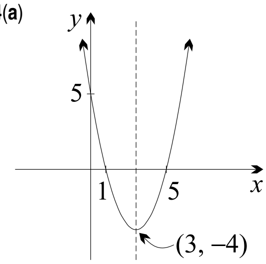
(b) $y = 3$

(c) $x = 2$

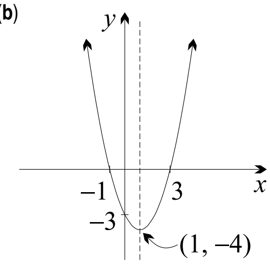
(d) $(2, -1)$



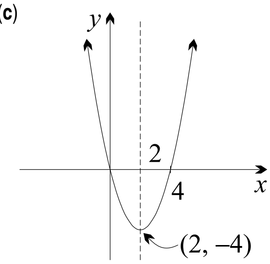
4(a)



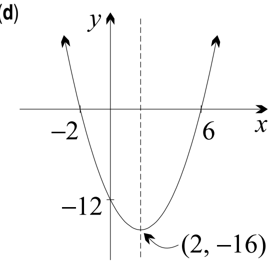
(b)



(c)



(d)

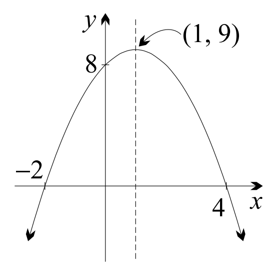


5(a) $x = -2$ and $x = 4$

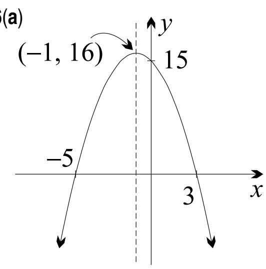
(b) $y = 8$

(c) $x = 1$

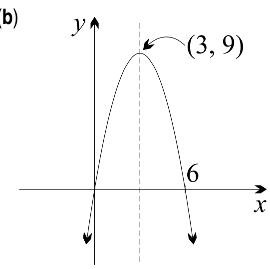
(d) $(1, 9)$



6(a)



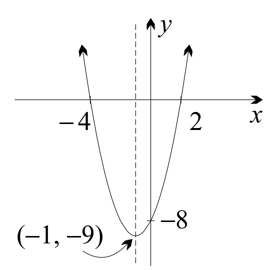
(b)



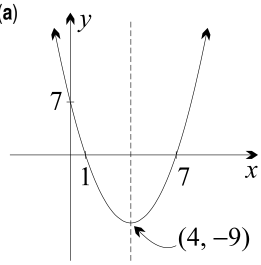
7(a) $y = -8$

(b) $x = -4$ and $x = 2$

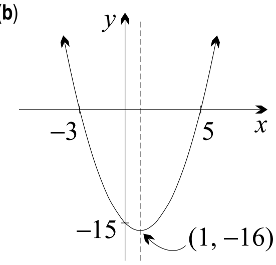
(c) $(-1, -9)$



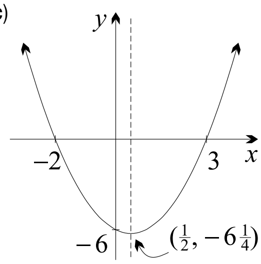
8(a)



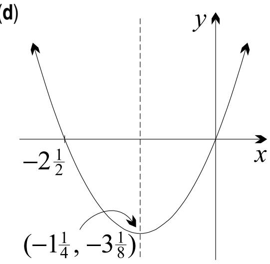
(b)



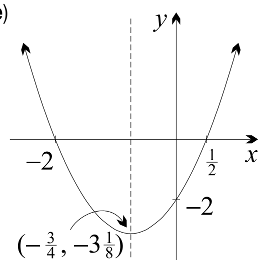
(c)



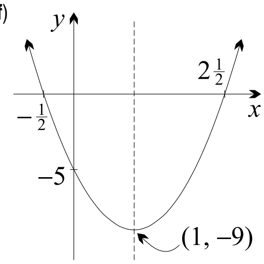
(d)



(e)



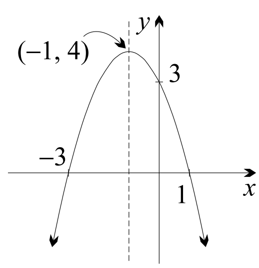
(f)



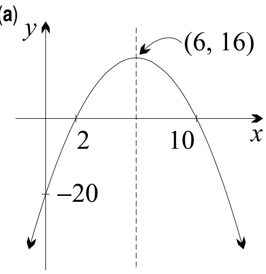
9(a) $y = 3$

(b) $x = -3$ and $x = 1$

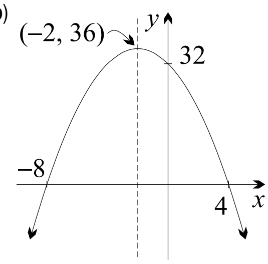
(c) $(-1, 4)$



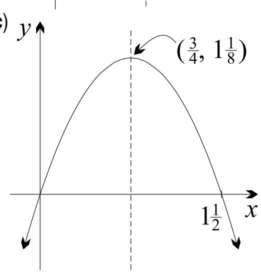
10(a)



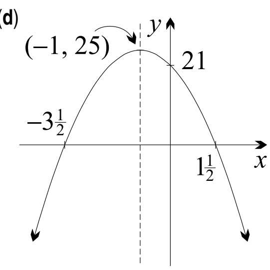
(b)



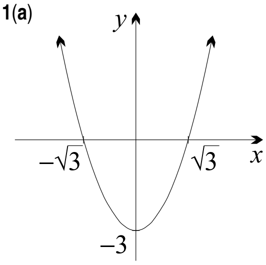
(c)



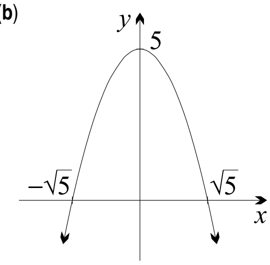
(d)



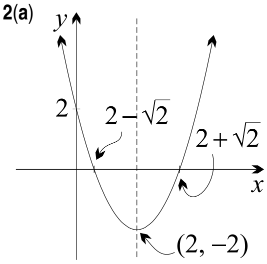
11(a)



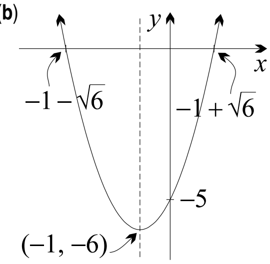
(b)



12(a)



(b)



13(a) $b = -11$, $c = 28$ **(b)** $\alpha = -4$

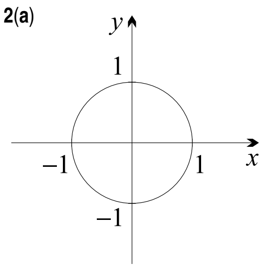
(c) $b = -12$, $c = 27$

Exercise **3D** (Page 55)

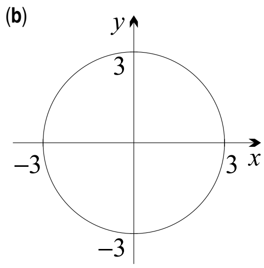
1(a) $(0, 0)$, 4 units **(b)** $(0, 0)$, 7 units

(c) $(0, 0)$, $\frac{1}{3}$ units **(d)** $(0, 0)$, $1{\cdot}2$ units

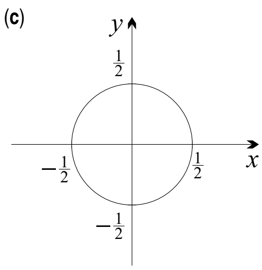
2(a)



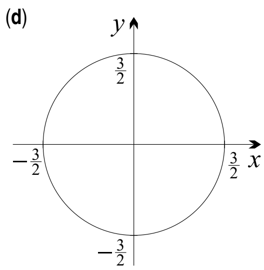
(b)



(c)



(d)



The domains and ranges are respectively:

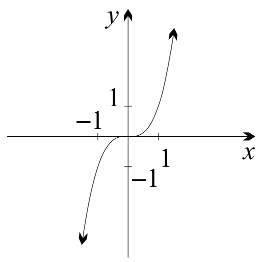
(a) $-1 \le x \le 1$, $-1 \le y \le 1$

(b) $-3 \le x \le 3$, $-3 \le y \le 3$

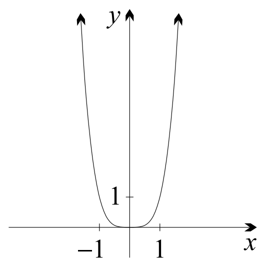
(c) $-\frac{1}{2} \le x \le \frac{1}{2}$, $-\frac{1}{2} \le y \le \frac{1}{2}$

(d) $-\frac{3}{2} \le x \le \frac{3}{2}$, $-\frac{3}{2} \le y \le \frac{3}{2}$

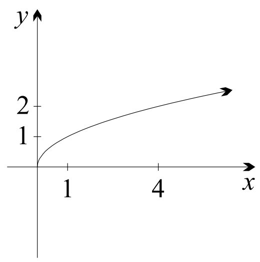
3(a) $-3{\cdot}375,\ -1,\ -0{\cdot}125,\ 0,\ 0{\cdot}125,\ 1,\ 3{\cdot}375$



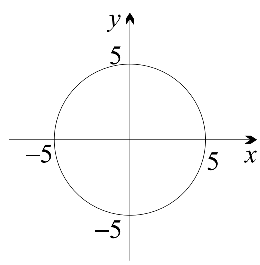
4 $5{\cdot}0625,\ 1,\ 0{\cdot}0625,\ 0,\ 0{\cdot}0625,\ 1,\ 5{\cdot}0625$



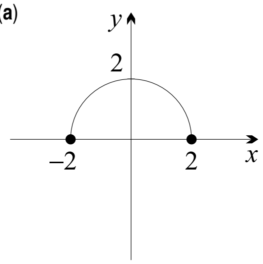
5(a) $0,\ 0{\cdot}5,\ 1,\ 1{\cdot}5,\ 2$



6(a) 5 or -5, $4{\cdot}9$ or $-4{\cdot}9$, $4{\cdot}6$ or $-4{\cdot}6$, 4 or -4, 3 or -3, 0

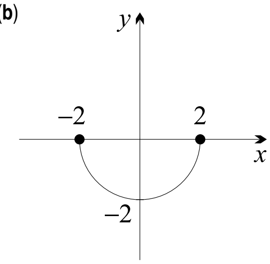


7(a)



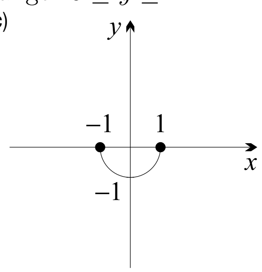
domain: $-2 \le x \le 2$, range: $0 \le y \le 2$

(b)



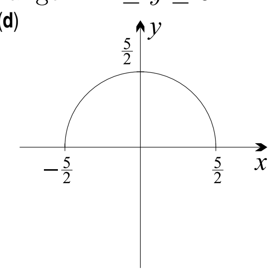
domain: $-2 \le x \le 2$, range: $-2 \le y \le 0$

(c)



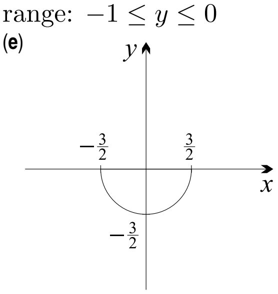
domain: $-1 \le x \le 1$, range: $-1 \le y \le 0$

(d)



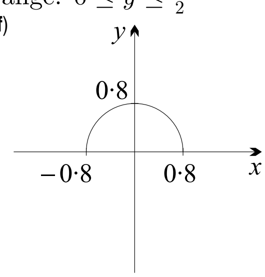
domain: $-\frac{5}{2} \le x \le \frac{5}{2}$, range: $0 \le y \le \frac{5}{2}$

(e)



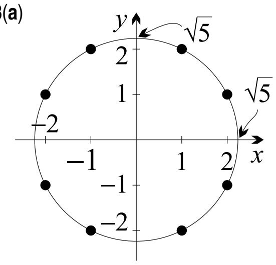
domain: $-\frac{3}{2} \le x \le \frac{3}{2}$, range: $-\frac{3}{2} \le y \le 0$

(f)

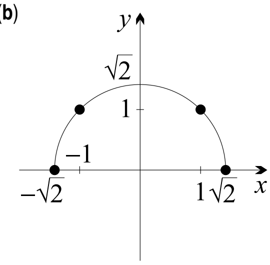


domain: $-0{\cdot}8 \le x \le 0{\cdot}8$, range: $0 \le y \le 0{\cdot}8$

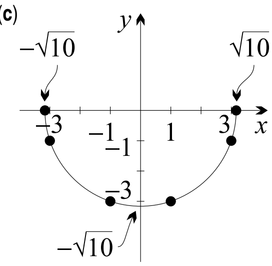
8(a)



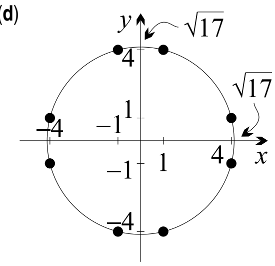
(b)



(c)



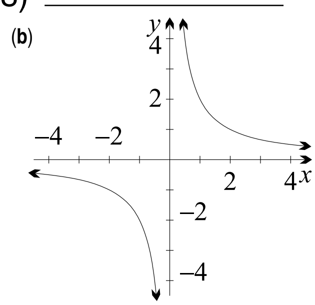
(d)



Exercise **3E** (Page 58)

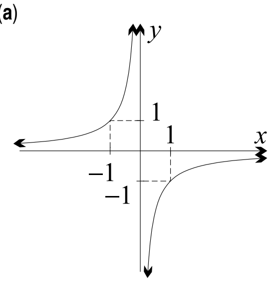
1(a) $-\frac{1}{2},\ -1,\ -2,\ -4,\ 4,\ 2,\ 1,\ \frac{1}{2}$
(c) the x-axis ($y = 0$) and the y-axis ($x = 0$)
(d) domain: $x \ne 0$, range: $y \ne 0$

(b)

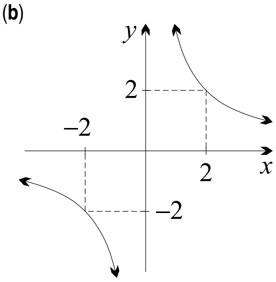


2 In each case, the domain is $x \ne 0$, the range is $y \ne 0$, and the asymptotes are $y = 0$ and $x = 0$.

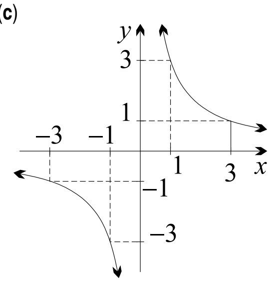
(a)



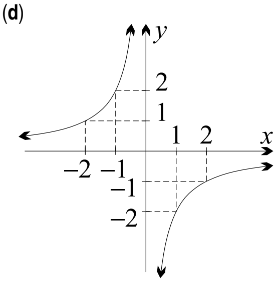
(b)



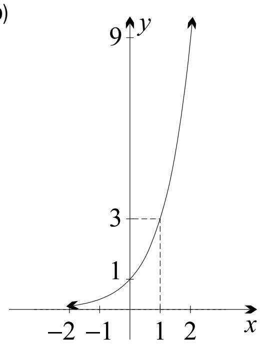
(c)



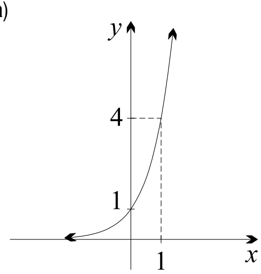
(d)

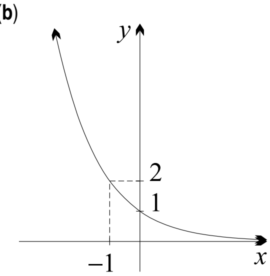


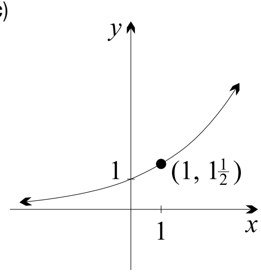
3(a) 0·1, 0·2, 0·3, 0·6, 1, 1·7, 3, 5·2, 9
(c) the x-axis ($y = 0$)
(d) domain: all real x, range: $y > 0$

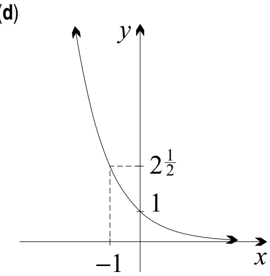
(b)


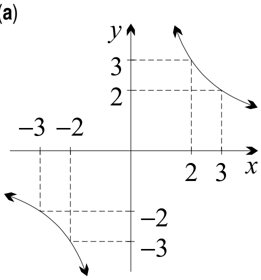
4 In each case, the domain is all real x, the range is $y > 0$, and the asymptote is $y = 0$.

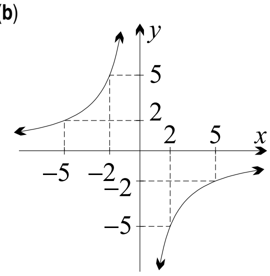
(a)


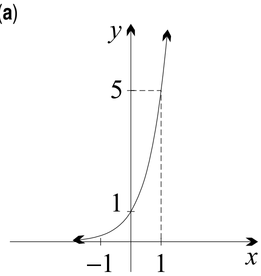
(b)


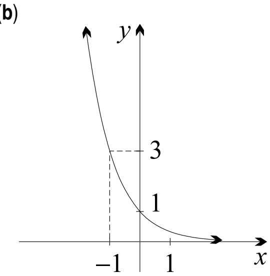
(c)


(d)


5(a)


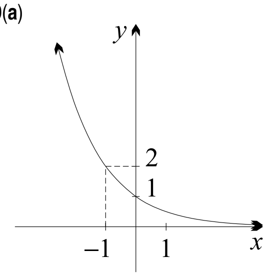
(b)


6(a)


(b)


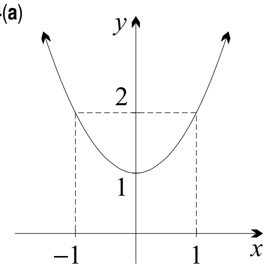
7(a) $y \to 0$ as $x \to -\infty$. **(b)** $y \to 0$ as $x \to \infty$.
(c) $y \to 0$ as $x \to \infty$ and as $x \to -\infty$, $y \to \infty$ as $x \to 0^+$, $y \to -\infty$ as $x \to 0^-$.

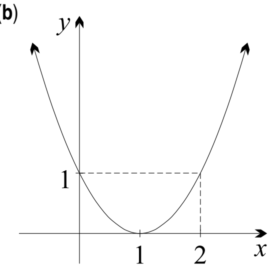
8 No, because the only points that satisfy the equation lie on the x and y axes.

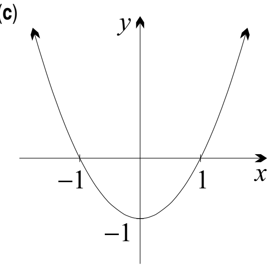
9(a)


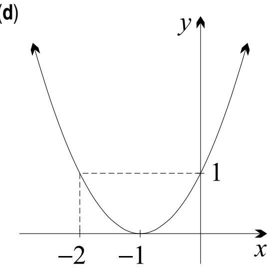
Exercise 3F (Page 61)

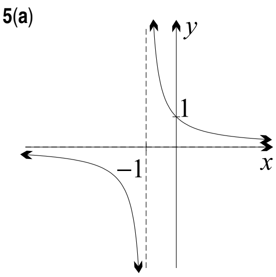
1(a) 2 units up **(b)** 3 units right **(c)** 5 units down **(d)** 4 units left
2(a) 2 units right **(b)** 4 units down **(c)** 3 units left **(d)** 5 units up
3(a) the y-axis **(b)** the x-axis

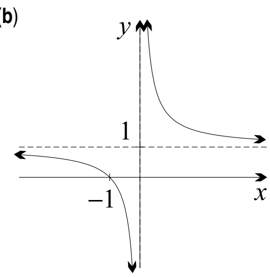
4(a)


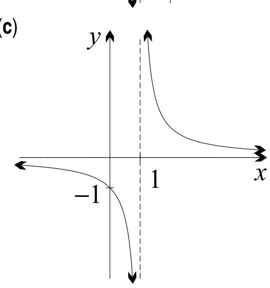
(b)


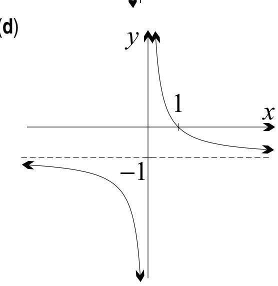
(c)


(d)


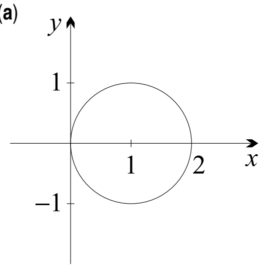
5(a)


(b)


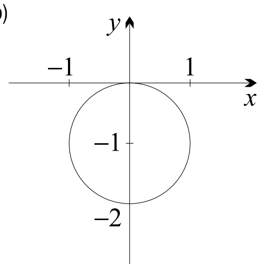
(c)


(d)


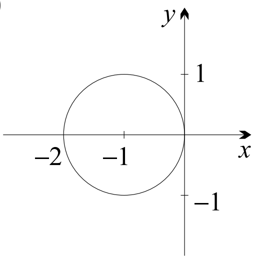
6(a)



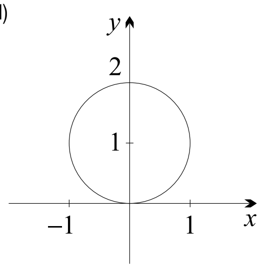
(b)



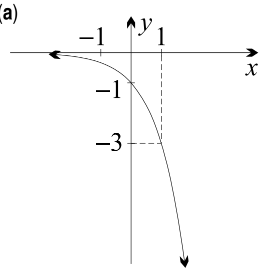
(c)



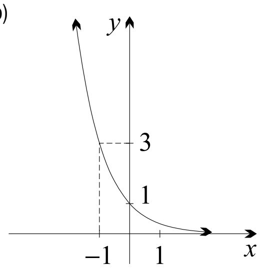
(d)



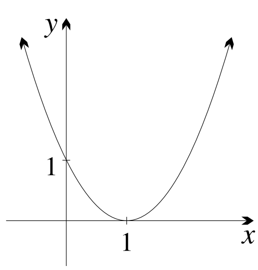
7(a)



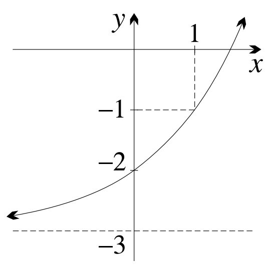
(b)



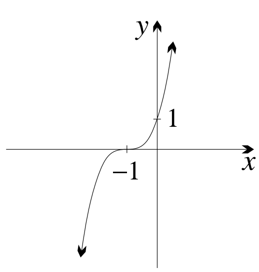
8(a) $y = (x-1)^2$



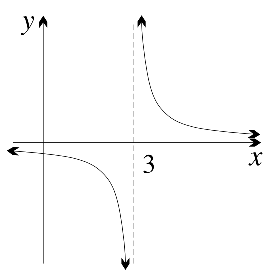
(b) $y = 2^x - 3$



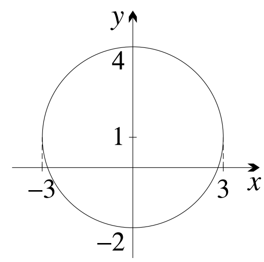
(c) $y = (x+1)^3$



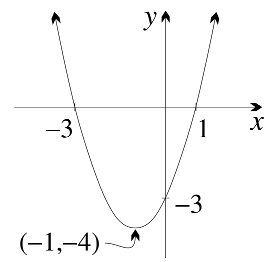
(d) $y = \frac{1}{x-3}$



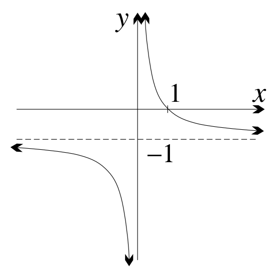
(e) $x^2 + (y-1)^2 = 9$



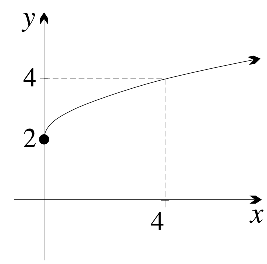
(f) $y = (x+1)^2 - 4$



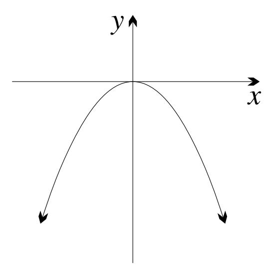
(g) $x(y+1) = 1$



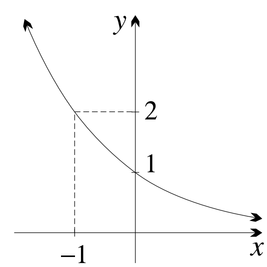
(h) $y = \sqrt{x} + 2$



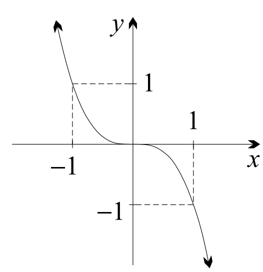
9(a) $y = -x^2$



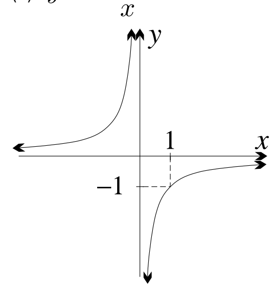
(b) $y = 2^{-x}$



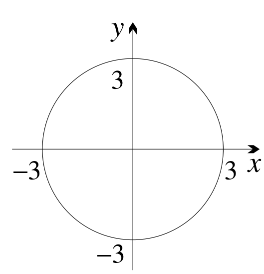
(c) $y = -x^3$



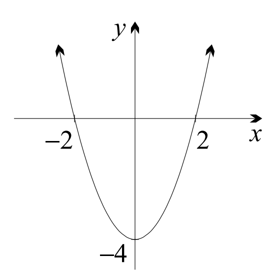
(d) $y = -\dfrac{1}{x}$



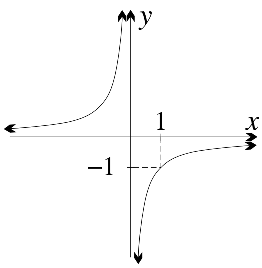
(e) $x^2 + y^2 = 9$



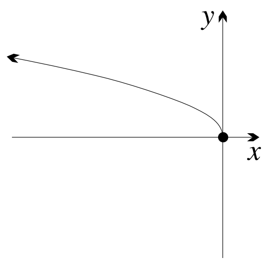
(f) $y = x^2 - 4$



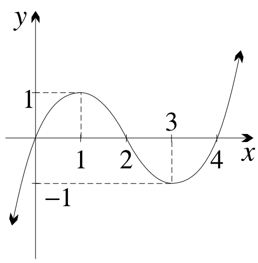
(g) $xy = -1$



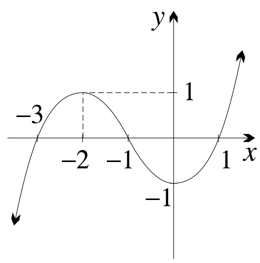
(h) $y = \sqrt{-x}$



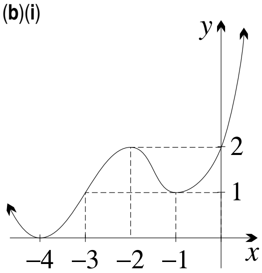
10(a)(i)



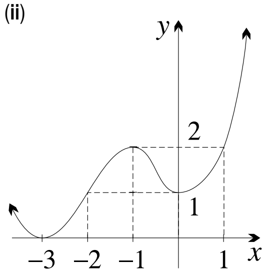
(ii)



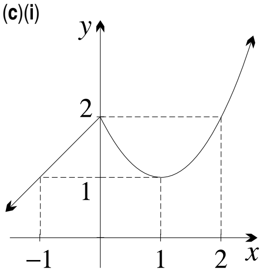
(b)(i)



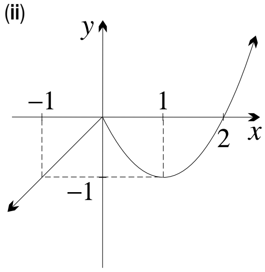
(ii)



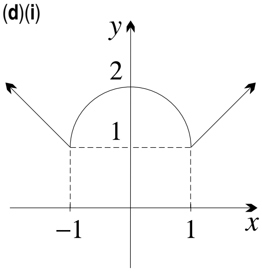
(c)(i)



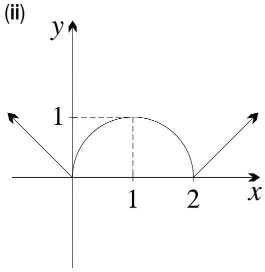
(ii)



(d)(i)



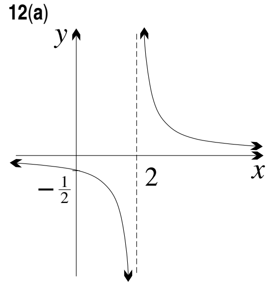
(ii)



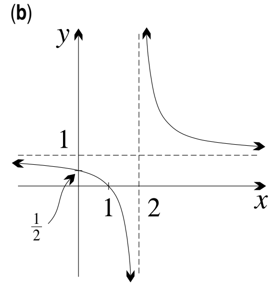
11(a) $x + 2y - 2 = 0$ **(b)** $x + 2y - 2 = 0$

(c) Both translations yield the same result.

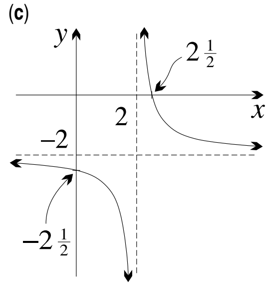
12(a)



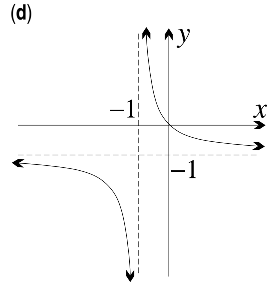
(b)



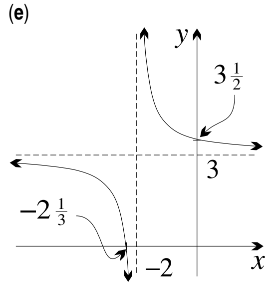
(c)



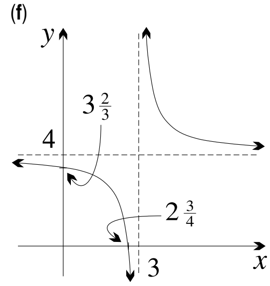
(d)



(e)



(f)

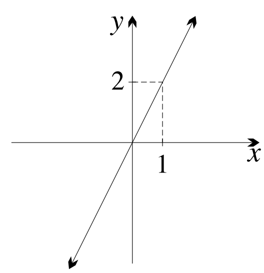


13(a) $r = 2,\ (-1, 0)$ **(b)** $r = 1,\ (1, 2)$

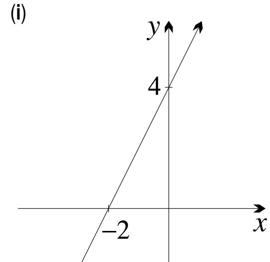
(c) $r = 3,\ (1, 2)$ **(d)** $r = 5,\ (-3, 4)$

(e) $r = 3,\ (5, -4)$ **(f)** $r = 6,\ (-7, 1)$

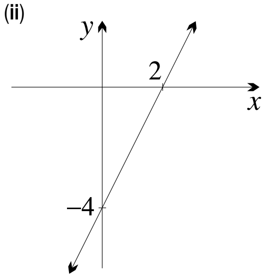
14(a) From $y = 2x$: **(i)** shift up 4 (or left 2) **(ii)** shift down 4 (or right 2) **(iii)** reflect in y-axis and shift up 4



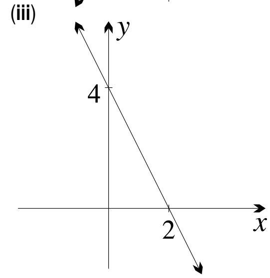
(i)



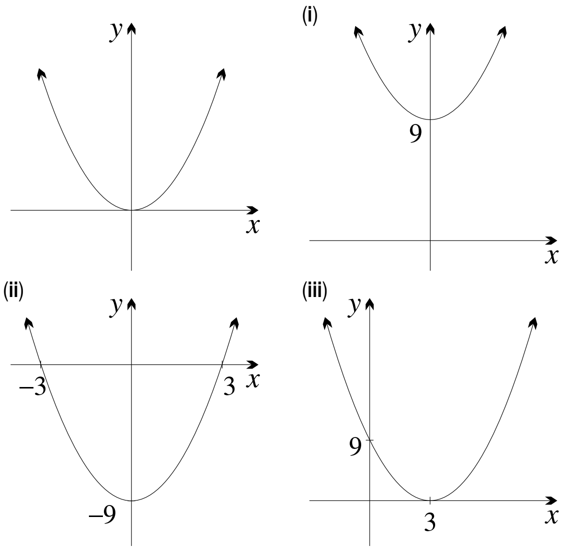
(ii)



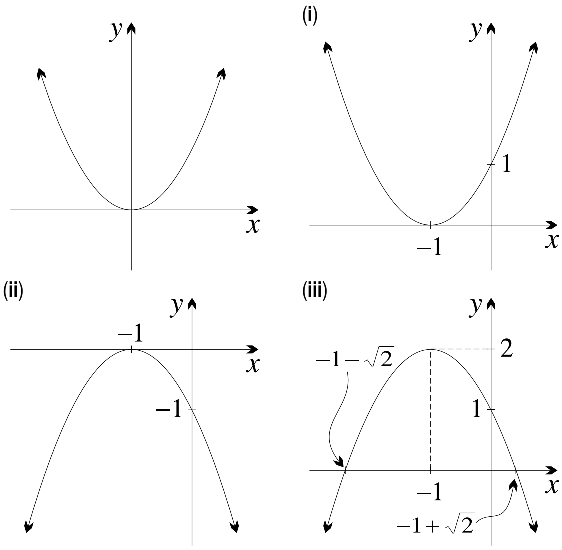
(iii)



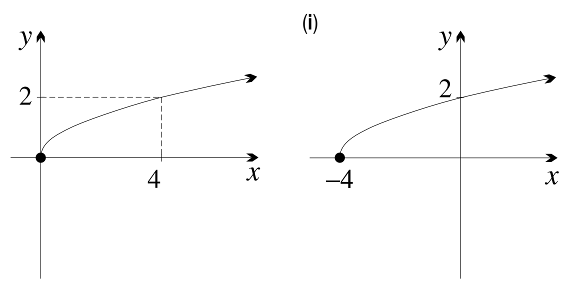
(b) From $y = x^2$: **(i)** shift 9 up **(ii)** shift 9 down **(iii)** shift 3 right

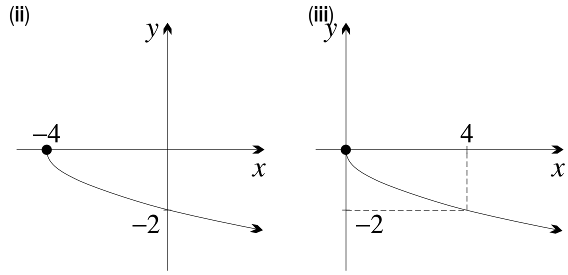


(c) From $y = x^2$: **(i)** shift 1 left
(ii) shift 1 left and reflect in x-axis
(iii) shift 1 left, reflect in x-axis and shift up 2

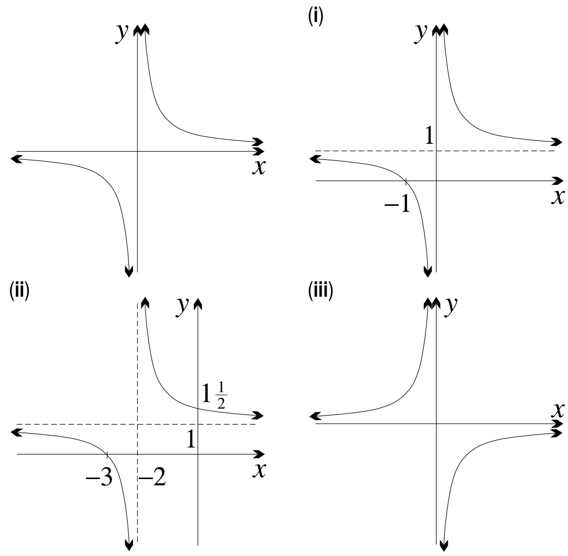


(d) From $y = \sqrt{x}$: **(i)** shift 4 left
(ii) shift 4 left and reflect in x-axis
(iii) reflect in x-axis



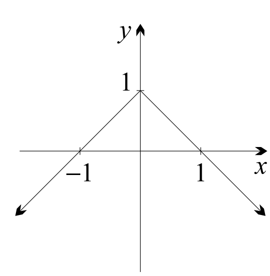


(e) From $y = \dfrac{1}{x}$: **(i)** shift up 1
(ii) shift up 1, left 2
(iii) reflect in the x-axis or in the y-axis

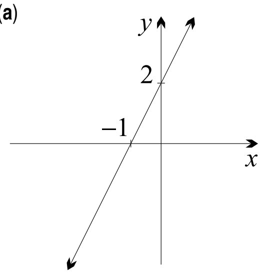


Review Exercise 3G (Page 63)

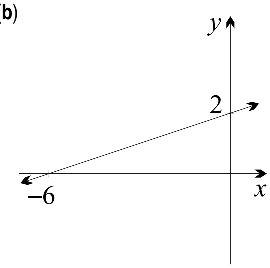
1(a) not a function **(b)** function **(c)** function
(d) not a function
2(a) $-2 \le x \le 0$, $-2 \le y \le 2$
(b) all real x, all real y **(c)** $x \ne 0$, $y \ne 0$
(d) $x = 2$, all real y
3(a) 21, -4 **(b)** 5, -15
4(a) $x \ne 2$ **(b)** $x \ge 1$ **(c)** $x \ge -\frac{2}{3}$ **(d)** $x < 2$
5(a) $2a + 2$, $2a + 1$ **(b)** $a^2 - 3a - 8$, $a^2 - 5a - 3$
6 -2, -1, 0, 1, 0, -1, -2



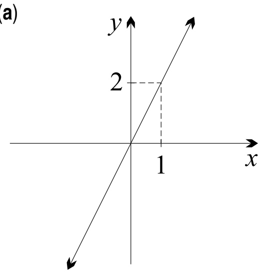
7(a)



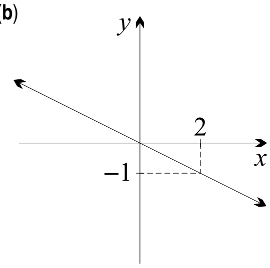
(b)



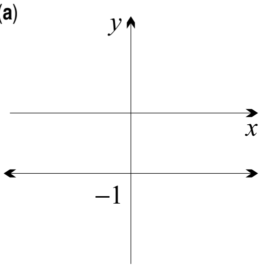
8(a)



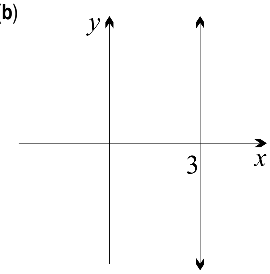
(b)



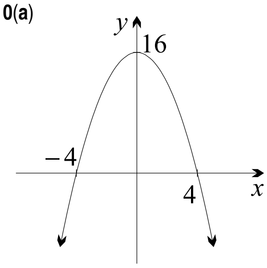
9(a)



(b)

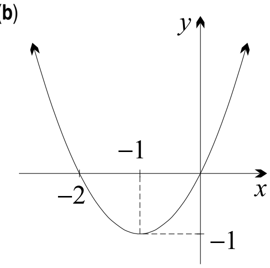


10(a)



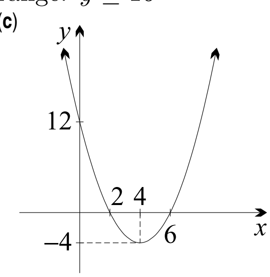
domain: all real x,
range: $y \le 16$

(b)



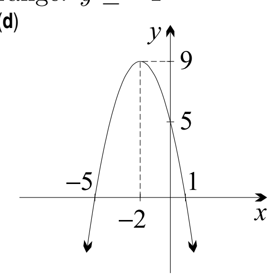
domain: all real x,
range: $y \ge -1$

(c)



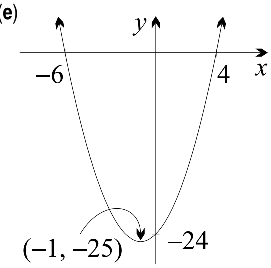
domain: all real x,
range: $y \ge -4$

(d)



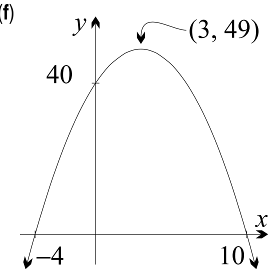
domain: all real x,
range: $y \le 9$

(e)



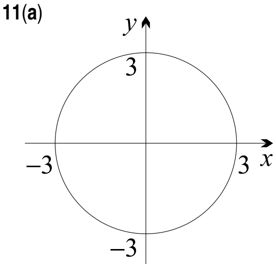
domain: all real x,
range: $y \ge -25$

(f)

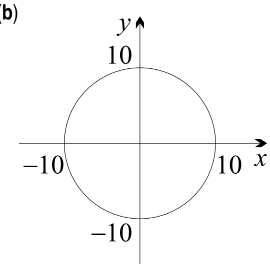


domain: all real x,
range: $y \le 49$

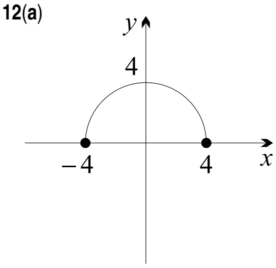
11(a)



(b)

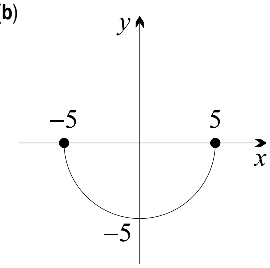


12(a)



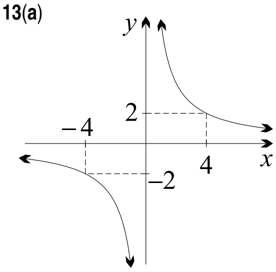
domain: $-4 \le x \le 4$,
range: $0 \le y \le 4$

(b)



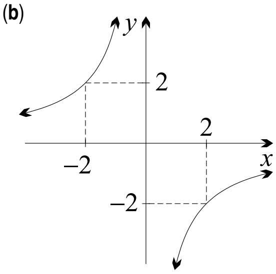
domain: $-5 \le x \le 5$,
range: $-5 \le y \le 0$

13(a)



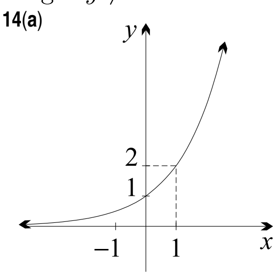
domain: $x \ne 0$,
range: $y \ne 0$

(b)



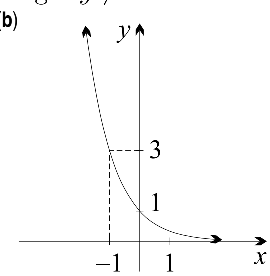
domain: $x \ne 0$,
range: $y \ne 0$

14(a)



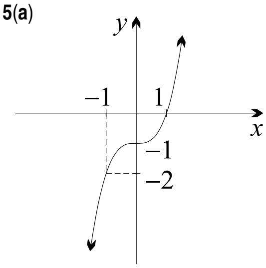
domain: all real x,
range: $y > 0$

(b)

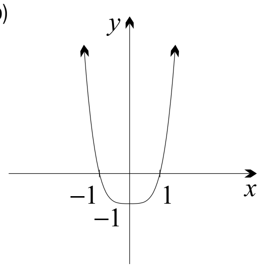


domain: all real x,
range: $y > 0$

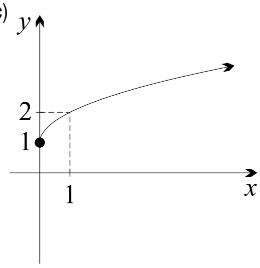
15(**a**)



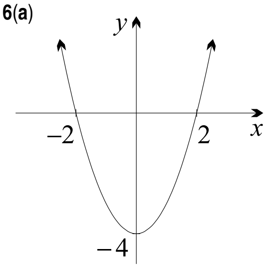
(**b**)



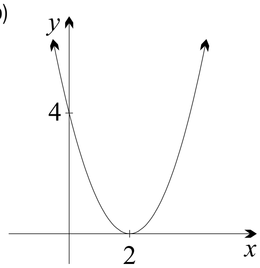
(**c**)



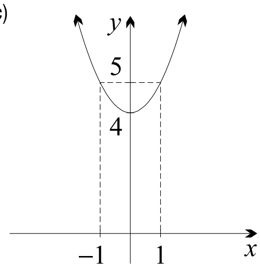
16(**a**)



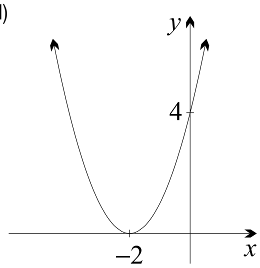
(**b**)



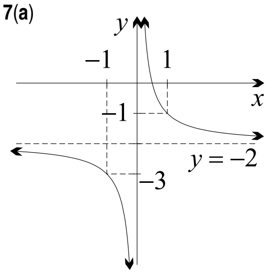
(**c**)



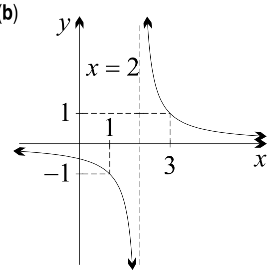
(**d**)



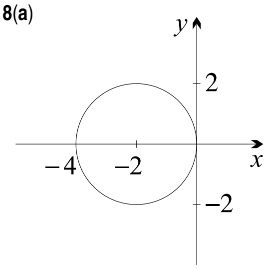
17(**a**)



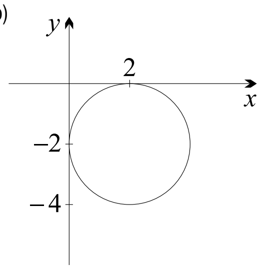
(**b**)



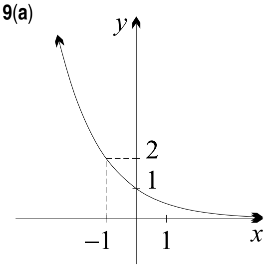
18(**a**)



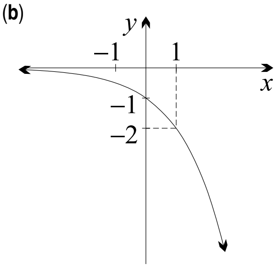
(**b**)



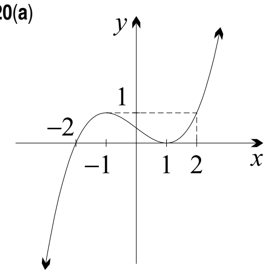
19(**a**)



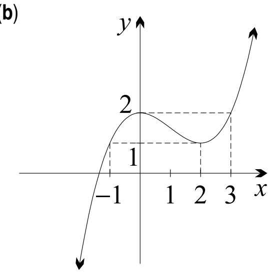
(**b**)



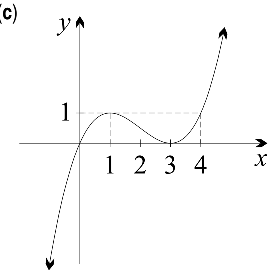
20(**a**)



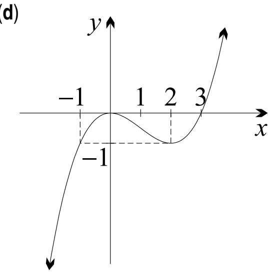
(**b**)



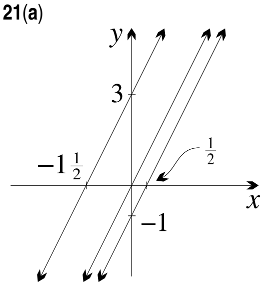
(**c**)



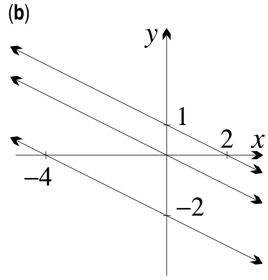
(**d**)



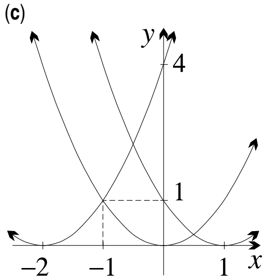
21(**a**)



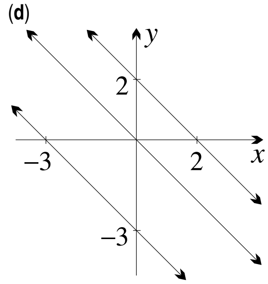
(**b**)



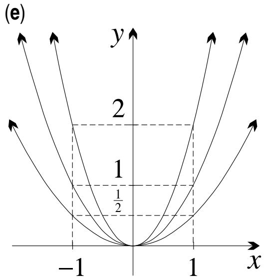
(**c**)



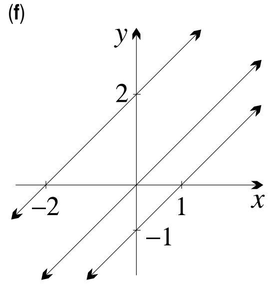
(**d**)



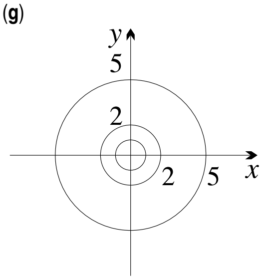
(**e**)



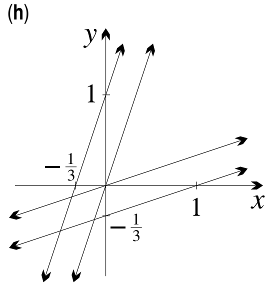
(**f**)



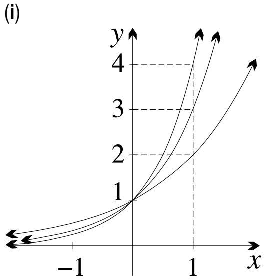
(g)



(h)



(i)



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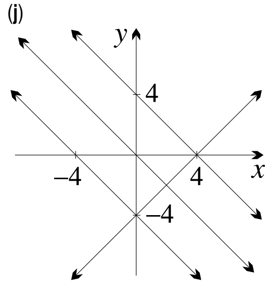

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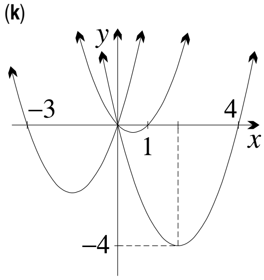

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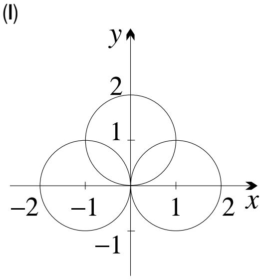
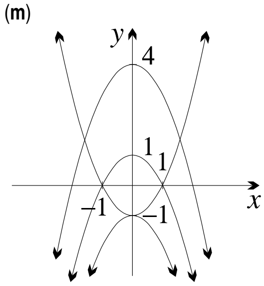
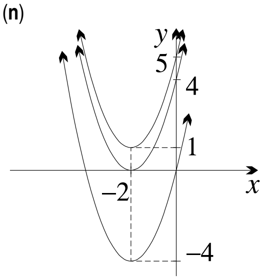


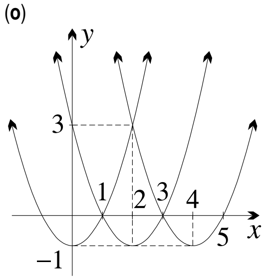
(m)



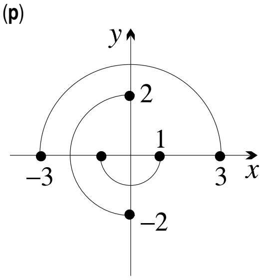
(n)



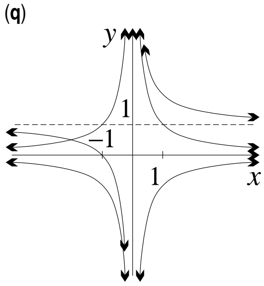
(o)



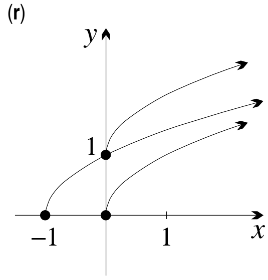
(p)



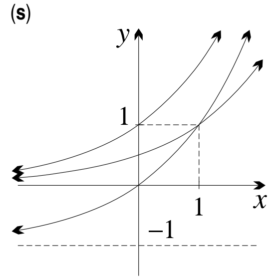
(q)



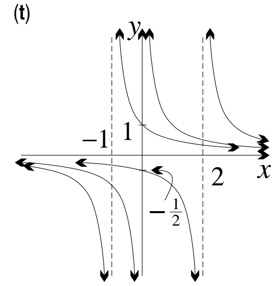
(r)



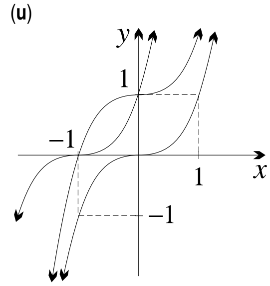
(s)



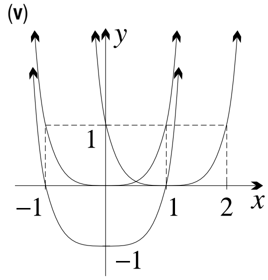
(t)



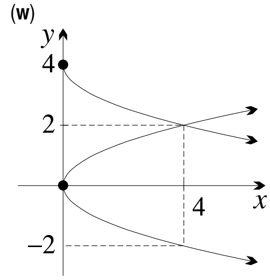
(u)



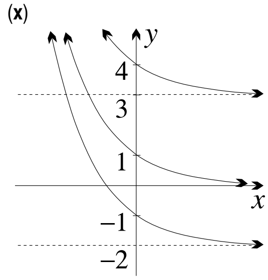
(v)



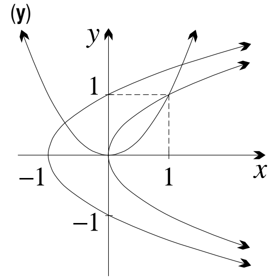
(w)



(x)

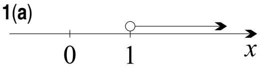


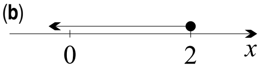
(y)

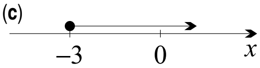


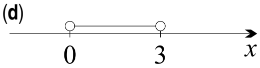
Chapter Four

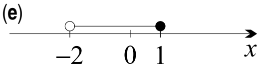
Exercise 4A (Page 68)

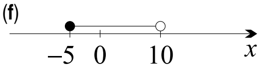
1(a)


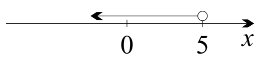
(b)


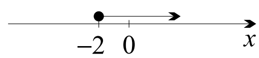
(c)


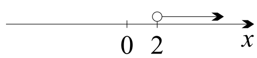
(d)


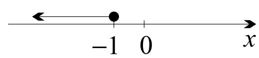
(e)


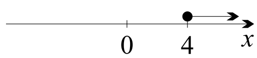
(f)


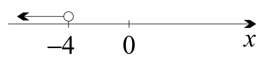
2(a) $x < 5$


(b) $x \geq -2$


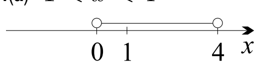
(c) $x > 2$


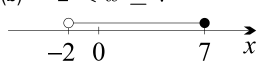
(d) $x \leq -1$


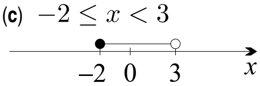
(e) $x \geq 4$


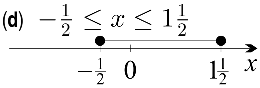
(f) $x < -4$


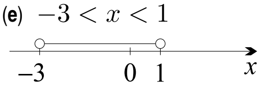
3(a) $x > -3$ **(b)** $x \leq 10$ **(c)** $x \geq -1$ **(d)** $x < -2$
(e) $x > 3$ **(f)** $x \leq -5$ **(g)** $x < 2$ **(h)** $x \geq -4$
(i) $x < 6$

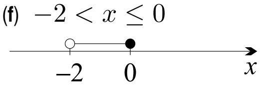
4(a) $1 < x < 4$


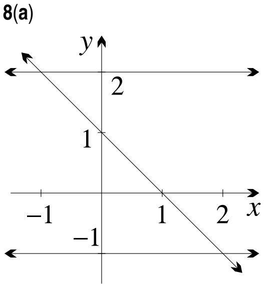
(b) $-2 < x \leq 7$


(c) $-2 \leq x < 3$


(d) $-\frac{1}{2} \leq x \leq 1\frac{1}{2}$


(e) $-3 < x < 1$


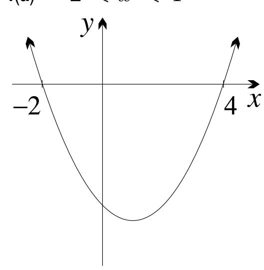
(f) $-2 < x \leq 0$


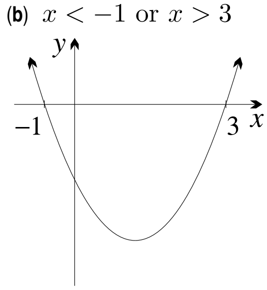
5(a) $-4 < x < 2$ **(b)** $-1 \leq x \leq 2$ **(c)** $\frac{1}{3} < x \leq 4$
(d) $-6 \leq x < 15$
6(a) $x > -10$ **(b)** $x \leq 4$ **(c)** $x \geq -1$ **(d)** $x < -4\frac{2}{3}$
7 $5x - 4 < 7 - \frac{1}{2}x$, with solution $x < 2$
8(a)


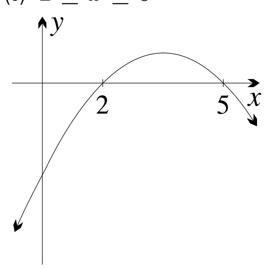
(b) $-1 \leq x < 2$. The solution to the inequation is where the diagonal line lies between the horizontal lines.

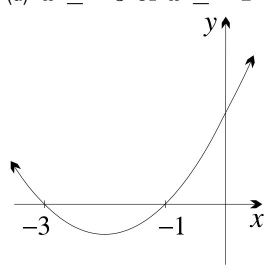
9(a) $-4 \leq 4t < 12$ **(b)** $-3 < -t \leq 1$
(c) $6 \leq t + 7 < 10$ **(d)** $-3 \leq 2t - 1 < 5$
(e) $0 \leq \frac{1}{2}(t + 1) < 2$ **(f)** $-2 \leq \frac{1}{2}(3t - 1) < 4$
10(a) false: $x = 0$ **(b)** false: $x = \frac{1}{2}$ **(c)** true
(d) false: $x = \frac{1}{2}$ or $x = -2$ **(e)** false: $x = -1$
(f) true **(g)** false: $x = -1$ **(h)** true
11(a) true **(b)** false: $a = -2$, $b = -1$ **(c)** true
(d) false: $a = -1$, $b = 1$ **(e)** true
(f) false: $a = 1$, $b = 2$

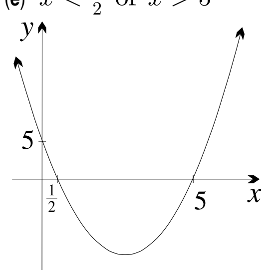
Exercise 4B (Page 69)

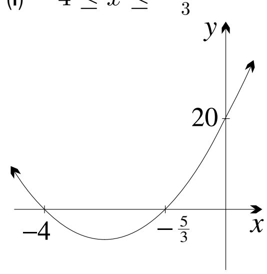
1(a)(i) $-1 < x < 1$ **(ii)** $x < -1$ or $x > 1$
(b)(i) $2 < x < 5$ **(ii)** $x < 2$ or $x > 5$
(c)(i) $x < -2$ or $x > 1$ **(ii)** $-2 < x < 1$
2(a)(i) $x = 1$ or 3 **(ii)** $x < 1$ or $x > 3$ **(iii)** $1 < x < 3$
(b)(i) $x = -4$ or 2 **(ii)** $x < -4$ or $x > 2$
(iii) $-4 < x < 2$ **(c)(i)** $x = 0$ or 5 **(ii)** $0 < x < 5$
(iii) $x < 0$ or $x > 5$
3(a) $0 < x < 4$ **(b)** $x \leq -1$ or $x \geq 3$
(c) $x \leq 0$ or $x \geq 2$
4(a) $-2 < x < 4$


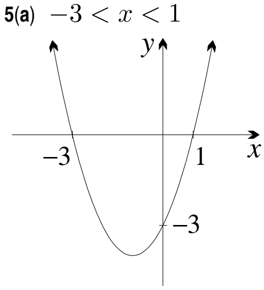
(b) $x < -1$ or $x > 3$


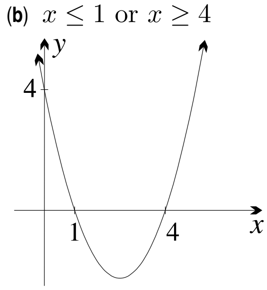
(c) $2 \leq x \leq 5$


(d) $x \leq -3$ or $x \geq -1$


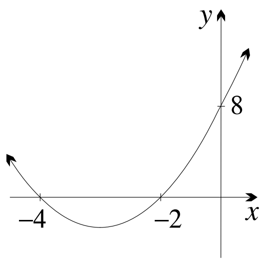
(e) $x < \frac{1}{2}$ or $x > 5$


(f) $-4 \leq x \leq -\frac{5}{3}$


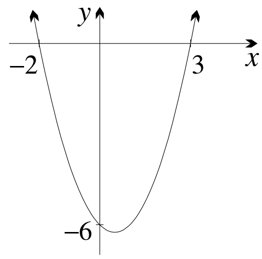
5(a) $-3 < x < 1$


(b) $x \leq 1$ or $x \geq 4$


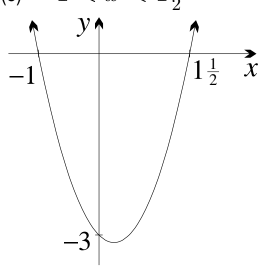
(c) $x < -4$ or $x > -2$



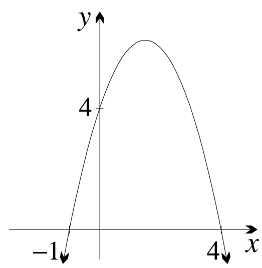
(d) $-2 \le x \le 3$



(e) $-1 < x < 1\frac{1}{2}$



(f) $-1 < x < 4$

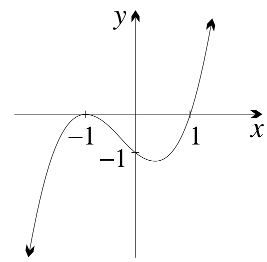


6(a) $-1 \le x \le 1$ **(b)** $x < 0$ or $x > 3$
(c) $x \le -12$ or $x \ge 12$
(d) $x < 0$ or $x > 0$ (or simply $x \neq 0$) **(e)** $x = 3$
(f) $1 \le x \le 3$
7(a) $7 < x^2 + 3 < 19$ **(b)** $3 \le x^2 + 3 \le 12$
8(a) $x \le -2$ or $x \ge 2$ **(b)** $x < -2$ or $x > 2$
9(a) $-2 \le x \le 2$ **(b)** $-2 < x < 2$ **(c)** $-5 \le x \le 5$
(d) $-5 < x < 5$ **(e)** $x \le -2$ or $x \ge 2$
(f) $x < -2$ or $x > 2$

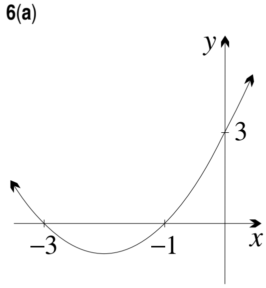
Exercise **4C** (Page 72)

1(a)(i) $x < -1$ or $0 < x < 1$
(ii) $-1 < x < 0$ or $x > 1$
(b)(i) $-5 < x < -2$ or $x > 1$
(ii) $x < -5$ or $-2 < x < 1$
2(a)(i) $x = -3, -1$ or 2 **(ii)** $-3 < x < -1$ or $x > 2$
(iii) $x < -3$ or $-1 < x < 2$ **(b)(i)** $x = -2, 3$ or 6
(ii) $x < -2$ or $3 < x < 6$ **(iii)** $-2 < x < 3$ or $x > 6$
3(a)(i) $x = 0$ or 2 **(ii)** $0 < x < 2$ or $x > 2$
(iii) $x < 0$ **(b)(i)** $x = -2, 0$ or 3
(ii) $x < -2$ or $x > 3$ **(iii)** $-2 < x < 0$ or $0 < x < 3$
4(a) $x \le 0$ or $1 \le x \le 2$
(b) $-2 < x < 0$ or $2 < x < 4$
(c) $0 < x < 3$ or $x > 3$ **(d)** $x = 0$ or $x \ge 4$
(e) $x = -3$ or $x = 3$ **(f)** $x = -3$ or $x \ge 0$

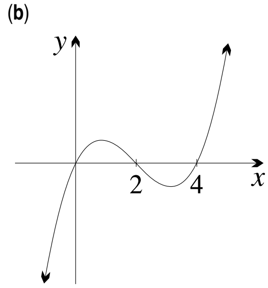
5



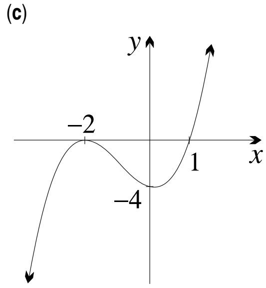
6(a)



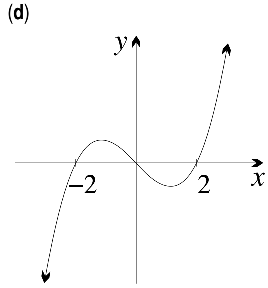
(b)



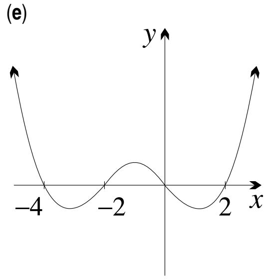
(c)



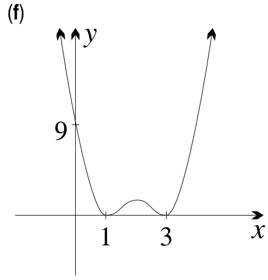
(d)



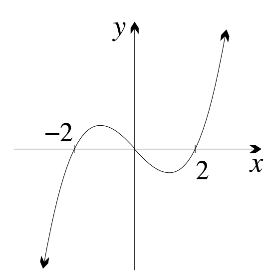
(e)



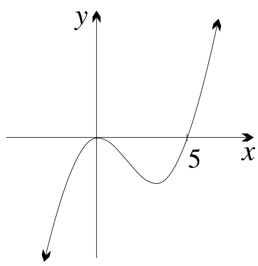
(f)



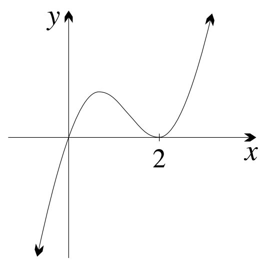
7(a) $f(x) = x(x-2)(x+2)$



(b) $f(x) = x^2(x-5)$



(c) $f(x) = x(x-2)^2$

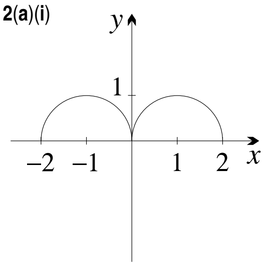


8(a) $-2 < x < 0$ or $x > 2$ **(b)** $x < 0$ or $0 < x < 5$
(c) $x \leq 0$ or $x = 2$
9(a) $x < 1$ or $3 < x < 5$
(b) $-3 \leq x < 1$ or $x > 4$
(c) $x \neq 1$ and $x \neq 3$
(alternatively, $x < 1$ or $1 < x < 3$ or $x > 3$)
(d) $x < -2$ or $0 < x < 2$ or $x > 4$
(e) $-3 < x < 0$ or $x > 3$ **(f)** $x = 0$ or $x \geq 5$

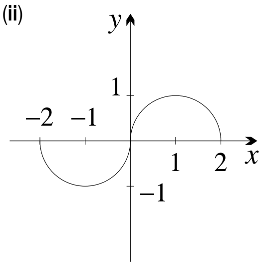
Exercise **4D** (Page 75)

1(a) even **(b)** neither **(c)** odd **(d)** neither **(e)** odd
(f) even

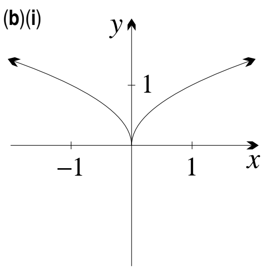
2(a)(i)



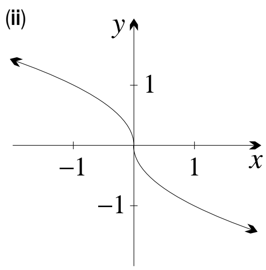
(ii)



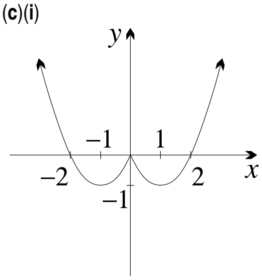
(b)(i)



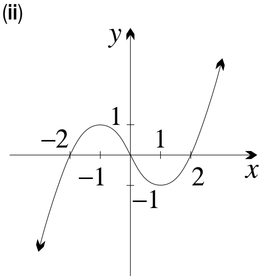
(ii)



(c)(i)

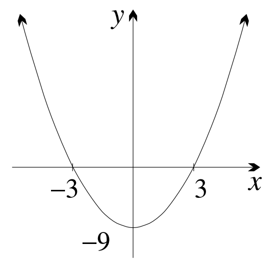


(ii)

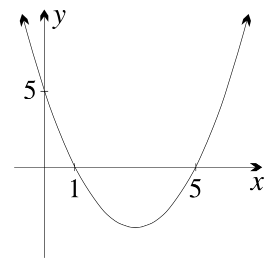


3(a) even **(b)** neither **(c)** odd **(d)** even
(e) neither **(f)** odd **(g)** odd **(h)** neither
4(a) ... if all powers of x are odd.
(b) ... if all powers of x are even.

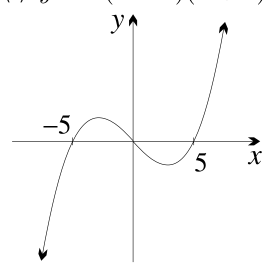
5(a) $y = (x+3)(x-3)$



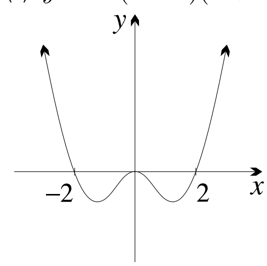
(b) $y = (x-1)(x-5)$



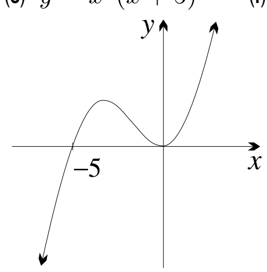
(c) $y = x(x-5)(x+5)$



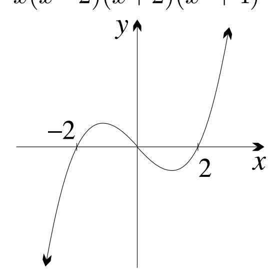
(d) $y = x^2(x-2)(x+2)$



(e) $y = x^2(x+5)$



(f) $y = x(x-2)(x+2)(x^2+4)$

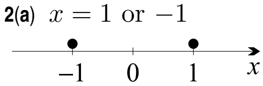


6(a) neither **(b)** neither **(c)** even **(d)** even **(e)** odd
(f) even **(g)** odd **(h)** neither
8(a) Suppose $f(0) = c$. Then since $f(x)$ is odd, $f(0) = -f(0) = -c$. So $c = -c$, and hence $c = 0$.
(b) No. A counter-example is $y = x^2 + 1$.

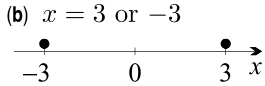
Exercise **4E** (Page 80)

1(a) 5 **(b)** 3 **(c)** 3 **(d)** 3 **(e)** 7 **(f)** 1 **(g)** 16 **(h)** -3

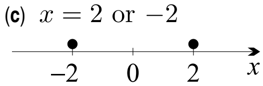
2(a) $x = 1$ or -1



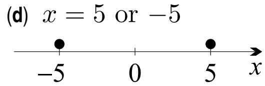
(b) $x = 3$ or -3



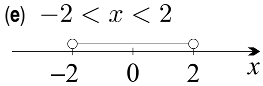
(c) $x = 2$ or -2



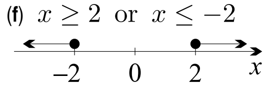
(d) $x = 5$ or -5



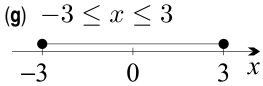
(e) $-2 < x < 2$



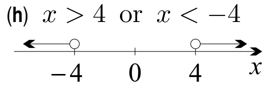
(f) $x \geq 2$ or $x \leq -2$



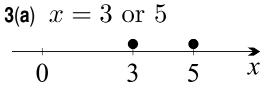
(g) $-3 \leq x \leq 3$



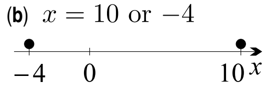
(h) $x > 4$ or $x < -4$



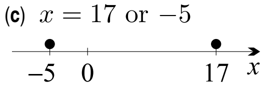
3(a) $x = 3$ or 5



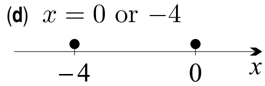
(b) $x = 10$ or -4



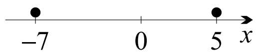
(c) $x = 17$ or -5



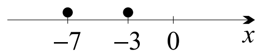
(d) $x = 0$ or -4



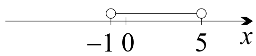
(e) $x = 5$ or -7



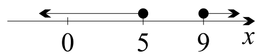
(f) $x = -3$ or -7



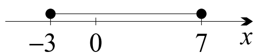
(g) $-1 < x < 5$



(h) $x \geq 9$ or $x \leq 5$

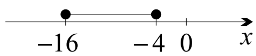


(i) $-3 \leq x \leq 7$



(j) $x > 1$ or $x < -7$

(k) $-16 \leq x \leq -4$

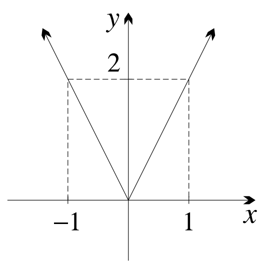


(l) $x \leq -12$ or $x \geq 0$

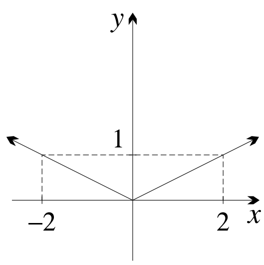
4(a) For $|x-2|$: 3, 2, 1, 0, 1.
For $|x| - 2$: $-1, -2, -1, 0, 1$.
(b) The first is $y = |x|$ shifted right 2 units, the second is $y = |x|$ shifted down 2 units.

7(a) $x = 5$ or -5 **(b)** $x = 6$ or -5
(c) $x = 2$ or $-\frac{8}{7}$ **(d)** $x = 2$ or $-3\frac{1}{3}$
(e) $x = \frac{7}{5}$ or $-\frac{11}{5}$ **(f)** $\frac{1}{3} \leq x \leq 3$
(g) $x > 2$ or $x < \frac{1}{3}$ **(h)** $-2 < x < 1$
(i) $x \geq \frac{2}{5}$ or $x \leq -2$

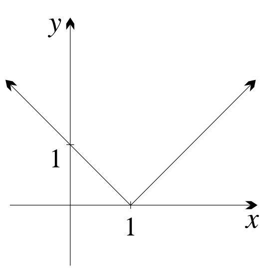
8(a)
$$y = \begin{cases} 2x, & \text{for } x \geq 0, \\ -2x, & \text{for } x < 0. \end{cases}$$



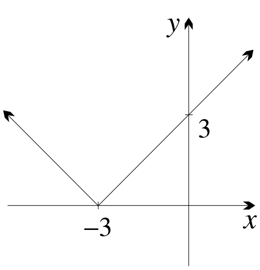
(b)
$$y = \begin{cases} \frac{1}{2}x, & \text{for } x \geq 0, \\ -\frac{1}{2}x, & \text{for } x < 0. \end{cases}$$



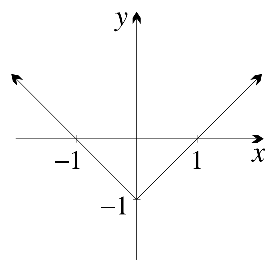
(c) $y =$
$$\begin{cases} x - 1, & \text{for } x \geq 1, \\ 1 - x, & \text{for } x < 1. \end{cases}$$



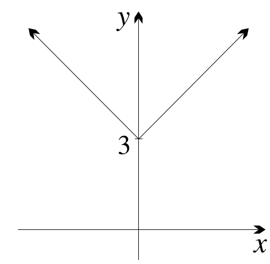
(d) $y =$
$$\begin{cases} x + 3, & \text{for } x \geq -3, \\ -x - 3, & \text{for } x < -3. \end{cases}$$



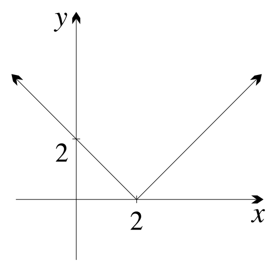
(e) $y =$
$$\begin{cases} x - 1, & \text{for } x \geq 0, \\ -x - 1, & \text{for } x < 0. \end{cases}$$



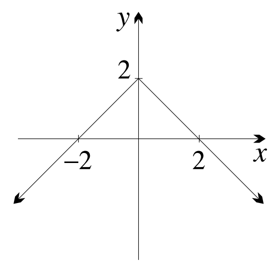
(f) $y =$
$$\begin{cases} x + 3, & \text{for } x \geq 0, \\ 3 - x, & \text{for } x < 0. \end{cases}$$



(g) $y =$
$$\begin{cases} x - 2, & \text{for } x \geq 2, \\ 2 - x, & \text{for } x < 2. \end{cases}$$

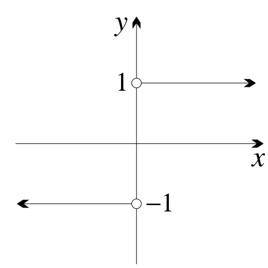


(h) $y =$
$$\begin{cases} 2 - x, & \text{for } x \geq 0, \\ 2 + x, & \text{for } x < 0. \end{cases}$$



9(a) $x = 0$

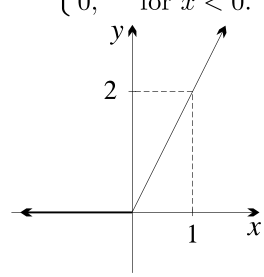
(b)



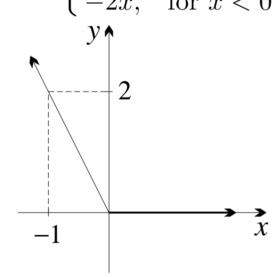
(c) $y = \begin{cases} 1, & \text{for } x > 0, \\ -1, & \text{for } x < 0. \end{cases}$

10(a) false: $x = 2$ and $y = -2$ **(b)** true
(c) false: $x = 2$ and $y = -2$ **(d)** true
(e) true **(f)** false: $x = -2$

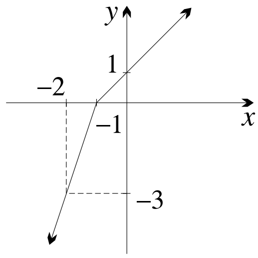
11(a)
$$y = \begin{cases} 2x, & \text{for } x \geq 0, \\ 0, & \text{for } x < 0. \end{cases}$$



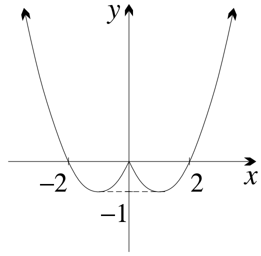
(b)
$$y = \begin{cases} 0, & \text{for } x \geq 0, \\ -2x, & \text{for } x < 0. \end{cases}$$



(c) $y = \begin{cases} x+1, & \text{for } x \ge -1, \\ 3x+3, & \text{for } x < -1. \end{cases}$



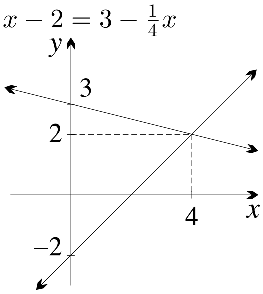
(d) $y = \begin{cases} x^2-2x, & \text{for } x \ge 0, \\ x^2+2x, & \text{for } x < 0. \end{cases}$



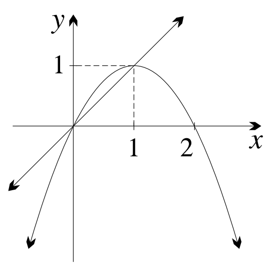
Exercise 4F (Page 83)

1(a)(i) $x > 1$ **(ii)** $x < 1$ **(b)(i)** $x < -3$ or $x > 2$
(ii) $-3 < x < 2$
2(a)(i) $x = 0$ or 3 **(ii)** $0 < x < 3$
(iii) $x < 0$ or $x > 3$ **(b)(i)** $x = -2$ or 1
(ii) $x < -2$ or $x > 1$ **(iii)** $-2 < x < 1$
3(a) $x \le -3$ **(b)** $0 \le x \le 2$ **(c)** $x = 1$
4(a) $x < -2$ or $x > 1$ **(b)** $0 \le x \le 1$
(c) $-1 < x < 0$ or $x > 1$
5(a) $\sqrt{2} \doteqdot 1{\cdot}4$, $\sqrt{3} \doteqdot 1{\cdot}7$ **(b)** $x = -1$ or $x = 2$
(c) $x < -1$ or $x > 2$
(d) $x = -2$ or $x = 1$, $-2 \le x \le 1$
(e) $x \doteqdot 1{\cdot}62$ or $x \doteqdot -0{\cdot}62$
(f)(i) Draw $y = -x$; $x = 0$ or $x = -1$.
(ii) Draw $y = x + \frac{1}{2}$; $x \doteqdot 1{\cdot}37$ or $x \doteqdot -0{\cdot}37$.
(iii) Draw $y = \frac{1}{2}x + \frac{1}{2}$; $x = 1$ or $x = -\frac{1}{2}$.

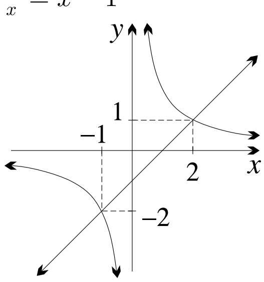
6(a) $(4, 2)$, $x - 2 = 3 - \frac{1}{4}x$



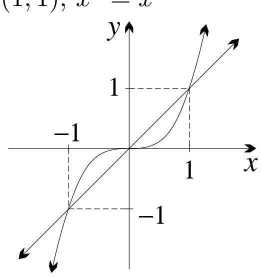
(b) $(0,0)$ and $(1,1)$, $x = 2x - x^2$



(c) $(-1,-2)$ and $(2,1)$, $\frac{2}{x} = x - 1$

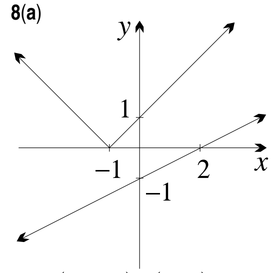


(d) $(-1,-1)$, $(0,0)$ and $(1,1)$, $x^3 = x$



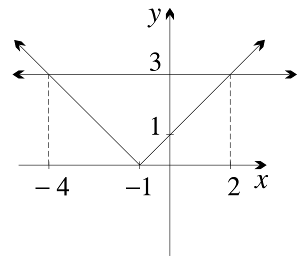
7(a) $x \ge 4$ **(b)** $0 < x < 1$ **(c)** $x < -1$ or $0 < x < 2$
(d) $-1 < x < 0$ or $x > 1$

8(a)

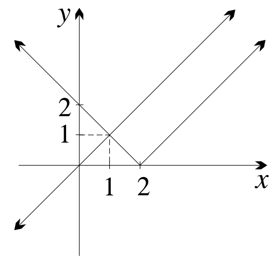


(b) The graph of $y = |x+1|$ is always above the graph of $y = \frac{1}{2}x - 1$.

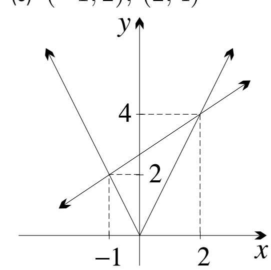
9(a) $(-4, 3)$, $(2, 3)$



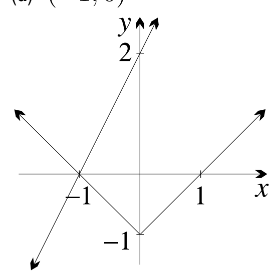
(b) $(1, 1)$



(c) $(-1, 2)$, $(2, 4)$

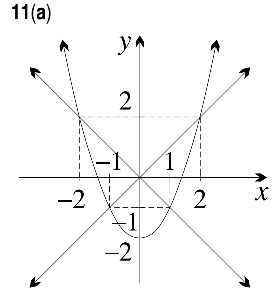


(d) $(-1, 0)$



10(a) $-4 \le x \le 2$ **(b)** $x < 1$ **(c)** $x \le -1$ or $x \ge 2$
(d) $x < -1$

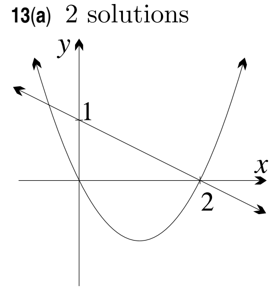
11(a)



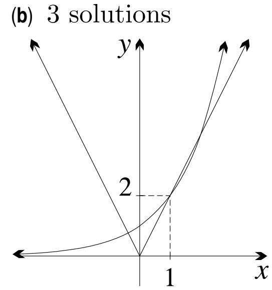
(b) $x = 2$ or -2
(c) $x < -2$ or $x > 2$

12(b) The right-hand branch is $y = x$, which gives the solution $x = 3$, and the left-hand branch is $y = -x$, which gives the solution $x = -3$.
(c) $-3 \le x \le 3$

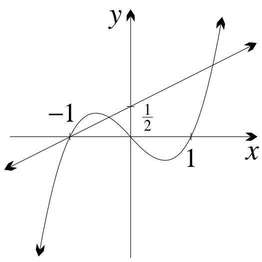
13(a) 2 solutions



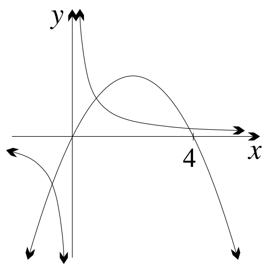
(b) 3 solutions



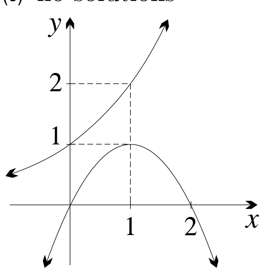
(c) 3 solutions



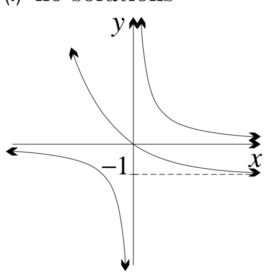
(d) 3 solutions



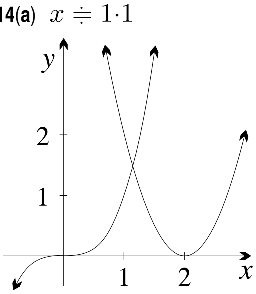
(e) no solutions



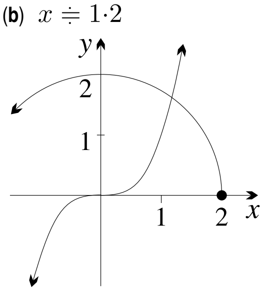
(f) no solutions



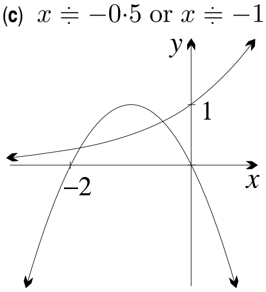
14(a) $x \doteqdot 1{\cdot}1$



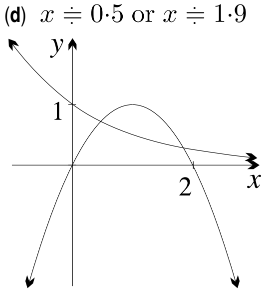
(b) $x \doteqdot 1{\cdot}2$



(c) $x \doteqdot -0{\cdot}5$ or $x \doteqdot -1{\cdot}9$

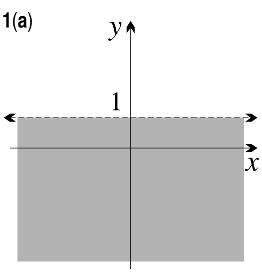


(d) $x \doteqdot 0{\cdot}5$ or $x \doteqdot 1{\cdot}9$

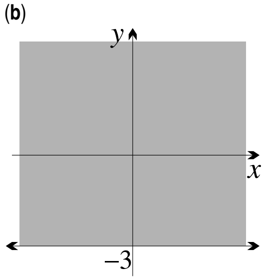


Exercise **4G** (Page 88)

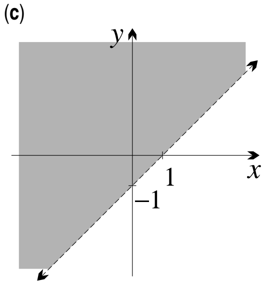
1(a)



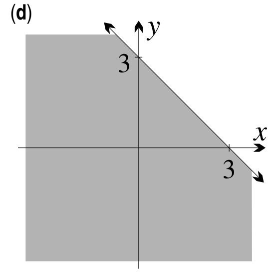
(b)



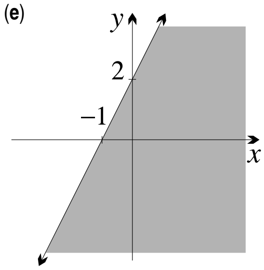
(c)



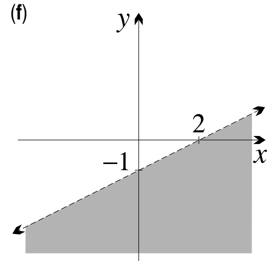
(d)



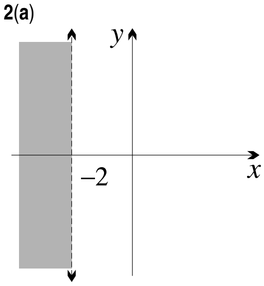
(e)



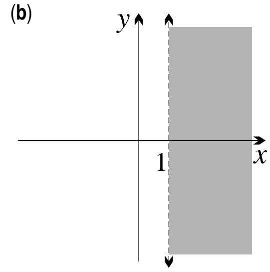
(f)



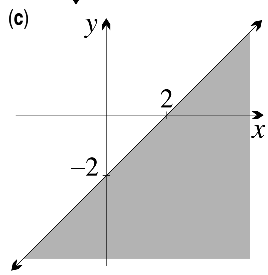
2(a)



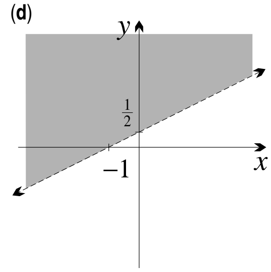
(b)



(c)



(d)



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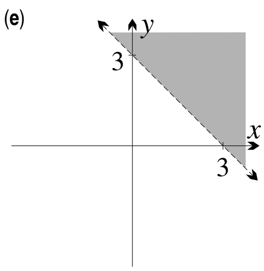

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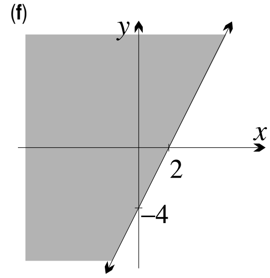
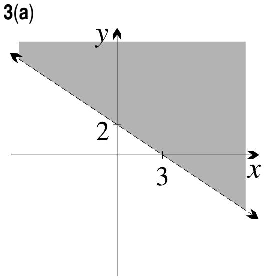


3(a)



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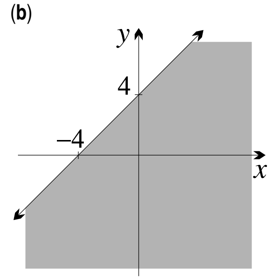

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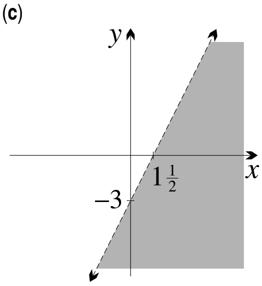
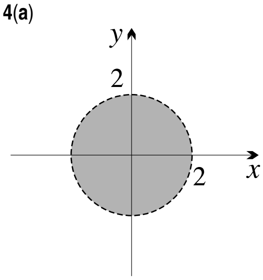


4(a)



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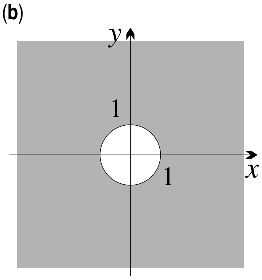
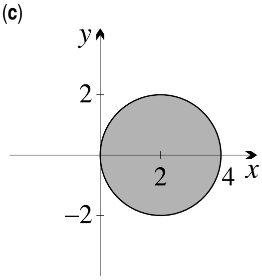
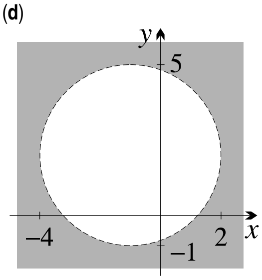


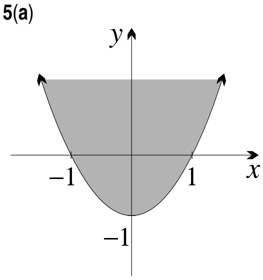
(c)



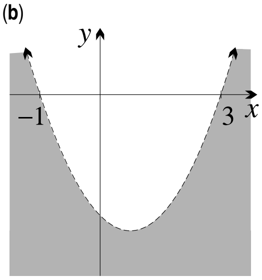
(d)



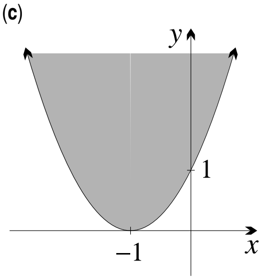
5(a)



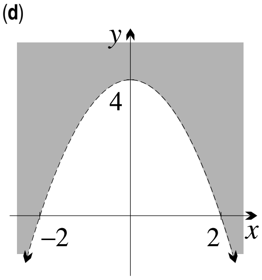
(b)



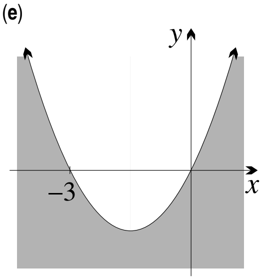
(c)



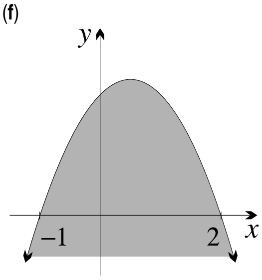
(d)



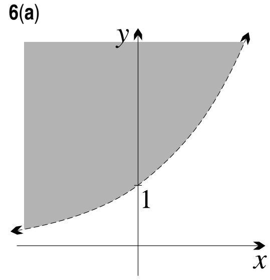
(e)



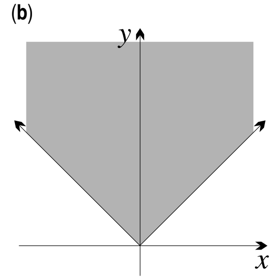
(f)



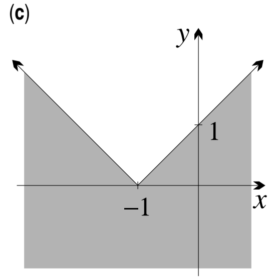
6(a)



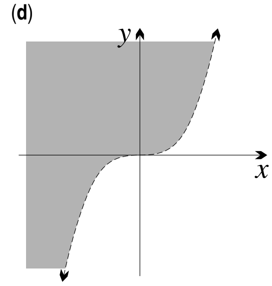
(b)



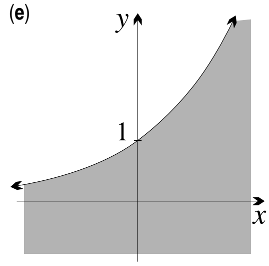
(c)



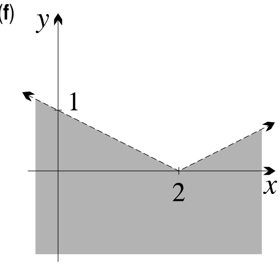
(d)



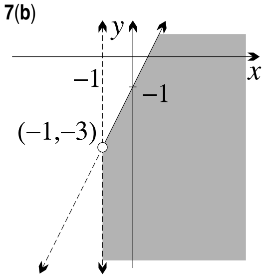
(e)



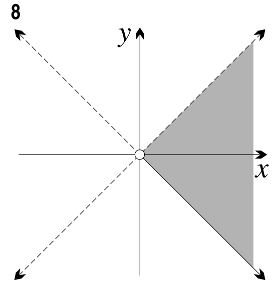
(f)



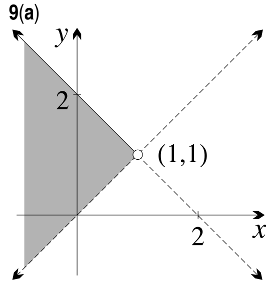
7(b)



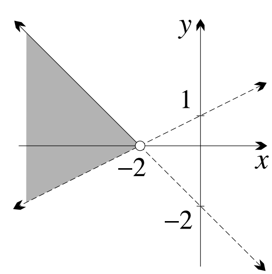
8



9(a)

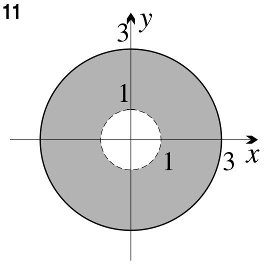


(b)

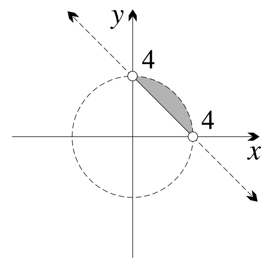


10(a) $x \geq 0$ and $y \geq 0$ **(b)** $x \leq 0$ and $y \geq 0$
(c) $x \leq 0$ and $y \leq 0$ **(d)** $x \geq 0$ and $y \leq 0$

11

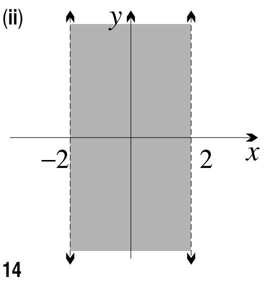


12(b)

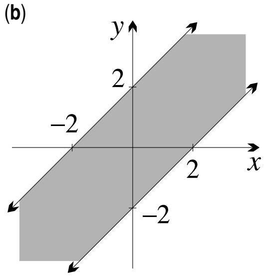


13(a)(i) $x < 2$ & $x > -2$

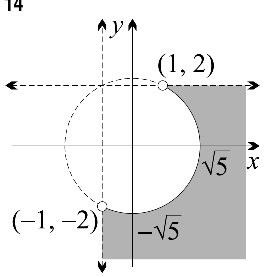
(ii)



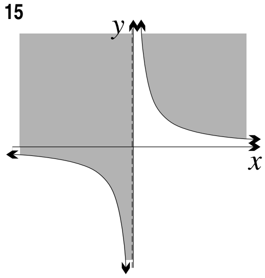
(b)



14

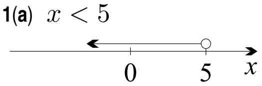


15

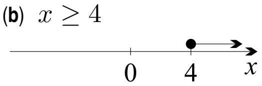


Review Exercise **4H** (Page 89)

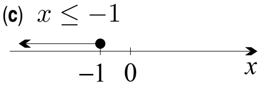
1(a) $x < 5$



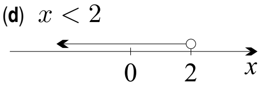
(b) $x \geq 4$



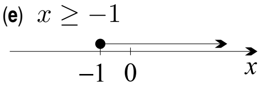
(c) $x \leq -1$



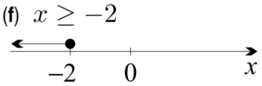
(d) $x < 2$



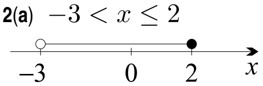
(e) $x \geq -1$



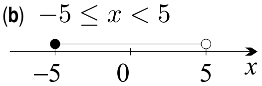
(f) $x \geq -2$



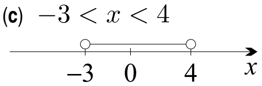
2(a) $-3 < x \leq 2$



(b) $-5 \leq x < 5$



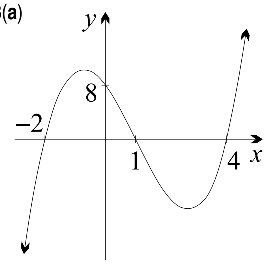
(c) $-3 < x < 4$



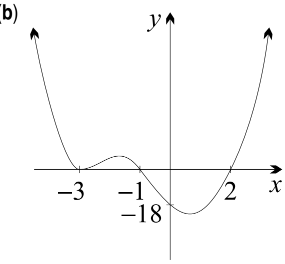
3(a) $x = -1$ or 3 **(b)** $x < -1$ or $x > 3$
(c) $-1 < x < 3$
4(a) $-2 < x < 0$ **(b)** $x < -3$ or $x > 5$
5(a) $-4 < x < 3$ **(b)** $x \leq -5$ or $x \geq 1$
(c) $x < 0$ or $x > 4$ **(d)** $-3 \leq x \leq 3$
(e) $x \leq -2$ or $x \geq 6$ **(f)** $-1 < x < \frac{1}{2}$
6(a) $x = -2,\ 0$ or 3 **(b)** $-2 < x < 0$ or $x > 3$
(c) $x < -2$ or $0 < x < 3$

7(a) $x < -3$ or $0 < x < 2$ **(b)** $x < -4$ or $x > 3$

8(a)

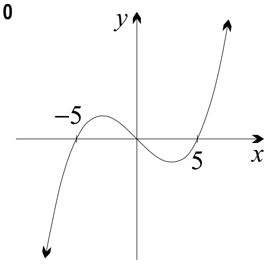


(b)



9(a) $x \leq -2$ or $1 \leq x \leq 4$ **(b)** $x \leq -1$ or $x \geq 2$

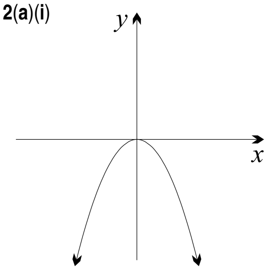
10



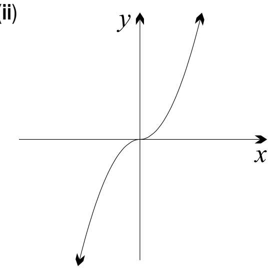
(a) $x(x-5)(x+5)$
(c) $-5 < x < 0$
or $x > 5$

11(a) neither **(b)** odd **(c)** even

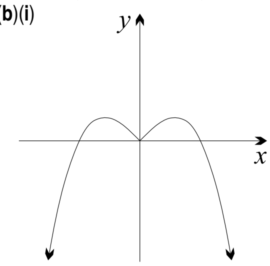
12(a)(i)



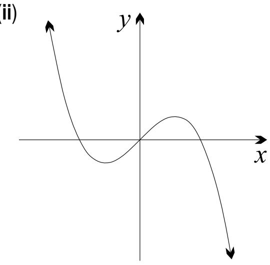
(ii)



(b)(i)

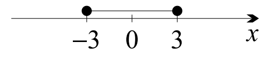


(ii)

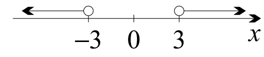


13(a) odd **(b)** even **(c)** neither **(d)** even
14(a) 7 **(b)** 4 **(c)** 5 **(d)** 3 **(e)** -3 **(f)** 12
15(a) $x = 5$ or -5 **(b)** $x = 6$ or -6
(c) $x = 6$ or -2 **(d)** $x = -1$ or -5

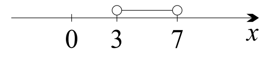
16(a) $-3 \leq x \leq 3$



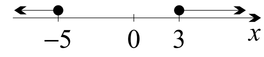
(b) $x > 3$ or $x < -3$



(c) $3 < x < 7$

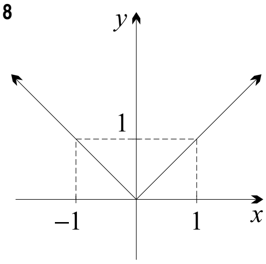


(d) $x \leq -5$ or $x \geq 3$

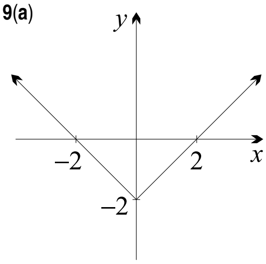


17(a) $x = 4$ or -1 **(b)** $x > 3$ or $x < -4$
(c) $-1 \leq x \leq 3\frac{2}{3}$

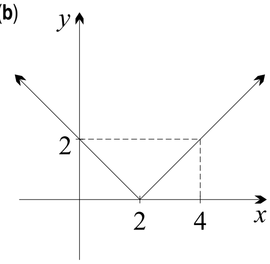
18



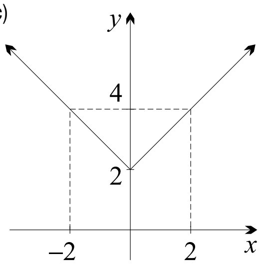
19(a)



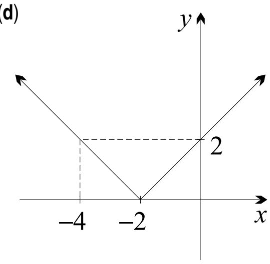
(b)



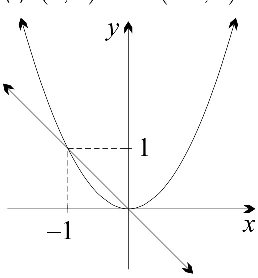
(c)

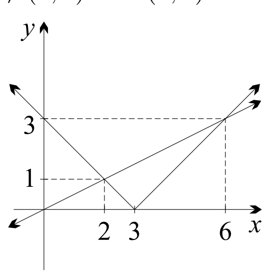


(d)



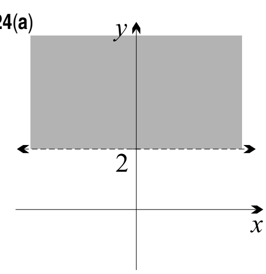
20(a) $x = -1$ or 2 **(b)** $x < -1$ or $x > 2$
(c) $-1 < x < 2$
21(a) $-3 < x < 2$ **(b)** $x < -2$ or $0 < x < 2$
22(a) $(0, 0)$ and $(-1, 1)$ **(b)** $(2, 1)$ and $(6, 3)$



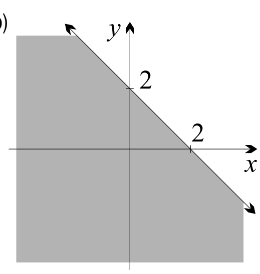


23(a) $x < -1$ or $x > 0$ **(b)** $2 < x < 6$

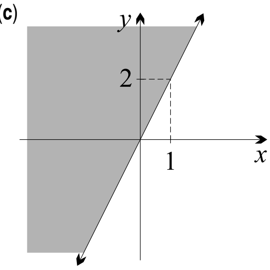
24(a)



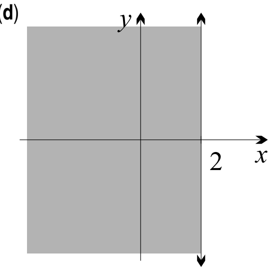
(b)



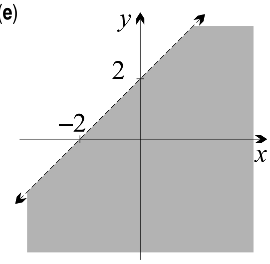
(c)



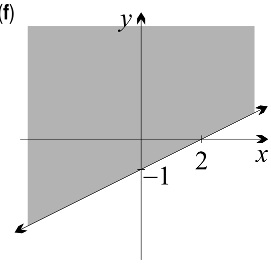
(d)



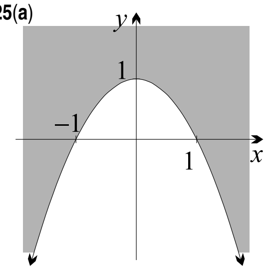
(e)



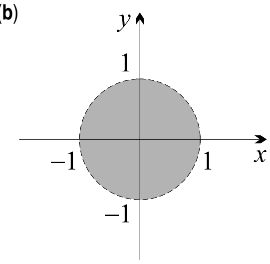
(f)



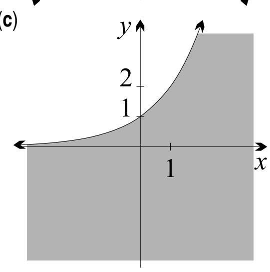
25(a)



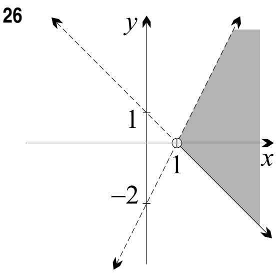
(b)



(c)



26



Chapter Five

Exercise 5A (Page 94)

1(a) $\frac{3}{5}$ **(b)** $\frac{3}{4}$ **(c)** $\frac{4}{5}$ **(d)** $\frac{4}{5}$ **(e)** $\frac{3}{5}$ **(f)** $\frac{4}{3}$
2(a) $0{\cdot}4067$ **(b)** $0{\cdot}4848$ **(c)** $0{\cdot}7002$ **(d)** $0{\cdot}9986$
(e) $0{\cdot}0349$ **(f)** $0{\cdot}8387$ **(g)** $0{\cdot}0175$ **(h)** $0{\cdot}9986$
3(a) $1{\cdot}5697$ **(b)** $0{\cdot}8443$ **(c)** $4{\cdot}9894$ **(d)** $0{\cdot}9571$
(e) $0{\cdot}6833$ **(f)** $0{\cdot}1016$ **(g)** $0{\cdot}0023$ **(h)** $0{\cdot}0166$
4(a) 76° **(b)** 46° **(c)** 12° **(d)** 27° **(e)** 39° **(f)** 60°
5(a) $41^\circ 25'$ **(b)** $63^\circ 26'$ **(c)** $5^\circ 44'$ **(d)** $16^\circ 42'$
(e) $46^\circ 29'$ **(f)** $57^\circ 25'$
6(a) 13 **(b)** 19 **(c)** 23 **(d)** 88
7(a) 53° **(b)** 41° **(c)** 67° **(d)** 59°
8(a) $\frac{12}{13}$ **(b)** $\frac{5}{12}$ **(c)** $\frac{13}{12}$ **(d)** $\frac{5}{12}$ **(e)** $\frac{13}{12}$ **(f)** $\frac{13}{5}$
9(a) 6 and 17 **(b)(i)** $\frac{15}{17}$ **(ii)** $\frac{4}{5}$ **(iii)** $\frac{3}{4}$ **(iv)** $\frac{17}{8}$ **(v)** $\frac{5}{3}$
(vi) $\frac{15}{8}$
10(a) $\frac{\sqrt{3}}{2}$ **(b)** $\frac{1}{\sqrt{3}}$ **(c)** $\frac{1}{\sqrt{2}}$ **(d)** 2 **(e)** $\sqrt{2}$ **(f)** $\sqrt{3}$
11(a) $2{\cdot}9238$ **(b)** $1{\cdot}4945$ **(c)** $0{\cdot}6745$ **(d)** $1{\cdot}0038$
(e) $1{\cdot}8418$ **(f)** $1{\cdot}0435$ **(g)** $2{\cdot}6372$ **(h)** $1{\cdot}0119$
12(a) 71° **(b)** 22° **(c)** 35° **(d)** 81° **(e)** 10° **(f)** 21°
13(a) $59^\circ 2'$ **(b)** $24^\circ 37'$ **(c)** $8^\circ 13'$ **(d)** $77^\circ 3'$
(e) $40^\circ 32'$ **(f)** $75^\circ 24'$
14(a) $19{\cdot}2$ **(b)** $21{\cdot}6$ **(c)** $30{\cdot}3$ **(d)** $8{\cdot}3$
15(a) $29{\cdot}78$ **(b)** $10{\cdot}14$ **(c)** $16{\cdot}46$ **(d)** $29{\cdot}71$
16(a) $36^\circ 2'$ **(b)** $68^\circ 38'$ **(c)** $34^\circ 44'$ **(d)** $38^\circ 40'$
(e) $54^\circ 19'$ **(f)** $70^\circ 32'$
17(a) $b \doteqdot 8{\cdot}452$ **(b)** $\ell \doteqdot 8{\cdot}476$
(c) $s \doteqdot 10{\cdot}534,\ h \doteqdot 17{\cdot}001$
(d) $a \doteqdot 16{\cdot}314,\ b \doteqdot 7{\cdot}607$
18(b) 3 **(c)** $\frac{1}{3}\sqrt{5}$, $\frac{2}{3}$
19(a)(i) $\frac{1}{2}\sqrt{22}$ **(ii)** $\frac{3}{2}\sqrt{2}$
20(a) 1 **(b)** $\frac{1}{2}$ **(c)** 4 **(d)** 1

Exercise 5B (Page 98)

1(a) $DF \doteqdot 2{\cdot}1, EF \doteqdot 4{\cdot}5, \angle D = 65^\circ$
(b) $AC \doteqdot 9{\cdot}5, BC \doteqdot 12{\cdot}4, \angle C = 40^\circ$
(c) $QR \doteqdot 9{\cdot}4, \angle Q \doteqdot 44^\circ, \angle P = 46^\circ$
(d) $XZ \doteqdot 31{\cdot}9, \angle X \doteqdot 58^\circ, \angle Z = 32^\circ$
2 $2{\cdot}65$ metres
3 63°
4 55 km
5 038°T
6 $13{\cdot}2$ metres
7 $2{\cdot}5$ metres
8 77 km
9 23 metres
10(a) 46° **(b)** 101°T
11 73°
12 $21{\cdot}3$ metres
13 11°
14(b) 67 km
15(a) $\angle PQR = 360^\circ - (200^\circ + 70^\circ) = 90^\circ$
(using cointerior angles on parallel lines and the fact that a revolution is 360°)
(b) $110^\circ + 39^\circ = 149^\circ$T
16(a) $5{\cdot}1$ cm **(b)** 16 cm **(c)** $PQ = 18\sin 40^\circ$, $63^\circ 25'$
18(a) $69^\circ 5'$, $69^\circ 5'$ and $41^\circ 51'$ **(b)** $60^\circ 31'$
(c) $64^\circ 1'$ and $115^\circ 59'$
19 457 metres
20 $1{\cdot}58$ nautical miles

Exercise 5C (Page 103)

3(a) -320° **(b)** -250° **(c)** -170° **(d)** -70°
(e) -300° **(f)** -220°
4(a) 310° **(b)** 230° **(c)** 110° **(d)** 10° **(e)** 280°
(f) 170°
5(a) $70^\circ, 430^\circ, -290^\circ, -650^\circ$
(b) $100^\circ, 460^\circ, -260^\circ, -620^\circ$
(c) $140^\circ, 500^\circ, -220^\circ, -580^\circ$
(d) $200^\circ, 560^\circ, -160^\circ, -520^\circ$
(e) $240^\circ, 600^\circ, -120^\circ, -480^\circ$
(f) $340^\circ, 700^\circ, -20^\circ, -380^\circ$
6(a) $\sin\theta = \frac{4}{5}$, $\cos\theta = \frac{3}{5}$, $\tan\theta = \frac{4}{3}$,
$\operatorname{cosec}\theta = \frac{5}{4}$, $\sec\theta = \frac{5}{3}$, $\cot\theta = \frac{3}{4}$
(b) $\sin\theta = \frac{3}{5}$, $\cos\theta = -\frac{4}{5}$, $\tan\theta = -\frac{3}{4}$,
$\operatorname{cosec}\theta = \frac{5}{3}$, $\sec\theta = -\frac{5}{4}$, $\cot\theta = -\frac{4}{3}$
(c) $\sin\theta = -\frac{2}{\sqrt{5}}$, $\cos\theta = -\frac{1}{\sqrt{5}}$, $\tan\theta = 2$,
$\operatorname{cosec}\theta = -\frac{\sqrt{5}}{2}$, $\sec\theta = -\sqrt{5}$, $\cot\theta = \frac{1}{2}$
(d) $\sin\theta = -\frac{5}{13}$, $\cos\theta = \frac{12}{13}$, $\tan\theta = -\frac{5}{12}$,
$\operatorname{cosec}\theta = -\frac{13}{5}$, $\sec\theta = \frac{13}{12}$, $\cot\theta = -\frac{12}{5}$
8(a)(i) $0{\cdot}5$ **(ii)** $-0{\cdot}5$ **(iii)** $0{\cdot}95$ **(iv)** $0{\cdot}95$ **(v)** $0{\cdot}59$
(vi) $0{\cdot}81$ **(vii)** $-0{\cdot}89$ **(viii)** $0{\cdot}45$ **(ix)** $-0{\cdot}81$ **(x)** $0{\cdot}59$
(b)(i) $30^\circ, 150^\circ$ **(ii)** $120^\circ, 240^\circ$
(iii) $64^\circ, 116^\circ$ **(iv)** $53^\circ, 307^\circ$ **(v)** $53^\circ, 127^\circ$
(vi) $143^\circ, 217^\circ$ **(vii)** $204^\circ, 336^\circ$ **(viii)** $107^\circ, 253^\circ$
(c) $45^\circ, 225^\circ$

Exercise **5D** (Page 109)

1(a) $+$ **(b)** $+$ **(c)** $-$ **(d)** $-$ **(e)** $+$ **(f)** $-$ **(g)** $-$ **(h)** $+$ **(i)** $-$ **(j)** $+$ **(k)** $-$ **(l)** $-$ **(m)** $-$ **(n)** $+$ **(o)** $+$ **(p)** $-$
2(a) 10° **(b)** 30° **(c)** 50° **(d)** 20° **(e)** 80° **(f)** 70° **(g)** 70° **(h)** 80° **(i)** 10° **(j)** 20°
3(a) $-\tan 50^\circ$ **(b)** $\cos 50^\circ$ **(c)** $-\sin 40^\circ$ **(d)** $\tan 80^\circ$ **(e)** $-\cos 10^\circ$ **(f)** $-\sin 40^\circ$ **(g)** $-\cos 5^\circ$ **(h)** $\sin 55^\circ$ **(i)** $-\tan 35^\circ$ **(j)** $\sin 85^\circ$ **(k)** $-\cos 85^\circ$ **(l)** $\tan 25^\circ$
4(a) 1 **(b)** -1 **(c)** 0 **(d)** 0 **(e)** 1 **(f)** 1 **(g)** -1 **(h)** undefined **(i)** 0 **(j)** 0 **(k)** undefined **(l)** 0
5(a) $\frac{\sqrt3}{2}$ **(b)** $\frac{\sqrt3}{2}$ **(c)** $-\frac{\sqrt3}{2}$ **(d)** $-\frac{\sqrt3}{2}$ **(e)** $\frac{1}{\sqrt2}$ **(f)** $-\frac{1}{\sqrt2}$ **(g)** $-\frac{1}{\sqrt2}$ **(h)** $\frac{1}{\sqrt2}$ **(i)** $\frac{1}{\sqrt3}$ **(j)** $-\frac{1}{\sqrt3}$ **(k)** $\frac{1}{\sqrt3}$ **(l)** $-\frac{1}{\sqrt3}$
6(a) $-\frac12$ **(b)** 1 **(c)** $-\frac12$ **(d)** $\frac{1}{\sqrt2}$ **(e)** $\sqrt3$ **(f)** $-\frac{\sqrt3}{2}$ **(g)** -1 **(h)** $\frac12$ **(i)** $-\frac{1}{\sqrt2}$ **(j)** $-\frac{\sqrt3}{2}$ **(k)** $-\frac12$ **(l)** $-\sqrt3$
7(a) 2 **(b)** $-\sqrt2$ **(c)** $-\frac{1}{\sqrt3}$ **(d)** $\sqrt3$ **(e)** $\frac{2}{\sqrt3}$ **(f)** $-\frac{2}{\sqrt3}$
8(a) 1 **(b)** -1 **(c)** undefined **(d)** undefined **(e)** 0 **(f)** undefined
9(a) 60° **(b)** 20° **(c)** 30° **(d)** 60° **(e)** 70° **(f)** 10° **(g)** 50° **(h)** 40°
10(a) $\frac12$ **(b)** $-\frac{\sqrt3}{2}$ **(c)** $\sqrt3$ **(d)** $\frac{1}{\sqrt2}$ **(e)** $-\frac{1}{\sqrt3}$ **(f)** $-\frac{1}{\sqrt2}$ **(g)** $\sqrt3$ **(h)** $-\frac{\sqrt3}{2}$ **(i)** $\frac{1}{\sqrt2}$ **(j)** $-\frac12$ **(k)** $-\frac12$ **(l)** 1
11(a) $0{\cdot}42$ **(b)** $-0{\cdot}91$ **(c)** $0{\cdot}91$ **(d)** $-0{\cdot}42$ **(e)** $0{\cdot}49$ **(f)** $0{\cdot}49$
12(a) $-0{\cdot}70$ **(b)** $-1{\cdot}22$ **(c)** $-0{\cdot}70$ **(d)** $-0{\cdot}52$ **(e)** $1{\cdot}92$ **(f)** $-0{\cdot}52$
14(a) $-\sin A$ **(b)** $\cos A$ **(c)** $-\tan A$ **(d)** $\sec A$ **(e)** $\sin A$ **(f)** $-\sin A$ **(g)** $-\cos A$ **(h)** $\tan A$ **(i)** $-\sec A$ **(j)** $-\operatorname{cosec} A$ **(k)** $-\cot A$ **(l)** $\sec A$

Exercise **5E** (Page 111)

1(a) $\sin\theta = \frac{15}{17}$, $\cos\theta = \frac{8}{17}$, $\tan\theta = \frac{15}{8}$
(b) $\sin\theta = \frac45$, $\cos\theta = -\frac35$, $\tan\theta = -\frac43$
(c) $\sin\theta = -\frac{7}{25}$, $\cos\theta = -\frac{24}{25}$, $\tan\theta = \frac{7}{24}$
(d) $\sin\theta = -\frac{21}{29}$, $\cos\theta = \frac{20}{29}$, $\tan\theta = -\frac{21}{20}$
2(a) $y = 12$, $\sin\alpha = \frac{12}{13}$, $\cos\alpha = \frac{5}{13}$, $\tan\alpha = \frac{12}{5}$
(b) $r = 3$, $\sin\alpha = \frac23$, $\cos\alpha = -\frac{\sqrt5}{3}$, $\tan\alpha = -\frac{2}{\sqrt5}$
(c) $x = -4$, $\sin\alpha = -\frac35$, $\cos\alpha = -\frac45$, $\tan\alpha = \frac34$
(d) $y = -3$, $\sin\alpha = -\frac{3}{\sqrt{13}}$, $\cos\alpha = \frac{2}{\sqrt{13}}$, $\tan\alpha = -\frac32$
3(a)(i) $\sin\theta = -\frac45$ **(ii)** $\tan\theta = -\frac43$
(b)(i) $\sin\theta = \frac{5}{13}$ **(ii)** $\cos\theta = -\frac{12}{13}$
4(a) $\cos\theta = -\frac34$ and $\tan\theta = \frac{\sqrt7}{3}$, or $\cos\theta = \frac34$ and $\tan\theta = -\frac{\sqrt7}{3}$
(b) $\sin\theta = \frac{\sqrt{15}}{4}$ and $\tan\theta = -\sqrt{15}$, or $\sin\theta = -\frac{\sqrt{15}}{4}$ and $\tan\theta = \sqrt{15}$
5 $-\frac34$
6 $-\frac{15}{17}$
7 $\frac{1}{\sqrt{10}}$ or $-\frac{1}{\sqrt{10}}$
8 $\frac{1}{\sqrt5}$ or $-\frac{1}{\sqrt5}$
9 $\frac{3}{2\sqrt2} = \frac{3\sqrt2}{4}$
10 $-\frac{3}{2\sqrt{10}} = -\frac{3\sqrt{10}}{20}$
11 $\frac{\sqrt3}{2}$ or $-\frac{\sqrt3}{2}$
12 $\frac65$ or $-\frac65$
13 $\cos\theta = -\dfrac{\sqrt{q^2-p^2}}{q}$, $\tan\theta = -\dfrac{p}{\sqrt{q^2-p^2}}$
14 $\sin\alpha = \dfrac{k}{\sqrt{1+k^2}}$ or $-\dfrac{k}{\sqrt{1+k^2}}$, $\sec\alpha = \sqrt{1+k^2}$ or $-\sqrt{1+k^2}$
15(b) $\sin x = \dfrac{2t}{1+t^2}$, $\tan x = \dfrac{2t}{1-t^2}$

Exercise **5F** (Page 115)

2(a) $\operatorname{cosec}\theta$ **(b)** $\cot\alpha$ **(c)** $\tan\beta$ **(d)** $\cot\phi$
3(a) 1 **(b)** 1 **(c)** 1
6(a) $\cos^2\alpha$ **(b)** $\sin^2\alpha$ **(c)** $\sin A$ **(d)** $\cos A$
7(a) $\cos^2\theta$ **(b)** $\tan^2\beta$ **(c)** $\cot^2 A$ **(d)** 1
8(a) $\cos\theta$ **(b)** $\operatorname{cosec}\alpha$ **(c)** $\cot\beta$ **(d)** $\tan\phi$
9(a) 1 **(b)** $\sin^2\beta$ **(c)** $\sec^2\phi$ **(d)** 1
10(a) $\cos^2\beta$ **(b)** $\operatorname{cosec}^2\phi$ **(c)** $\cot^2 A$ **(d)** -1

Exercise **5G** (Page 121)

1(a) $\theta = 60^\circ$ or 120° **(b)** $\theta = 30^\circ$ or 150° **(c)** $\theta = 45^\circ$ or 225° **(d)** $\theta = 60^\circ$ or 240° **(e)** $\theta = 135^\circ$ or 225° **(f)** $\theta = 120^\circ$ or 300° **(g)** $\theta = 210^\circ$ or 330° **(h)** $\theta = 150^\circ$ or 210°
2(a) $\theta = 90^\circ$ **(b)** $\theta = 0^\circ$ or 360° **(c)** $\theta = 90^\circ$ or 270° **(d)** $\theta = 180^\circ$ **(e)** $\theta = 0^\circ$ or 180° or 360° **(f)** $\theta = 270^\circ$
3(a) $x \doteqdot 65^\circ$ or 295° **(b)** $x \doteqdot 7^\circ$ or 173° **(c)** $x \doteqdot 82^\circ$ or 262° **(d)** $x \doteqdot 222^\circ$ or 318° **(e)** $x \doteqdot 114^\circ$ or 294° **(f)** $x \doteqdot 140^\circ$ or 220°
4(a) $\alpha \doteqdot 5^\circ 44'$ or $174^\circ 16'$ **(b)** $\alpha \doteqdot 95^\circ 44'$ or $264^\circ 16'$ **(c)** $\alpha = 135^\circ$ or 315° **(d)** $\alpha = 270^\circ$ **(e)** no solutions **(f)** $\alpha = 120^\circ$ or 240° **(g)** $\alpha = 150^\circ$ or 330° **(h)** $\alpha \doteqdot 18^\circ 26'$ or $198^\circ 26'$
5(a) $x \doteqdot -16^\circ 42'$ or $163^\circ 18'$ **(b)** $x = 90^\circ$ or -90° **(c)** $x = 45^\circ$ or -45°

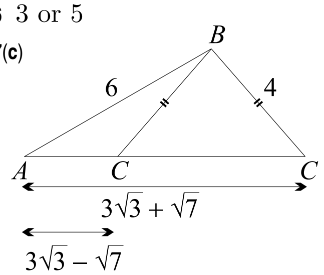
(d) $x \doteqdot -135°34'$ or $-44°26'$
6(a) $\theta = 60°,\ 300°,\ 420°$ or $660°$
(b) $\theta = 90°,\ 270°,\ 450°$ or $630°$
(c) $\theta = 210°,\ 330°,\ 570°$ or $690°$
(d) $\theta = 22°30',\ 202°30',\ 382°30'$ or $562°30'$
7(a) $\theta = 90°$ or $270°$
(b) $\theta = 45°,\ 135°,\ 225°$ or $315°$
(c) $\theta = 60°,\ 120°,\ 240°$ or $300°$
(d) $\theta = 30°,\ 150°,\ 210°$ or $330°$
8(a) $x = 15°,\ 75°,\ 195°$ or $255°$
(b) $x = 30°,\ 120°,\ 210°$ or $300°$
(c) $x = 67°30',\ 112°30',\ 247°30'$ or $292°30'$
(d) $x = 135°$ or $315°$
9(a) $\alpha = 75°$ or $255°$ **(b)** $\alpha = 210°$ or $270°$
(c) $\alpha = 300°$ **(d)** $\alpha = 210°$ or $300°$
10(a) $\theta = 45°$ or $225°$ **(b)** $\theta = 150°$ or $330°$
(c) $\theta = 60°,\ 120°,\ 240°$ or $300°$
(d) $\theta = 45°,\ 135°,\ 225°$ or $315°$
11(a) $\theta = 0°,\ 90°,\ 270°$ or $360°$
(b) $\theta = 30°,\ 90°,\ 210°$ or $270°$
(c) $\theta = 0°,\ 60°,\ 180°,\ 300°$ or $360°$
(d) $\theta = 135°$ or $315°$, or $\theta \doteqdot 63°26'$ or $243°26'$
(e) $\theta = 90°,\ 210°$ or $330°$
(f) $\theta = 60°$ or $300°$, or $\theta \doteqdot 104°29'$ or $255°31'$
(g) $\theta \doteqdot 70°32'$ or $289°28'$
(h) $\theta \doteqdot 23°35',\ 156°25',\ 221°49'$ or $318°11'$
(i) $\theta = 0°,\ 60°,\ 120°,\ 180°,\ 240°,\ 300°$ or $360°$
12(a) $x = 60°,\ 90°,\ 270°$ or $300°$
(b) $x = 135°$ or $315°$, or $x \doteqdot 71°34'$ or $251°34'$
(c) $x = 210°$ or $330°$, or $x \doteqdot 14°29'$ or $165°31'$
(d) $x \doteqdot 48°11'$ or $311°49'$
(e) $x \doteqdot 56°19',\ 116°34',\ 236°19'$ or $296°34'$
13(a) $\alpha = 90°$, or $\alpha \doteqdot 199°28'$ or $340°32'$
(b) $\alpha \doteqdot 63°26',\ 161°34',\ 243°26'$ or $341°34'$
14(a) $0°,\ 180°,\ 360°,\ 135°$ or $315°$
(b) $63°26',\ 243°26',\ 71°34'$ or $251°34'$

Exercise **5H** (Page 126)

1(a) 8·2 **(b)** 4·4 **(c)** 4·9 **(d)** 1·9 **(e)** 9·2 **(f)** 3·5
2(a) 14·72 **(b)** 46·61 **(c)** 5·53
3(a) $49°$ **(b)** $53°$ **(c)** $43°$ **(d)** $20°$ **(e)** $29°$ **(f)** $42°$
4(a) $5\,\text{cm}^2$ **(b)** $19\,\text{cm}^2$ **(c)** $22\,\text{cm}^2$
5(a) 16 metres **(b)** 11·35 metres **(c)** 3·48 metres
6(a) $30°$ or $150°$ **(b)** $17°27'$ or $162°33'$
(c) No solutions, because $\sin\theta = 1{\cdot}2$ is impossible.
7 $42°, 138°$
8 $62°, 118°$
9(b) $b \doteqdot 10{\cdot}80\,\text{cm}$, $c \doteqdot 6{\cdot}46\,\text{cm}$
10(b) 97 cm
11(a) $49°46'$ **(b)** $77°53'$ **(c)** $3{\cdot}70\,\text{cm}^2$
12(a) $69°2'$ or $110°58'$ **(b)** 16·0 cm or 11·0 cm
13(b) Either $B \doteqdot 62°38', C \doteqdot 77°22', c \doteqdot 11{\cdot}5$
or $B \doteqdot 117°22', C \doteqdot 22°38', c \doteqdot 4{\cdot}6$.
14 317 km
15(b) 9 metres
16(b) 32
17(b) $\frac{5}{7}$
18(a) $3\sqrt{6}$ **(b)** $3\sqrt{2}$ **(c)** $2\sqrt{6}$ **(d)** $6\sqrt{2}$
19 11·0 cm
20(a) $\sin\angle BMA = \sin(180° - \theta) = \sin\theta$

Exercise **5I** (Page 130)

1(a) 3·3 **(b)** 4·7 **(c)** 4·0 **(d)** 15·2 **(e)** 21·9 **(f)** 24·6
2(a) $39°$ **(b)** $56°$ **(c)** $76°$ **(d)** $94°$ **(e)** $117°$ **(f)** $128°$
3(a) $\sqrt{13}$ **(b)** $\sqrt{7}$
4(a) $\sqrt{10}$ **(b)** $\sqrt{21}$
5(c) $44°25'$ $101°32'$ $-\frac{7}{32}$
6 11·5 km
7 167 nautical miles
8 $20°$
9(a) $94°48'$ **(b)** $84°33'$
10(a) $101°38'$ **(b)** $78°22'$
12 $13°10', 120°$
13(a) 19 cm **(b)** $\frac{37}{38}$
14 $\cos A = \frac{3}{4}$, $\cos B = \frac{9}{16}$, $\cos C = \frac{1}{8}$
15(a) $\angle DAP = \angle DPA = 60°$ (angle sum of isosceles triangle), so $\triangle ADP$ is equilateral.
Hence $AP = 3\,\text{cm}$. **(b)** $3\sqrt{7}\,\text{cm}$
16 3 or 5
17(c)



Exercise **5J** (Page 134)

1(a) $28{\cdot}3$ **(b)** $17{\cdot}3$ **(c)** $12{\cdot}5$ **(d)** $36{\cdot}2$ **(e)** $12{\cdot}6$ **(f)** $23{\cdot}2$

2(a) 59° **(b)** 55° **(c)** 40° **(d)** 37° **(e)** 52° **(f)** 107°

3(b) 28 metres

4(a) $\angle ACP + 31^\circ = 68^\circ$ (exterior angle of $\triangle ACP$) **(c)** 6 cm

5(a) 11·6 cm **(b)** 49°

6(a) $44^\circ 25'$ **(b)** $10\,\text{cm}^2$

7(b) $C = 50^\circ 24'$, $B = 128^\circ 4'$

8(b) 36 cm

10(a) 26 cm **(b)** 28 cm **(c)** 52° **(d)** 62°

11(a) 8·048 cm **(b)** 16·16 cm **(c)** $51^\circ 24'$ or $128^\circ 36'$ **(d)** $76^\circ 32'$ **(e)** $25{\cdot}35\,\text{cm}^2$ **(f)** $51^\circ 47'$ or $128^\circ 13'$ **(g)** 11·03 cm

12(a) 9·85 metres **(b)** 5·30 metres **(c)** 12·52 metres

13(b) 10·61 metres

14(a) PQ is inclined at 26° to a north–south line through Q, because of alternate angles on parallel lines. Then $\angle PQR = 26^\circ + 90^\circ$. **(b)** 112 nautical miles

15(a) $\dfrac{86 \sin 60^\circ 45'}{\sin 65^\circ 45'}$ **(b)** 66 metres

16(a) $46^\circ 59'$ or $133^\circ 1'$ **(b)** 66·4 metres or 52·7 metres

17(b) 108 km **(c)** $\angle ACB \doteqdot 22^\circ$, bearing $\doteqdot 138^\circ$T

18(a) 42 km **(b)** 078°T

19(c) 34 metres

20(a) $\angle PJK = \angle PBQ = 20^\circ$ (corresponding angles on parallel lines), but $\angle PJK = \angle PAJ + \angle APJ$ (exterior angle of triangle), so $\angle APJ = 20^\circ - 5^\circ = 15^\circ$. **(d)** 53 metres

21(a) $\angle QSM = 36^\circ$ (angle sum of $\triangle QRS$) and $\angle PSM = 48^\circ$ (angle sum of $\triangle PSM$), so $\angle PSQ = 48^\circ - 36^\circ = 12^\circ$ and $\angle SPQ = 24^\circ$. Hence $\angle PQS = 180^\circ - 24^\circ - 12^\circ = 144^\circ$ (angle sum of $\triangle PQS$). **(c)** 473 metres

22(a) $34^\circ 35'$ **(b)** $\angle PDA = \angle ABP$ (base angles of isosceles $\triangle ABD$) and $\angle ABP = \angle PDC$ (alternate angles on parallel lines), so $\angle PDA = \angle PDC$ and $\angle PDC = \frac{1}{2}\angle ADC$. **(c)** $65^\circ 35'$

23 50·4 metres

24 P_1 by 2·5 min

25(b) 12 km **(c)** 9:52 am

26(a) $-\cos\theta$

Review Exercise **5K** (Page 138)

1(a) 0·2924 **(b)** 0·9004 **(c)** 0·6211 **(d)** 0·9904

2(a) $17^\circ 27'$ **(b)** $67^\circ 2'$ **(c)** $75^\circ 31'$ **(d)** $53^\circ 8'$

3(a) 10·71 **(b)** 5·23 **(c)** 10·36 **(d)** 15·63

4(a) $45^\circ 34'$ **(b)** $59^\circ 2'$ **(c)** $58^\circ 43'$ **(d)** $36^\circ 14'$

5(a) $\sqrt{3}$ **(b)** $\frac{1}{\sqrt{2}}$ **(c)** $\frac{\sqrt{3}}{2}$ **(d)** 1 **(e)** 2 **(f)** $\frac{2}{\sqrt{3}}$

6 6·25 metres

7 65°

8(b) 114 km **(c)** 108°T

9 All six trigonometric graphs are drawn just before Exercise 5D.

10(a) $-\cos 55^\circ$ **(b)** $-\sin 48^\circ$ **(c)** $\tan 64^\circ$ **(d)** $\sin 7^\circ$

11(a) $\sqrt{3}$ **(b)** $-\frac{1}{\sqrt{2}}$ **(c)** $\frac{\sqrt{3}}{2}$ **(d)** $-\frac{1}{\sqrt{3}}$

12(a) 0 **(b)** -1 **(c)** undefined **(d)** -1

13(a) $y = 3$, $\sin\theta = \frac{3}{5}$, $\cos\theta = -\frac{4}{5}$, $\tan\theta = -\frac{3}{4}$ **(b)** $x = -2\sqrt{5}$, $\sin\theta = -\frac{\sqrt{5}}{5}$, $\cos\theta = -\frac{2\sqrt{5}}{5}$, $\tan\theta = \frac{1}{2}$

14(a) $\sin\alpha = -\frac{9}{41}$, $\cos\alpha = \frac{40}{41}$ **(b)** $\cos\beta = -\frac{5}{7}$, $\tan\beta = -\frac{2\sqrt{6}}{5}$

15(a) $\sec\theta$ **(b)** $\tan\theta$ **(c)** $\tan\theta$ **(d)** $\cos^2\theta$ **(e)** 1 **(f)** $\cot^2\theta$

17(a) $x = 60^\circ$ or 300° **(b)** $x = 90^\circ$ **(c)** $x = 135^\circ$ or 315° **(d)** $x = 90^\circ$ or 270° **(e)** $x = 30^\circ$ or 210° **(f)** $x = 0^\circ$, 180° or 360° **(g)** $x = 225^\circ$ or 315° **(h)** $x = 150^\circ$ or 210° **(i)** $x = 45^\circ$, 135°, 225° or 315° **(j)** $x = 30^\circ$, 150°, 210° or 330° **(k)** $\tan x = -1$, $x = 135^\circ$ or 315° **(l)** $\sin x = 0$ or -1, $x = 0^\circ$, 180°, 270° or 360°

18(a) 8·5 **(b)** 10·4 **(c)** 7·6 **(d)** 8·9

19(a) $27\,\text{cm}^2$ **(b)** $56\,\text{cm}^2$

20(a) $57^\circ 55'$ **(b)** $48^\circ 33'$ **(c)** $24^\circ 29'$ **(d)** $150^\circ 26'$

21 $28\,\text{cm}^2$

22(a) $\frac{5\sqrt{3}}{3}$ cm **(b)** 30° or 150°

23(b) 48 metres

24(b) 31·5 metres

25(b) 316 nautical miles **(c)** 104°T

Chapter Six

Exercise 6A (Page 145)

1(a) $(2,7)$ **(b)** $(5,6)$ **(c)** $(2,-2)$
(d) $(0,3\frac{1}{2})$ **(e)** $(-5\frac{1}{2},-10)$ **(f)** $(4,0)$
2(a) 5 **(b)** 13 **(c)** 10 **(d)** $\sqrt{8}=2\sqrt{2}$
(e) $\sqrt{80}=4\sqrt{5}$ **(f)** 13
3(a) $M(1,5)$ **(b)** $PM=MQ=5$
4(a) $PQ=QR=\sqrt{17}$, $PR=\sqrt{50}=5\sqrt{2}$
5(a) $AB=\sqrt{58}$, $BC=\sqrt{72}=6\sqrt{2}$, $CA=\sqrt{10}$
(b) AB: $(1\frac{1}{2},1\frac{1}{2})$, BC: $(0,1)$, CA: $(-1\frac{1}{2},4\frac{1}{2})$
6(a) 13 **(b)** $\sqrt{41}$ **(c)** $(5,-3)$ **(d)** 5
7(a) $C(-2,-3)$ **(b)** $B(0,-4)$, $D(-4,-2)$
8(a) $(1,6)$ **(b)** $(1,6)$ **(c)** The diagonals bisect each other. **(d)** parallelogram
9(a) All sides are $5\sqrt{2}$. **(b)** rhombus
10(a) $XY=YZ=\sqrt{52}=2\sqrt{13}$, $ZX=\sqrt{104}=2\sqrt{26}$
(b) $XY^2+YZ^2=104=ZX^2$ **(c)** 26 square units
11(a) Both midpoints are $M(2\frac{1}{2},2\frac{1}{2})$.
(b) It must be a parallelogram, since its diagonals bisect each other. **(c)** $AB=AD=\sqrt{5}$. $ABCD$ is a rhombus, since it is a parallelogram with a pair of adjacent sides equal.
12(a) Each point is $\sqrt{17}$ from the origin.
(b) $\sqrt{17}$, $2\sqrt{17}$, $2\pi\sqrt{17}$, 17π
13 $(5,2)$
14(a) $S(-5,-2)$ **(b)(i)** $P=(4,-14)$
(ii) $P=(-1,-17)$ **(iii)** $P=(7,-7)$
(c) $B=(0,7)$ **(d)** $R=(12,-9)$
15(a) $A(3,5)$ and $B(5,7)$ will do.
(b) $C(0,0)$ and $D(6,8)$ will do.
16(a) ABC is an equilateral triangle.
(b) PQR is a right triangle.
(c) DEF is none of these.
(d) XYZ is an isosceles triangle.
17(a) $(x-5)^2+(y+2)^2=45$
(b) $(x+2)^2+(y-2)^2=74$

Exercise 6B (Page 151)

1(a)(i) 2 **(ii)** -1 **(iii)** $\frac{3}{4}$ **(iv)** $-1\frac{1}{2}$ **(b)(i)** $-\frac{1}{2}$ **(ii)** 1
(iii) $-\frac{4}{3}$ **(iv)** $\frac{2}{3}$
2(a) -1, 1 **(b)** 2, $-\frac{1}{2}$ **(c)** $\frac{1}{2}$, -2 **(d)** $-\frac{1}{2}$, 2
(e) 3, $-\frac{1}{3}$ **(f)** $-\frac{7}{10}$, $\frac{10}{7}$
3(a) 3 **(b)** $\frac{1}{2}$ **(c)** parallelogram
4(a) $m_{AB}=m_{CD}=\frac{1}{2}$, $m_{BC}=m_{DA}=-\frac{1}{5}$.
(b) $m_{AB}=m_{CD}=-\frac{6}{5}$, $m_{BC}=m_{DA}=\frac{3}{11}$.
(c) $m_{AB}=2$, $m_{CD}=-3$
5(a) 0·27 **(b)** $-1{\cdot}00$ **(c)** 0·41 **(d)** 3·08
6(a) 45° **(b)** 120° **(c)** 76° **(d)** 30°
7(a) $m_{AB}=m_{CD}=-\frac{1}{2}$, $m_{BC}=m_{DA}=2$
(b) $m_{AB}=m_{BC}=-1$ **(c)** $AB=BC=2\sqrt{5}$
8 In each case, show that each pair of opposite sides is parallel. **(a)** Show also that two adjacent sides are equal. **(b)** Show also that two adjacent sides are perpendicular. **(c)** Show that it is both a rhombus and a rectangle.
9(a) $m_{WZ}=m_{XY}=-\frac{4}{3}$ $WZ=5$, $XY=10$
(b) trapezium **(c)** $m_{WY}=2$ and $m_{XZ}=-\frac{1}{2}$
10(a) -2, $-\frac{7}{3}$, non-collinear **(b)** $\frac{2}{3}$, $\frac{2}{3}$, collinear
11 The gradients of AB, BC and CD are all $\frac{1}{3}$.
12 $m_{AB}=\frac{1}{2}$, $m_{BC}=-2$ and $m_{AC}=0$, so $AB\perp BC$.
13(a) $m_{PQ}=4$, $m_{QR}=-\frac{1}{4}$ and $m_{PR}=-\frac{5}{3}$, so $PQ\perp QR$. Area $=8\frac{1}{2}$ square units
(b) $m_{XY}=\frac{7}{3}$, $m_{YZ}=\frac{2}{5}$ and $m_{XZ}=-\frac{5}{2}$, so $XZ\perp YZ$. Area $=14\frac{1}{2}$ square units
14(a) $A(0,0)$ and $B(1,3)$ will do.
(b) $A(1,1)$ and $B(1,4)$ will do.
(c) $A(3,4)$ and $B(5,8)$ will do.
15(a) -5 **(b)** 5
16(a) $A(-2,0)$, $B(0,6)$ $m=3$, $\alpha\doteqdot 72^\circ$
(b) $A(2,0)$, $B(0,1)$, $m=-\frac{1}{2}$, $\alpha\doteqdot 153^\circ$
(c) $A(-4,0)$, $B(0,-3)$, $m=-\frac{3}{4}$, $\alpha\doteqdot 143^\circ$
(d) $A(3,0)$, $B(0,-2)$, $m=\frac{2}{3}$, $\alpha\doteqdot 34^\circ$
(e) $A(5,0)$, $B(0,-4)$, $m=\frac{4}{5}$, $\alpha\doteqdot 39^\circ$
(f) $A(2,0)$, $B(0,5)$, $m=-\frac{5}{2}$, $\alpha\doteqdot 112^\circ$
17(a) $P=(2,-1)$, $Q=(-1,4)$, $R=(-3,2)$, $S=(0,-3)$
(b) $m_{PQ}=m_{RS}=-\frac{5}{3}$ and $m_{PS}=m_{QR}=1$
18(a) $P=(3,2)$, $Q=(5,6)$
(b) $m_{PQ}=m_{BC}=2$ and $PQ=2\sqrt{5}=\frac{1}{2}BC$
19(a) They all satisfy the equation, or they all lie 5 units from O.
(b) The centre $O(0,0)$ lies on AB.
(c) $m_{AC}=\frac{1}{2}$, $m_{BC}=-2$
20(a) 3·73 **(b)** 1 **(c)** 2·41 **(d)** 0·32
21 $a=-\frac{1}{2}$
22 $k=2$ or -1

Exercise 6C (Page 155)

1(a) not on the line **(b)** on the line **(c)** on the line
2 Check the points in your answer by substitution. $(0,8)$, $(3,7)$ and $(6,6)$ will do.
3(a) $x=1$, $y=2$ **(b)** $x=-1$, $y=1$
(c) $x=0$, $y=-4$ **(d)** $x=5$, $y=0$
(e) $x=-2$, $y=-3$
4(a) $m=4$, $b=-2$ **(b)** $m=\frac{1}{5}$, $b=-3$
(c) $m=-1$, $b=2$ **(d)** $m=-\frac{5}{7}$, $b=0$
5(a) $y=-3x+5$ **(b)** $y=-3x-\frac{2}{3}$
(c) $y=-3x+17\frac{1}{2}$ **(d)** $y=-3x$
6(a) $y=5x-4$ **(b)** $y=-\frac{2}{3}x-4$
(c) $y=\frac{35}{2}x-4$ **(d)** $y=-4$
7(a) $x-y+3=0$ **(b)** $2x+y-5=0$
(c) $x-5y-5=0$ **(d)** $x+2y-6=0$
8(a) $m=1$, $b=3$ **(b)** $m=-1$, $b=2$
(c) $m=2$, $b=-5$ **(d)** $m=\frac{1}{3}$, $b=0$
(e) $m=-\frac{3}{4}$, $b=\frac{5}{4}$ **(f)** $m=1\frac{1}{2}$, $b=-2$
9(a) $m=1$, $\alpha=45^\circ$ **(b)** $m=-1$, $\alpha=135^\circ$
(c) $m=2$, $\alpha \doteqdot 63^\circ 26'$ **(d)** $m=-\frac{3}{4}$, $\alpha \doteqdot 143^\circ 8'$
10 The sketches required are clear from the intercepts. **(a)** $A(3,0)$, $B(0,5)$ **(b)** $A(-3,0)$, $B(0,6)$
(c) $A(-4,0)$, $B(0,2\frac{2}{5})$
11(a) $y=2x+4$, $2x-y+4=0$
(b) $y=-x$, $x+y=0$
(c) $y=-\frac{1}{3}x-4$, $x+3y+12=0$
(d) $y=-\frac{5}{2}x+11$, $5x+2y-22=0$
12(a) $y=-2x+3$, $y=\frac{1}{2}x+3$
(b) $y=\frac{5}{2}x+3$, $y=-\frac{2}{5}x+3$
(c) $y=-\frac{3}{4}x+3$, $y=\frac{4}{3}x+3$
13(a) -3, $\frac{1}{2}$, -3, $\frac{1}{2}$, parallelogram
(b) $\frac{4}{3}$, $-\frac{3}{4}$, $\frac{4}{3}$, $-\frac{3}{4}$, rectangle
14 The gradients are $\frac{5}{7}$, $\frac{2}{5}$ and $-\frac{7}{5}$, so the first and last are perpendicular.
(a) $A(-3,0)$, $B(0,3)$ **(b)** $A(2,0)$, $B(0,2)$
(c) $A(2\frac{1}{2},0)$, $B(0,-5)$ **(d)** $A(-6,0)$, $B(0,2)$
(e) $A(1\frac{2}{3},0)$, $B(0,1\frac{1}{4})$ **(f)** $A(1\frac{1}{3},0)$, $B(0,-2)$
15(a) $x=3$, $x=0$, $y=-7$, $y=-2$
(b) $y=0$, $y=-4x+12$, $y=2x+12$
16(a) $x-y+3=0$ **(b)** $-\sqrt{3}x+y+1=0$
(c) $x-\sqrt{3}y-2\sqrt{3}=0$ **(d)** $x+y-1=0$
17(a) They are about 61° and 119°.
(b) It is isosceles. (The two interior angles with the x-axis are equal.)
18(a) $k=-\frac{1}{3}$ **(b)** $k=3$

Exercise 6D (Page 158)

1 $3x-y-4=0$
2(a) $6x-y+19=0$ **(b)** $2x+y-3=0$
(c) $2x-3y+25=0$ **(d)** $7x+2y=0$
3(a) $3x+5y-13=0$ **(b)** $3x+5y-18=0$
(c) $3x+5y=0$ **(d)** $3x+5y+20=0$
4(a) $2x-y-1=0$ **(b)** $x+y-4=0$ **(c)** $3x-y+8=0$
(d) $5x+y=0$ **(e)** $x+3y-8=0$ **(f)** $4x+5y+8=0$
5(a) $y=2x+1$ **(b)** $y=-\frac{1}{2}x+6$ **(c)** $y=-5x+18$
(d) $y=\frac{1}{5}x-8$ **(e)** $y=\frac{3}{7}x+9$ **(f)** $y=\frac{5}{2}x+10$
6(a) 3 **(b)** $3x-y-5=0$
7(a) 2, $2x-y-2=0$ **(b)** -2, $2x+y-1=0$
(c) $-\frac{1}{2}$, $x+2y+6=0$ **(d)** $\frac{1}{3}$, $x-3y+13=0$
(e) 2, $2x-y+2=0$ **(f)** $-\frac{3}{2}$, $3x+2y-6=0$
(g) $-\frac{1}{4}$, $x+4y+4=0$ **(h)** 1, $x-y-3=0$
8(a) $-\frac{3}{2}$ **(b)(i)** $3x+2y+1=0$ **(ii)** $2x-3y-8=0$
9(a) $2x-3y+2=0$ **(b)** $2x-3y-9=0$
10(a) $4x-3y-8=0$ **(b)** $4x-3y+11=0$
11(c)(i) No, the first two intersect at $(-4,7)$, which does not lie on the third.
(ii) They all meet at $(5,4)$.
12(a) $y=-2x+5$, $y=\frac{1}{2}x+6$
(b) $y=2\frac{1}{2}x-8\frac{1}{2}$, $y=-\frac{2}{5}x+4\frac{1}{5}$
(c) $y=-1\frac{1}{3}x+3$, $y=\frac{3}{4}x+6\frac{1}{2}$
13(a) $x-y-1=0$ **(b)** $\sqrt{3}x+y+\sqrt{3}=0$
(c) $x-y\sqrt{3}-4-3\sqrt{3}=0$
(d) $x+\sqrt{3}y+2+5\sqrt{3}=0$
14(a)(i) $x-3=0$ **(ii)** $y+1=0$
(b) $3x+2y-6=0$ **(c)(i)** $x-y+4=0$
(ii) $\sqrt{3}x+y-4=0$ **(d)** $x\sqrt{3}+y+6\sqrt{3}=0$
15 $\ell_1 \parallel \ell_2$, and $\ell_3 \parallel \ell_4$, so there are two pairs of parallel sides. The vertices are $(-2,-1)$, $(-4,-7)$, $(1,-2)$, $(3,4)$.
16 $m_{BC} \times m_{AC} = -1$ so $BC \perp AC$.
AB: $y=x-1$, BC: $y=\frac{1}{2}x+2$,
AC: $y=2-2x$
17(a) $m_{AC}=\frac{2}{3}$, $\theta \doteqdot 34^\circ$ **(b)** $2x-3y-2=0$
(c) $D(4,2)$ **(d)** $m_{AC} \times m_{BD} = \frac{2}{3} \times -\frac{3}{2} = -1$, hence they are perpendicular. **(e)** isosceles
(f) area $= \frac{1}{2} \times AC \times BD = \frac{1}{2} \times \sqrt{52} \times \sqrt{52} = 26$
(g) $E(8,-4)$
18(b) $3x-4y-12=0$ **(c)** OB and AC are both vertical and so are parallel; OA and BC both have gradient $\frac{3}{4}$ and so are parallel.
(d) $12\,\text{units}^2$, $AB=2\sqrt{13}$
19(a) $4y=3x+12$ **(b)** $ML=MP=5$ **(c)** $N(4,6)$
(e) $x^2+(y-3)^2=25$

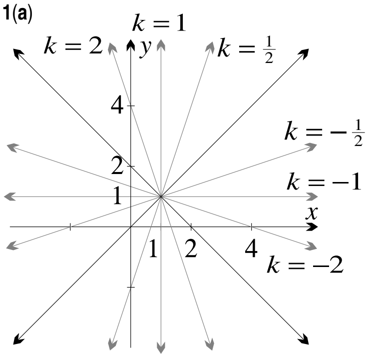
20(a) $(0, 2)$
(d) gradient $= \tan(180° - \theta) = -\tan\theta = -2$ so $2x + y - 6 = 0$. **(e)** $R(3, 0)$, hence area $= 8\,\text{units}^2$.
(f) $QR = 2\sqrt{5}$, $PS = \frac{8}{5}\sqrt{5}$
21 $k = 2\frac{1}{2}$
22(a)(i) $\mu = 4$ **(ii)** $\mu = -9$ **(b)** $\mu \neq 4$
(c)(i) $\lambda = 8$ **(ii)** $\lambda = 0$ or 16

Exercise **6E** (Page 162)

1(a) $\frac{1}{2}\sqrt{10}$ **(b)** $\frac{4}{\sqrt{5}} = \frac{4}{5}\sqrt{5}$ **(c)** $\frac{5}{\sqrt{20}} = \frac{1}{2}\sqrt{5}$
2(a) 1 **(b)** 2 **(c)** $\sqrt{17}$ **(d)** $\frac{1}{2}\sqrt{10}$
(e) 0 (The point is on the line.) **(f)** $\frac{3}{2}\sqrt{5}$
3(a) D is distant $\frac{3}{10}$. **(b)** C is distant $4\frac{1}{10}$.
4(a) ℓ_3 is distant $\frac{3}{5}\sqrt{10}$. **(b)** ℓ_1 is distant $\frac{17}{13}\sqrt{13}$.
5(a) They do not intersect.
(b) Once; the line is tangent to the circle.
(c) Once; the line is tangent to the circle.
(d) They intersect twice.
6(a) $\frac{7}{\sqrt{10}} = \frac{7}{10}\sqrt{10}$ **(b)** $\frac{10}{\sqrt{17}} = \frac{10}{17}\sqrt{17}$
7(a) $k = 15$ or -5 **(b)** $\ell = \frac{1}{2}$ or -1
8(a) $k > -4$ or $k < -6$ **(b)** $-6 \le k \le 4$
9(a) $x - 2y - 1 = 0$ **(b)** $2\sqrt{5}$
(c) $AB = 3\sqrt{5}$ so the area is 15 square units.
(d) 10 square units
10(e) AC is common, $AO = AB$ and both triangles are right-angled, thus they are congruent by the RHS test. **(f)** 50 units^2 **(g)** $2\frac{2}{5}$
11(a) centre $(-2, -3)$ and $r = 2$, distance $\frac{4}{\sqrt{5}}$
(b) $\frac{4}{\sqrt{5}}$
12(a) $y = mx$ **(b)** $\frac{|3m-1|}{\sqrt{m^2+1}}$
(d) $y = \frac{1}{5}(3 + 2\sqrt{6})x$ or $y = \frac{1}{5}(3 - 2\sqrt{6})x$

Exercise **6F** (Page 165)

1(a)



(b) $k = 2$: $3x + y - 4 = 0$, $k = 1$: $x = 1$,
$k = \frac{1}{2}$: $3x - y - 2 = 0$
(c) $k = -\frac{1}{2}$: $x - 3y + 2 = 0$, $k = -1$: $y = 1$,
$k = -2$: $x + 3y - 4 = 0$

2(a) $x + 2y + 9 + k(2x - y + 3) = 0$
(b) $k = -3$ gives $y = x$.
3(a) $2x - 3y + 6 + k(x + 3y - 15) = 0$ **(b)(i)** $x = 3$
(ii) $4x + 3y - 24 = 0$ **(iii)** $x - 6y + 21 = 0$ **(iv)** $3y = 4x$
4(a) $x - 2y - 4 = 0$ **(b)** $2x + y - 3 = 0$ **(c)** $y = x - 3$
5(b) $k = -2$ gives $y = 3$. **(c)** $h = 1$ gives $x = 1$.
(d) $(1, 3)$
6(b)(i) $3x + 4y + 5 = 0$ **(ii)** $3x + 2y + 7 = 0$
(iii) $2y + 5x + 13 = 0$ **(iv)** $x - y + 4 = 0$
7(b)(i) concurrent **(ii)** non-concurrent
8(a) $(1, 1)$ **(c)** $3y + x - 4 = 0$
9(a) $2x - y = 0$
(b) Using $(x + 2y + 10) + k(2x - y) = 0$ yields $k = -1$, hence the line is $3y - x + 10 = 0$.

Exercise **6G** (Page 167)

1(a)(i) $1, -1$ **(ii)** The product of their gradients is -1. **(b)(i)** $1, -1$ **(ii)** The product of their gradients is -1.
2(a)(i) $M = (4, 5)$ **(ii)** $OM = PM = QM = \sqrt{41}$
(iii) OM, PM and QM are three radii of the circle.
(b) $M = (p, q)$, $OM = PM = QM = \sqrt{p^2 + q^2}$
3(a)(ii) $PQ^2 = 5$, $RS^2 = 25$, $PS^2 = 17$, $QR^2 = 13$
(b) $PQ^2 = p^2 + q^2$, $RS^2 = r^2 + s^2$, $PS^2 = p^2 + s^2$, $QR^2 = q^2 + r^2$
4(a)(i) $P(2, 0)$, $Q(0, 2)$
(ii) $m_{PQ} = m_{AC} = -1$ and $AC = 4\sqrt{2}$
(b) $P(a + b, c)$, $Q(b, c)$, $m_{PQ} = m_{AC} = 0$ and $PQ = a$
5(a) $P = \left(\frac{1}{2}(a_1 + b_1), \frac{1}{2}(a_2 + b_2)\right)$,
$Q = \left(\frac{1}{2}(b_1 + c_1), \frac{1}{2}(b_2 + c_2)\right)$,
$R = \left(\frac{1}{2}(c_1 + d_1), \frac{1}{2}(c_2 + d_2)\right)$,
$S = \left(\frac{1}{2}(d_1 + a_1), \frac{1}{2}(d_2 + a_2)\right)$ **(b)** Both midpoints are $\left(\frac{1}{4}(a_1 + a_2 + a_3 + a_4), \frac{1}{4}(b_1 + b_2 + b_3 + b_4)\right)$.
(c) Part (b) shows that its diagonals bisect each other, so (using Box 4) it is a parallelogram.
7(a) $\frac{x}{3} + \frac{y}{4} = 1$ and $4y = 3x$, thus $C = (\frac{48}{25}, \frac{36}{25})$.
(b) $OA = 3$, $AB = 5$, $OC = \frac{12}{5}$, $BC = \frac{16}{5}$, $AC = \frac{9}{5}$
8(a) $AB = BC = CA = 2a$ **(b)** $AB = AD = 2a$
(c) $BD = 2a\sqrt{3}$
9(a) AB and DC have gradient $\frac{b}{a}$; AD and BC have gradient $\frac{d}{c}$. **(b)** Both the midpoints are $(a + c, b + d)$. **(c)** The midpoints coincide.
10(a)(i) $P = (1, 4)$, $Q = (-1, 0)$ and $R = (3, 2)$,
BQ: $x - y + 1 = 0$, CR: $y - 2 = 0$, AP: $x = 1$

(ii) The medians intersect at $(1, 2)$.
(b)(i) $P(-3a, 3c-3b)$, $Q(3a, 3c+3b)$, $R(0, 0)$
(ii) The median passing through B is
$3a(y+6b) = (c+3b)(x+6a)$.
The median passing through A is
$-3a(y-6b) = (c-3b)(x-6a)$.
(iii) The medians intersect at $(0, 2c)$.
11(b) perpendicular bisector of AB: $x = 0$,
of BC: $c(c-y) = (b+a)(x-b+a)$,
of AC: $c(c-y) = (b-a)(x-b-a)$
(c) They all meet at $(0, \frac{c^2+b^2-a^2}{c})$.
(d) Any point on the perpendicular bisector of an interval is equidistant from the endpoints of that interval.

Review Exercise **6H** (Page 170)

1(a) $(8, 6\frac{1}{2})$ **(b)** $-\frac{5}{12}$ **(c)** 13
2(a) $AB = 5$, $BC = \sqrt{2}$, $CA = 5$ **(b)** isosceles
3(a) $P(3, 7)$, $Q(6, 5)$, $R(3, -3)$, $S(0, -1)$
(b) PQ and RS have gradient $-\frac{2}{3}$, QR and SP have gradient $\frac{8}{3}$. **(c)** parallelogram
4(a) $C = (-1, 1)$, $r = \sqrt{45} = 3\sqrt{5}$
(b) $PC = \sqrt{53}$, no
5(a) $m_{LM} = -2$, $m_{MN} = -\frac{8}{9}$, $m_{NL} = \frac{1}{2}$
(b) $m_{LM} \times m_{NL} = -1$
6(a) -1 **(b)** $a = 8$ **(c)** $Q = (7, -4)$
(d) $d^2 = 16$, so $d = 4$ or -4.
7(a) $2x+y-5 = 0$ **(b)** $2x-3y+9 = 0$ **(c)** $x+7y = 0$
(d) $3x+y+8 = 0$ **(e)** $x\sqrt{3} - y - 2 = 0$
8(a) $b = -\frac{7}{6}$, $m = \frac{5}{6}$, $\alpha \doteqdot 39°48'$
(b) $b = \frac{3}{4}$, $m = -1$, $\alpha = 135°$
9(a) $8x - y - 24 = 0$ **(b)** $5x + 2y - 21 = 0$
10(a) No; $m_{LM} = -\frac{1}{3}$ and $m_{MN} = -\frac{5}{12}$.
(b) Yes; they all pass through $(2, 5)$.
11(a) Yes; the 2nd and 3rd lines have gradients $\frac{3}{2}$ and $-\frac{2}{3}$ and so are perpendicular.
(b) Trapezium; the 1st and 3rd lines are parallel.
12(a) $A = (6, 0)$, $B = (0, 7\frac{1}{2})$ **(b)** $22\frac{1}{2}$ square units
13(a) $k = -12$ **(b)** $k = \frac{1}{3}$ **(c)** $k = -27$ **(d)** $k = -18$
(e) $k = \frac{1}{2}$ **(f)** The y-intercept is always -4.
14(a) $m_{AB} = -\frac{3}{4}$, $AB = 10$, $M = (6, 5)$
(c) $C = (15, 17)$ **(d)** $AC = BC = 5\sqrt{10}$ **(e)** $75\,\text{u}^2$
(f) $\sin\theta = \frac{3}{5}$, $\theta \doteqdot 36°52'$
15(a) $\frac{11\sqrt{26}}{26}$ **(b)** $\sqrt{10}$ **(c)** $k = 10\frac{2}{3}$ or $k = \frac{2}{3}$
16(a) $m_{ST} = -\frac{3}{4}$, $3x + 4y + 14 = 0$ **(b)** $\frac{7}{5}$
(c) $ST = 5$, area of $\triangle RST = 3\frac{1}{2}$ square units
17(a) $m_{JK} = m_{ML} = -\frac{4}{3}$ **(b)** $JK = 10$, $ML = 15$
(c) $4x + 3y + 7 = 0$ **(d)** 10 **(e)** 125 square units
18 The general line through M has equation
$(3x - 4y - 5) + k(4x + y + 7) = 0$.
(a) $k = \frac{1}{2}$, $10x - 7y - 3 = 0$
(b) $k = -15$, $57x + 19y + 110 = 0$
(c) $k = 4$, $19x + 23 = 0$ **(d)** $k = -\frac{3}{4}$, $19y + 41 = 0$
19(a) $PO^2 = a^2 + b^2$ by the distance formula, and $PO^2 = r^2$ because P lies on the circle.
(b) $m_{AP} = \dfrac{b}{a-r}$ and $m_{BP} = \dfrac{b}{a+r}$
(d) Use the fact that $a^2 - r^2 = -b^2$.

Chapter Seven

Exercise 7A (Page 177)

1(a) $1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096$ **(b)(i)** $1, 3, 9, 27, 81, 243, 729$ **(ii)** $1, 5, 25, 125, 625, 3125$ **(iii)** $1, 6, 36, 216$ **(iv)** $1, 7, 49, 343$ **(v)** $1, 10, 100, 1000, 10\,000, 100\,000, 1\,000\,000$ **(vi)** $1, 20, 400, 8000, 160\,000, 3\,200\,000, 64\,000\,000$ **(c)(i)** $1, 4, 16, 64, 256, 1024, 4096$ **(ii)** $1, 8, 64, 512, 4096$ **(d)** $1, 9, 81, 729$ **(e)** $1, 25, 625$

2(a) 8 **(b)** 64 **(c)** 81 **(d)** 729 **(e)** $\frac{4}{9}$ **(f)** $\frac{8}{27}$ **(g)** $\frac{81}{10\,000}$ **(h)** $\frac{16}{49}$ **(i)** $\frac{5}{9}$ **(j)** 1

3(a) 1 **(b)** 1 **(c)** $\frac{1}{5}$ **(d)** $\frac{1}{11}$ **(e)** $\frac{1}{36}$ **(f)** $\frac{1}{100}$ **(g)** $\frac{1}{27}$ **(h)** $\frac{1}{125}$ **(i)** $\frac{1}{32}$ **(j)** $\frac{1}{1\,000\,000}$

4(a) 5 **(b)** 11 **(c)** $\frac{7}{2}$ or $3\frac{1}{2}$ **(d)** $\frac{2}{7}$ **(e)** $\frac{4}{3}$ or $1\frac{1}{3}$ **(f)** $\frac{23}{10}$ or $2\frac{3}{10}$ **(g)** $\frac{1}{10}$ or $0{\cdot}1$ **(h)** 10 **(i)** 100 **(j)** 50

5(a) $\frac{1}{25}$ **(b)** 25 **(c)** 125 **(d)** 16 **(e)** $1\,000\,000$ **(f)** $\frac{9}{4}$ **(g)** $\frac{81}{16}$ **(h)** $\frac{16}{81}$ **(i)** $\frac{25}{4}$ **(j)** 1

6(a) 2^{14} **(b)** a^{15} **(c)** 7^{-8} **(d)** x^2 **(e)** $9^0 = 1$ **(f)** $a^0 = 1$ **(g)** 5^{-3} **(h)** 8

7(a) 7^5 **(b)** a^{-2} **(c)** x^{12} **(d)** x^{-12} **(e)** 2^{16} **(f)** 1 **(g)** y^{11} **(h)** y^{-11}

8(a) x^{15} **(b)** x^{15} **(c)** z^{14} **(d)** a^{-6} **(e)** a^{-6} **(f)** 5^{-28} **(g)** y^{10} **(h)** 2^{16}

9(a) $x = 2$ **(b)** $x = 4$ **(c)** $x = 3$ **(d)** $x = 6$ **(e)** $x = -1$ **(f)** $x = -1$ **(g)** $x = -2$ **(h)** $x = -3$ **(i)** $x = -1$ **(j)** $x = -1$ **(k)** $x = 0$ **(l)** $x = 0$

10(a) $9x^2$ **(b)** $125a^3$ **(c)** $64c^6$ **(d)** $81s^4t^4$ **(e)** $49x^2y^2z^2$ **(f)** $\frac{1}{x^5}$ **(g)** $\frac{9}{x^2}$ **(h)** $\frac{y^2}{25}$ **(i)** $\frac{49a^2}{25}$ **(j)** $\frac{27x^3}{8y^3}$

11(a) $\frac{1}{9}$ **(b)** $\frac{1}{x}$ **(c)** $\frac{1}{b^2}$ **(d)** $-\frac{1}{a^4}$ **(e)** $\frac{1}{7x}$ **(f)** $\frac{7}{x}$ **(g)** $-\frac{9}{x}$ **(h)** $\frac{1}{9a^2}$ **(i)** $\frac{3}{a^2}$ **(j)** $\frac{4}{x^3}$

12(a) x^{-1} **(b)** $-x^{-2}$ **(c)** $-12x^{-1}$ **(d)** $9x^{-2}$ **(e)** $-x^{-3}$ **(f)** $12x^{-5}$ **(g)** $7x^{-3}$ **(h)** $-6x^{-1}$ **(i)** $\frac{1}{6}x^{-1}$ **(j)** $-\frac{1}{4}x^{-2}$

13(a) $\frac{2}{3}$ **(b)** $\frac{3}{7}$ **(c)** $\frac{3}{8}$ **(d)** $\frac{4}{25}$ **(e)** $\frac{27}{1000}$ **(f)** $\frac{9}{400}$ **(g)** 5 **(h)** $\frac{5}{12}$ **(i)** $\frac{4}{9}$ **(j)** $\frac{4}{25}$ **(k)** $\frac{8}{125}$ **(l)** 400

14(a) $x = 2$ **(b)** $x = -1$ **(c)** $x = -2$ **(d)** $x = -3$ **(e)** $x = \frac{10}{13}$ **(f)** $x = 2$ **(g)** $x = \frac{1}{3}$ **(h)** $x = \frac{9}{8}$

15(a) x^6y^4 **(b)** $\frac{y}{x^2}$ **(c)** $\frac{21a^3}{x}$ **(d)** $\frac{1}{3st^2}$ **(e)** $\frac{7x}{y^2}$ **(f)** $\frac{5b^{10}}{4a^6}$ **(g)** $\frac{s^6}{y^9}$ **(h)** $\frac{c^2}{5d^3}$ **(i)** $27x^8y^{17}$ **(j)** $\frac{2a^7}{y^{15}}$ **(k)** $5s^5$ **(l)** $\frac{250x^8}{y^{12}}$

16(a) 2^{x+3} **(b)** 3^{x+1} **(c)** 7^{-x} **(d)** 5^{2x-3} **(e)** 10^{6x} **(f)** 5^{-8x} **(g)** 6^{14x} **(h)** 2^{3x-4}

17(a) $x^2 + 2 + \frac{1}{x^2}$ **(b)** $x^2 - 2 + \frac{1}{x^2}$ **(c)** $x^4 - 2 + \frac{1}{x^4}$

18(a) 2^{x+1} **(b)** 2^{x+1} **(c)** 3^{x+1} **(d)** 3^{x+1} **(e)** 2^{x+2} **(f)** 2^{x+5} **(g)** 5^{x+3} **(h)** 3^{x+4} **(i)** 2^{x-1} **(j)** 3^{x-2}

19(a) $x = -1$ **(b)** $x = 6$ **(c)** $x = 8$ **(d)** $x = -1$ **(e)** $x = -4$ **(f)** $x = 2$

Exercise 7B (Page 180)

1(a) 5 **(b)** 6 **(c)** 10 **(d)** 3 **(e)** 4 **(f)** 10 **(g)** 3 **(h)** 2 **(i)** 10 **(j)** 1000

2(a) 125 **(b)** 27 **(c)** 9 **(d)** 4 **(e)** 8 **(f)** 27 **(g)** 81 **(h)** 32 **(i)** 8 **(j)** 16

3(a) $\frac{1}{7}$ **(b)** $\frac{1}{2}$ **(c)** $\frac{5}{7}$ **(d)** $\frac{3}{2}$ **(e)** $\frac{1}{8}$ **(f)** $\frac{1}{125}$ **(g)** $\frac{8}{27}$ **(h)** $\frac{27}{1000}$

4(a) $28\,561$ **(b)** $109{\cdot}5$ **(c)** $1{\cdot}126 \times 10^{15}$ **(d)** 15 **(e)** $2{\cdot}154$ **(f)** $2{\cdot}031$ **(g)** $7{\cdot}225 \times 10^{-11}$ **(h)** $0{\cdot}1969$

5(a) x **(b)** x^6 **(c)** $x^{3\frac{1}{2}}$ **(d)** x **(e)** $x^{\frac{1}{2}}$ **(f)** $x^{-4\frac{1}{2}}$ **(g)** x^2 **(h)** x^{-4} **(i)** x^6

6(a) $2^1 = 2$ **(b)** $2^0 = 1$ **(c)** $2^3 = 8$ **(d)** $3^{-1} = \frac{1}{3}$ **(e)** $25^{\frac{1}{2}} = 5$ **(f)** $7^0 = 1$ **(g)** $3^{-3} = \frac{1}{27}$ **(h)** $3^{-2} = \frac{1}{9}$ **(i)** $9^2 = 81$

7(a) $x = \frac{1}{2}$ **(b)** $x = \frac{1}{2}$ **(c)** $x = \frac{1}{4}$ **(d)** $x = \frac{1}{6}$ **(e)** $x = \frac{1}{2}$ **(f)** $x = \frac{1}{3}$

8(a) $\sqrt{x}$ **(b)** $7\sqrt{x}$ **(c)** $\sqrt{7x}$ **(d)** $\sqrt[3]{x}$ **(e)** $15\sqrt[4]{x}$ **(f)** $\sqrt{x^3}$ or $\left(\sqrt{x}\right)^3$ **(g)** $6\sqrt{x^5}$ or $6\left(\sqrt{x}\right)^5$ **(h)** $\sqrt[3]{x^4}$ or $\left(\sqrt[3]{x}\right)^4$

9(a) $x^{\frac{1}{2}}$ **(b)** $3x^{\frac{1}{2}}$ **(c)** $(3x)^{\frac{1}{2}}$ **(d)** $12x^{\frac{1}{3}}$ **(e)** $9x^{\frac{1}{6}}$ **(f)** $x^{\frac{3}{2}}$ **(g)** $x^{\frac{9}{2}}$ **(h)** $25x^{\frac{6}{5}}$

10(a) $\frac{1}{5}$ **(b)** $\frac{1}{10}$ **(c)** $\frac{1}{5}$ **(d)** $\frac{1}{2}$ **(e)** $\frac{1}{8}$ **(f)** $\frac{1}{9}$ **(g)** $\frac{1}{27}$ **(h)** $\frac{1}{343}$

11(a) 2 **(b)** 5 **(c)** 7 **(d)** 3 **(e)** 8 **(f)** 27 **(g)** $\frac{27}{8}$ **(h)** $\frac{4}{25}$

12(a) $9xy^3$ **(b)** $35b$ **(c)** $3s^{\frac{1}{2}}$ **(d)** $x^{1\frac{1}{2}}y^{2\frac{1}{2}}$ **(e)** a **(f)** $a^{-1}b^2$ **(g)** $2xy^{-2}$ **(h)** p^2q^{-6} **(i)** x^7

13(a) $x^{-\frac{1}{2}}$ **(b)** $12x^{-\frac{1}{2}}$ **(c)** $-5x^{-\frac{1}{2}}$ **(d)** $15x^{-\frac{1}{3}}$ **(e)** $-4x^{-\frac{2}{3}}$ **(f)** $x^{1\frac{1}{2}}$ **(g)** $5x^{-1\frac{1}{2}}$ **(h)** $8x^{2\frac{1}{2}}$

14(a) 9 **(b)** -3 **(c)** $\frac{1}{20}$ **(d)** $\frac{3}{10}$

15(a) $x + 2 + x^{-1}$ **(b)** $x - 2 + x^{-1}$ **(c)** $x^5 - 2 + x^{-5}$

16(a) $x = -\frac{1}{2}$ **(b)** $x = -\frac{1}{4}$ **(c)** $x = \frac{2}{3}$ **(d)** $x = -\frac{2}{3}$ **(e)** $x = \frac{3}{2}$ **(f)** $x = -\frac{3}{2}$ **(g)** $x = \frac{3}{4}$ **(h)** $x = -\frac{4}{3}$ **(i)** $x = -\frac{1}{2}$ **(j)** $x = -\frac{2}{3}$

17(a) $3^{\frac{1}{3}} > 2^{\frac{1}{2}}$ **(b)** $2^{\frac{1}{2}} > 5^{\frac{1}{5}}$ **(c)** $7^{\frac{3}{2}} < 20$ **(d)** $5^{\frac{1}{5}} < 3^{\frac{1}{3}}$

Exercise 7C (Page 184)

1(a) because $2^3 = 8$. **(b)** because $5^2 = 25$. **(c)** because $10^3 = 1000$. **(d)** so $\log_7 49 = 2$. **(e)** so $\log_3 81 = 4$. **(f)** so $\log_{10} 100\,000 = 5$.

2(a) $\ldots\ x = a^y$ **(b)** $\ldots\ x = \log_a y$

3(a) $10^x = 10\,000,\ x = 4$ **(b)** $10^x = 1000,\ x = 3$ **(c)** $10^x = 100,\ x = 2$ **(d)** $10^x = 10,\ x = 1$ **(e)** $10^x = 1,\ x = 0$ **(f)** $10^x = \frac{1}{10},\ x = -1$ **(g)** $10^x = \frac{1}{100},\ x = -2$ **(h)** $10^x = \frac{1}{1000},\ x = -3$

4(a) $3^x = 9$, $x = 2$ **(b)** $5^x = 125$, $x = 3$
(c) $7^x = 49$, $x = 2$ **(d)** $2^x = 64$, $x = 6$
(e) $4^x = 64$, $x = 3$ **(f)** $8^x = 64$, $x = 2$
(g) $8^x = 8$, $x = 1$ **(h)** $8^x = 1$, $x = 0$
(i) $7^x = \frac{1}{7}$, $x = -1$ **(j)** $12^x = \frac{1}{12}$, $x = -1$
(k) $11^x = \frac{1}{121}$, $x = -2$ **(l)** $6^x = \frac{1}{36}$, $x = -2$
(m) $4^x = \frac{1}{64}$, $x = -3$ **(n)** $8^x = \frac{1}{64}$, $x = -2$
(o) $2^x = 64$, $x = -6$ **(p)** $5^x = \frac{1}{125}$, $x = -3$
5(a) $x = 7^2 = 49$ **(b)** $x = 9^2 = 81$
(c) $x = 5^3 = 125$ **(d)** $x = 2^5 = 32$
(e) $x = 4^3 = 64$ **(f)** $x = 100^3 = 1\,000\,000$
(g) $x = 7^1 = 7$ **(h)** $x = 11^0 = 1$
(i) $x = 13^{-1} = \frac{1}{13}$ **(j)** $x = 7^{-1} = \frac{1}{7}$
(k) $x = 10^{-2} = \frac{1}{100}$ **(l)** $x = 12^{-2} = \frac{1}{144}$
(m) $x = 5^{-3} = \frac{1}{125}$ **(n)** $x = 7^{-3} = \frac{1}{343}$
(o) $x = 2^{-5} = \frac{1}{32}$ **(p)** $x = 3^{-4} = \frac{1}{81}$
6(a) $x^2 = 49$, $x = 7$ **(b)** $x^3 = 8$, $x = 2$
(c) $x^3 = 27$, $x = 3$ **(d)** $x^4 = 10\,000$, $x = 10$
(e) $x^2 = 10\,000$, $x = 100$ **(f)** $x^6 = 64$, $x = 2$
(g) $x^2 = 64$, $x = 8$ **(h)** $x^1 = 125$, $x = 125$
(i) $x^1 = 11$, $x = 11$ **(j)** $x^{-1} = \frac{1}{17}$, $x = 17$
(k) $x^{-1} = \frac{1}{6}$, $x = 6$ **(l)** $x^{-1} = \frac{1}{7}$, $x = 7$
(m) $x^{-2} = \frac{1}{9}$, $x = 3$ **(n)** $x^{-2} = \frac{1}{49}$, $x = 7$
(o) $x^{-3} = \frac{1}{8}$, $x = 2$ **(p)** $x^{-2} = \frac{1}{81}$, $x = 9$
7(a) $a^x = a$, $x = 1$ **(b)** $x = a^1 = a$
(c) $x^1 = a$, $x = a$ **(d)** $a^x = \frac{1}{a}$, $x = -1$
(e) $x = a^{-1} = \frac{1}{a}$ **(f)** $x^{-1} = \frac{1}{a}$, $x = a$
(g) $a^x = 1$, $x = 0$ **(h)** $x = a^0 = 1$
(i) $x^0 = 1$, where x can be any positive number.
8(a) 1 **(b)** -1 **(c)** 3 **(d)** -2 **(e)** -5 **(f)** $\frac{1}{2}$ **(g)** $-\frac{1}{2}$
(h) 0
9(a) 1 & 2 **(b)** 2 & 3 **(c)** 0 & 1 **(d)** 3 & 4 **(e)** 5 & 6
(f) 9 & 10 **(g)** -1 & 0 **(h)** -2 & -1
10(a) 1 & 2 **(b)** 0 & 1 **(c)** 3 & 4 **(d)** 0 & 1 **(e)** 3 & 4
(f) 4 & 5 **(g)** 2 & 3 **(h)** 1 & 2 **(i)** -1 & 0
(j) -2 & -1
11(a) 0·301 **(b)** 1·30 **(c)** 2·00 **(d)** 20·0 **(e)** 3·16
(f) 31·6 **(g)** 0·500 **(h)** 1·50 **(i)** 3 **(j)** 6 **(k)** 1000
(l) 1 000 000 **(m)** $-0·155$ **(n)** $-2·15$ **(o)** 0·700
(p) 0·00708
12(a) $7^x = \sqrt{7}$, $x = \frac{1}{2}$ **(b)** $11^x = \sqrt{11}$, $x = \frac{1}{2}$
(c) $x = 9^{\frac{1}{2}} = 3$ **(d)** $x = 144^{\frac{1}{2}} = 12$
(e) $x^{\frac{1}{2}} = 3$, $x = 9$ **(f)** $x^{\frac{1}{2}} = 13$, $x = 169$
(g) $6^x = \sqrt[3]{6}$, $x = \frac{1}{3}$ **(h)** $9^x = 3$, $x = \frac{1}{2}$
(i) $x = 64^{\frac{1}{3}} = 4$ **(j)** $x = 16^{\frac{1}{4}} = 2$
(k) $x^{\frac{1}{3}} = 2$, $x = 8$ **(l)** $x^{\frac{1}{6}} = 2$, $x = 64$
(m) $8^x = 2$, $x = \frac{1}{3}$ **(n)** $125^x = 5$, $x = \frac{1}{3}$
(o) $x = 7^{\frac{1}{2}}$ or $\sqrt{7}$ **(p)** $x = 7^{-\frac{1}{2}}$ or $\frac{1}{\sqrt{7}}$
(q) $x^{-\frac{1}{2}} = \frac{1}{7}$, $x = 49$ **(r)** $x^{-\frac{1}{2}} = \frac{1}{20}$, $x = 400$
(s) $4^x = \frac{1}{2}$, $x = -\frac{1}{2}$ **(t)** $27^x = \frac{1}{3}$, $x = -\frac{1}{3}$
(u) $x = 121^{-\frac{1}{2}} = \frac{1}{11}$ **(v)** $x = 81^{-\frac{1}{4}} = \frac{1}{3}$
(w) $x^{-\frac{1}{4}} = \frac{1}{2}$, $x = 16$ **(x)** $x^{-\frac{1}{4}} = 2$, $x = \frac{1}{16}$

Exercise 7D (Page 187)

1(a) $\log_6 6 = 1$ **(b)** $\log_{15} 15 = 1$ **(c)** $\log_{10} 100 = 2$
(d) $\log_{12} 144 = 2$ **(e)** $\log_{10} 1000 = 3$
(f) $\log_6 36 = 2$
2(a) $\log_3 3 = 1$ **(b)** $\log_4 4 = 1$ **(c)** $\log_2 8 = 3$
(d) $\log_5 25 = 2$ **(e)** $\log_3 81 = 4$ **(f)** $\log_2 32 = 5$
3(a) 1 **(b)** 2 **(c)** 3 **(d)** 2 **(e)** 0 **(f)** -2 **(g)** -3
(h) 2 **(i)** 0
4(a) $3\log_a 2$ **(b)** $4\log_a 2$ **(c)** $6\log_a 2$ **(d)** $-\log_a 2$
(e) $-3\log_a 2$ **(f)** $-5\log_a 2$ **(g)** $\frac{1}{2}\log_a 2$
(h) $-\frac{1}{2}\log_a 2$
5(a) $2\log_2 3$ **(b)** $2\log_2 5$ **(c)** $1+\log_2 3$ **(d)** $1+\log_2 5$
(e) $1 + 2\log_2 3$ **(f)** $2 + \log_2 5$ **(g)** $1 - \log_2 3$
(h) $-1 + \log_2 5$
6(a) 3·90 **(b)** 3·16 **(c)** 4·64 **(d)** 3·32 **(e)** 5·64
(f) 4·58 **(g)** 0·58 **(h)** $-0·74$ **(i)** 0·84 **(j)** $-0·58$
(k) 5·74 **(l)** 6·22
7(a) 2 **(b)** 15 **(c)** -1 **(d)** 6
8(a) $3\log_a x$ **(b)** $-\log_a x$ **(c)** $\frac{1}{2}\log_a x$ **(d)** $-2\log_a x$
(e) $-2\log_a x$ **(f)** $2\log_a x$ **(g)** $8-8\log_a x$ **(h)** $\log_a x$
9(a) $\log_a y + \log_a z$ **(b)** $\log_a z - \log_a y$ **(c)** $4\log_a y$
(d) $-2\log_a x$ **(e)** $\log_a x + 3\log_a y$
(f) $2\log_a x + \log_a y - 3\log_a z$ **(g)** $\frac{1}{2}\log_a y$
(h) $\frac{1}{2}\log_a x + \frac{1}{2}\log_a z$
10(a) 1·30 **(b)** $-0·70$ **(c)** 2·56 **(d)** 0·15 **(e)** 0·45
(f) $-0·50$ **(g)** 0·54 **(h)** $-0·35$
11(a) $6x$ **(b)** $-x-y-z$ **(c)** $3y+5$ **(d)** $2x+2z-1$
(e) $y-x$ **(f)** $x+2y-2z-1$ **(g)** $-2z$ **(h)** $3x-y-z-2$
12(a) 5 **(b)** 7 **(c)** n **(d)** y
13(a) $10 = 3^{\log_3 10}$ **(b)** $3 = 10^{\log_{10} 3}$
(c) $0·1 = 2^{\log_2 0·1}$

Exercise 7E (Page 189)

1(a) 2·807 **(b)** 4·700 **(c)** $-3·837$ **(d)** 7·694
(e) 0·4307 **(f)** 1·765 **(g)** 0·6131 **(h)** 0·2789
(i) $-2·096$ **(j)** $-7·122$ **(k)** 2·881 **(l)** 7·213
(m) 0·03323 **(n)** 578·0 **(o)** $-687·3$
2(a) $x = \log_2 15 \doteqdot 3·907$ **(b)** $x = \log_2 5 \doteqdot 2·322$
(c) $x = \log_2 1·45 \doteqdot 0·5361$
(d) $x = \log_2 0·1 \doteqdot -3·322$
(e) $x = \log_2 0·0007 \doteqdot -10·48$

(f) $x = \log_3 10 \doteqdot 2{\cdot}096$ **(g)** $x = \log_3 0{\cdot}01 \doteqdot -4{\cdot}192$
(h) $x = \log_5 10 \doteqdot 1{\cdot}431$ **(i)** $x = \log_{12} 150 \doteqdot 2{\cdot}016$
(j) $x = \log_8 \frac{7}{9} \doteqdot -0{\cdot}1209$
(k) $x = \log_6 1{\cdot}4 \doteqdot 0{\cdot}1878$ **(l)** $x = \log_{30} 2 \doteqdot 0{\cdot}2038$
(m) $x = \log_{0\cdot7} 0{\cdot}1 \doteqdot 6{\cdot}456$
(n) $x = \log_{0\cdot98} 0{\cdot}03 \doteqdot 173{\cdot}6$
(o) $x = \log_{0\cdot99} 0{\cdot}01 \doteqdot 458{\cdot}2$
3(a) $x > 5$ **(b)** $x \le 5$ **(c)** $x < 6$ **(d)** $x \ge 4$
(e) $x > 1$ **(f)** $x \le 0$ **(g)** $x < -1$ **(h)** $x \le -3$
4(a) $0 < x < 8$ **(b)** $x \ge 8$ **(c)** $x > 1000$ **(d)** $x \ge 10$
(e) $x > 1$ **(f)** $0 < x < 6$ **(g)** $0 < x \le 125$
(h) $x > 36$
5(a) $x > \log_2 12 \doteqdot 3{\cdot}58$ **(b)** $x < \log_2 100 \doteqdot 6{\cdot}64$
(c) $x < \log_2 0{\cdot}02 \doteqdot -5{\cdot}64$
(d) $x > \log_2 0{\cdot}1 \doteqdot -3{\cdot}32$ **(e)** $x < \log_5 100 \doteqdot 2{\cdot}86$
(f) $x < \log_3 0{\cdot}007 \doteqdot -4{\cdot}52$
(g) $x > \log_{1\cdot2} 10 \doteqdot 12{\cdot}6$
(h) $x > \log_{1\cdot001} 100 \doteqdot 4610$
6(a) $x = 3$ **(b)** $x = 2$ **(c)** $x < 1$ **(d)** $x \le 9$
(e) $x = 0$ **(f)** $x = \frac{1}{5}$ **(g)** $x < 4{\cdot}81$ **(h)** $x > -2{\cdot}90$
7(a) $x < 33{\cdot}2$, 33 powers **(b)** $x < 104{\cdot}8$, 104 powers
8(a) $10^2 < 300 < 10^3$ **(b)** $1 \le \log_{10} x < 2$
(c) 5 digits **(d)** 27·96, 28 digits
(e) $1000 \log_{10} 2 = 301{\cdot}03$, 302 digits

Exercise 7F (Page 191)

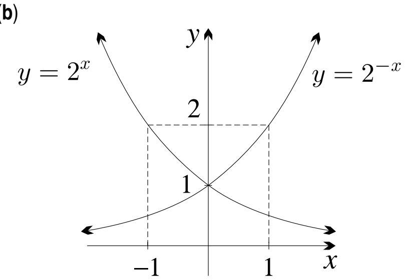
1(a)(i)

x	-3	-2	-1	0	1	2	3
2^x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

(ii)

x	-3	-2	-1	0	1	2	3
2^{-x}	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

(b)



(c) The values of $y = 2^{-x}$ are the values of $y = 2^x$ in reverse order.
(d) The two graphs are reflections of each other in the y-axis.
(e) For both, domain: all real x, range: $y > 0$
(f) For both, the asymptote is $y = 0$ (the x-axis).
(g)(i) 'As $x \to -\infty$, $2^x \to 0$.'
(ii) 'As $x \to \infty$, $2^x \to \infty$.'
(h)(i) 'As $x \to -\infty$, $2^{-x} \to \infty$.'
(ii) 'As $x \to \infty$, $2^{-x} \to 0$.'

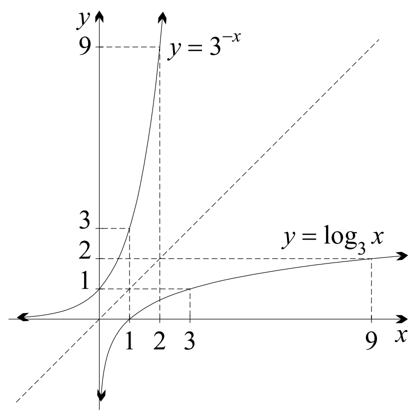
2(a)(i)

x	-2	-1	0	1	2
3^x	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9

(ii)

x	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9
$\log_3 x$	-2	-1	0	1	2

(b)



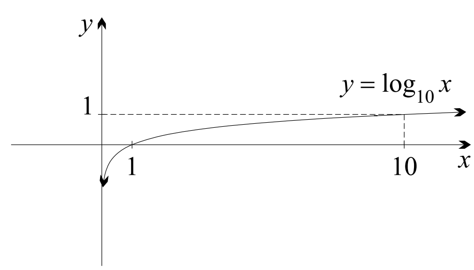
(c) The two rows have been exchanged.
(d) The two graphs are reflections of each other in the diagonal line $y = x$.
(e)(i) domain: all real x, range: $y > 0$
(ii) domain: $x > 0$, range: all real y
(f)(i) $y = 0$ (the x-axis) **(ii)** $x = 0$ (the y-axis)
(g)(i) 'As $x \to -\infty$, $3^x \to 0$'.
(ii) 'As $x \to 0^+$, $\log_3 x \to -\infty$'.

3(a)

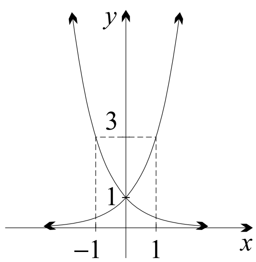
x	$0{\cdot}1$	$0{\cdot}25$	$0{\cdot}5$	$0{\cdot}75$	1	2
$\log_{10} x$	-1	$-0{\cdot}60$	$-0{\cdot}30$	$-0{\cdot}12$	0	$0{\cdot}30$

x	3	4	5	6	7	8	9	10
$\log_{10} x$	$0{\cdot}48$	$0{\cdot}60$	$0{\cdot}70$	$0{\cdot}78$	$0{\cdot}85$	$0{\cdot}90$	$0{\cdot}95$	1

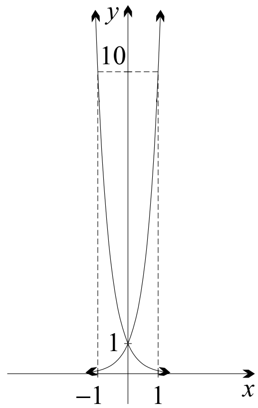
(b)



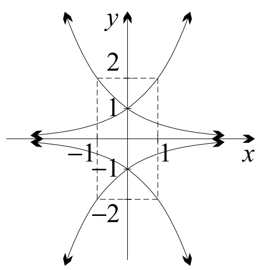
4(a)



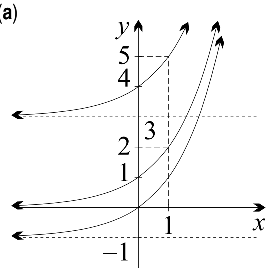
(b)



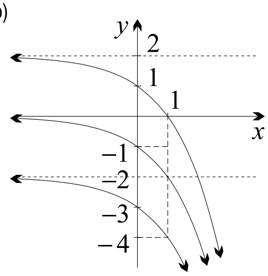
5



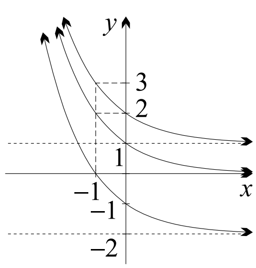
6(a)



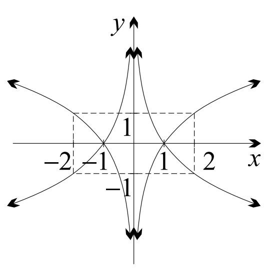
(b)



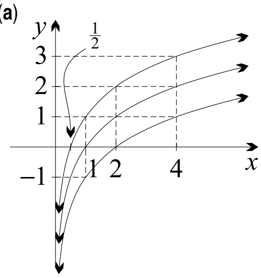
(c)



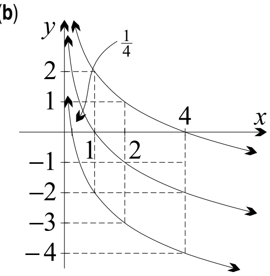
7



8(a)

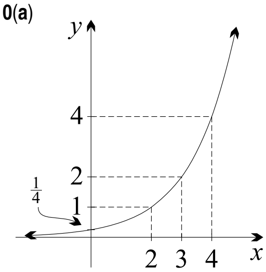


(b)

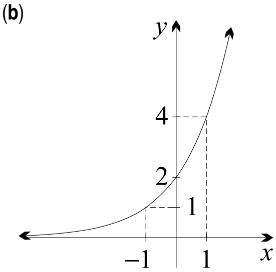


9(a)(i) 4 **(ii)** $\frac{1}{4}$ **(iii)** $2{\cdot}83$ **(iv)** $1{\cdot}32$ **(v)** $0{\cdot}66$
(b)(i) 1 **(ii)** $1{\cdot}58$ **(iii)** $0{\cdot}26$ **(iv)** $-1{\cdot}32$
(c)(i) $0 \le x \le 2$ **(ii)** $0 \le x \le 1$ **(iii)** $0{\cdot}58 \le x \le 1{\cdot}58$
(iv) $-1 \le x \le 1$ **(d)(i)** 2 **(ii)** $1{\cdot}58$ **(iii)** $0{\cdot}49$
(iv) $-0{\cdot}32$

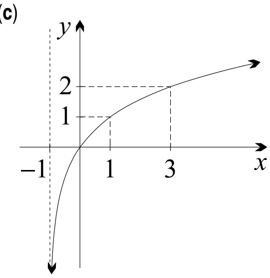
10(a)



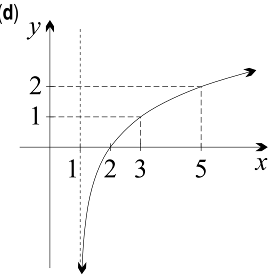
(b)



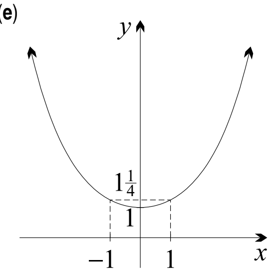
(c)



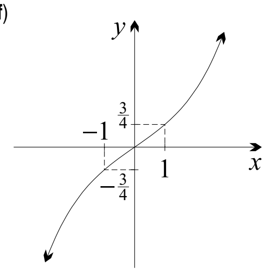
(d)



(e)



(f)

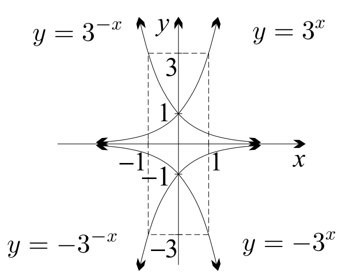


Review Exercise **7G** (Page 193)

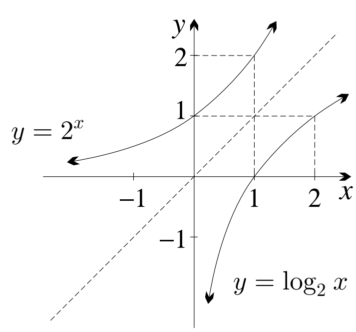
1(a) 125 **(b)** 256 **(c)** $1\,000\,000\,000$ **(d)** $\frac{1}{17}$ **(e)** $\frac{1}{81}$
(f) $\frac{1}{8}$ **(g)** $\frac{1}{81}$ **(h)** 1 **(i)** $\frac{8}{27}$ **(j)** $\frac{12}{7}$ **(k)** $\frac{36}{25}$ **(l)** 6
(m) 3 **(n)** 4 **(o)** 243 **(p)** $\frac{2}{7}$ **(q)** 1 **(r)** $\frac{5}{3}$ **(s)** $\frac{4}{9}$
(t) $\frac{1000}{27}$
2(a) x^{-1} **(b)** $7x^{-2}$ **(c)** $-\frac{1}{2}x^{-1}$ **(d)** $x^{\frac{1}{2}}$ **(e)** $30x^{\frac{1}{2}}$
(f) $4x^{-\frac{1}{2}}$ **(g)** yx^{-1} **(h)** $2yx^{\frac{1}{2}}$
3(a) x^{20} **(b)** $\frac{81}{a^{12}}$ **(c)** $5x^3$ **(d)** $\frac{2r}{t^2}$
4(a) x^3y^3 **(b)** $60xy^3z^5$ **(c)** $18x^{-1}y^{-2}$ **(d)** $4a^3b^3c^{-1}$
(e) x^2y^{-2} **(f)** $2x^{-3}y$ **(g)** m^2n^{-1} **(h)** $72s^9t^3$
(i) $8x^3y^{-3}$
5(a) 4 **(b)** 2 **(c)** -1 **(d)** -5 **(e)** 2 **(f)** 3 **(g)** $\frac{1}{2}$
(h) $\frac{1}{3}$
6(a) $2^x = 8,\ x = 3$ **(b)** $3^x = 9,\ x = 2$
(c) $10^x = 10\,000,\ x = 4$ **(d)** $5^x = \frac{1}{5},\ x = -1$
(e) $7^x = \frac{1}{49},\ x = -2$ **(f)** $13^x = 1,\ x = 0$
(g) $9^x = 3,\ x = \frac{1}{2}$ **(h)** $2^x = \sqrt{2},\ x = \frac{1}{2}$
(i) $7^2 = x,\ x = 49$ **(j)** $11^{-1} = x,\ x = \frac{1}{11}$
(k) $16^{\frac{1}{2}} = x,\ x = 4$ **(l)** $27^{\frac{1}{3}} = x,\ x = 3$
(m) $x^2 = 36,\ x = 6$ **(n)** $x^3 = 1000,\ x = 10$
(o) $x^{-1} = \frac{1}{7},\ x = 7$ **(p)** $x^{\frac{1}{2}} = 4,\ x = 16$
7(a) 1 **(b)** 2 **(c)** 2 **(d)** -2 **(e)** 2 **(f)** 0

8(a) $\log_a x + \log_a y + \log_a z$ **(b)** $\log_a x - \log_a y$
(c) $3\log_a x$ **(d)** $-2\log_a z$ **(e)** $2\log_a x + 5\log_a y$
(f) $2\log_a y - \log_a x - 2\log_a z$ **(g)** $\frac{1}{2}\log_a x$
(h) $\frac{1}{2}\log_a x + \frac{1}{2}\log_a y + \frac{1}{2}\log_a z$
9(a) 1 & 2 **(b)** 2 & 3 **(c)** 4 & 5 **(d)** 5 & 6 **(e)** -1 & 0
(f) -3 & -2 **(g)** -4 & -3 **(h)** -2 & -1
10(a) $2{\cdot}332$ **(b)** $-2{\cdot}347$ **(c)** $2{\cdot}010$ **(d)** $9{\cdot}966$
(e) $-0{\cdot}9551$ **(f)** $69{\cdot}66$ **(g)** -3 **(h)** $687{\cdot}3$
11(a) $3{\cdot}459$ **(b)** $-4{\cdot}644$ **(c)** $3{\cdot}010$ **(d)** $-0{\cdot}3645$
(e) $161{\cdot}7$ **(f)** $-161{\cdot}7$ **(g)** $10{\cdot}32$ **(h)** $458{\cdot}2$

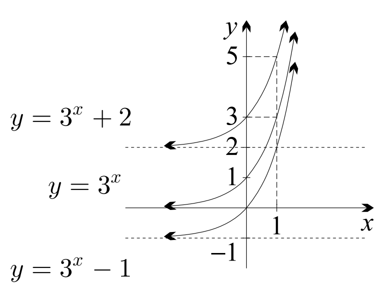
12(a)



(b)



(c)



Chapter Eight

Exercise 8A (Page 197)

1(a) $20, 25, 30, 35$ **(b)** $36, 46, 56, 66$
(c) $16, 32, 64, 128$ **(d)** $24, 48, 96, 192$
(e) $26, 22, 18, 14$ **(f)** $12, 3, -6, -15$
(g) $3, 1\frac{1}{2}, \frac{3}{4}, \frac{3}{8}$ **(h)** $3, 1, \frac{1}{3}, \frac{1}{9}$ **(i)** $1, -1, 1, -1$
(j) $16, 25, 36, 49$ **(k)** $\frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}$ **(l)** $-2, 1, -\frac{1}{2}, \frac{1}{4}$
2(a) $6, 12, 18, 24$ **(b)** $3, 8, 13, 18$ **(c)** $2, 4, 8, 16$
(d) $5, 25, 125, 625$ **(e)** $19, 18, 17, 16$
(f) $4, 2, 0, -2$ **(g)** $6, 12, 24, 48$
(h) $70, 700, 7000, 70\,000$ **(i)** $1, 8, 27, 64$
(j) $2, 6, 12, 20$ **(k)** $-1, 1, -1, 1$ **(l)** $-3, 9, -27, 81$
3(a) $6, 8, 10, 12$ **(b)** $11, 61, 111, 161$
(c) $15, 12, 9, 6$ **(d)** $12, 4, -4, -12$
(e) $5, 10, 20, 40$ **(f)** $\frac{1}{3}, 1, 3, 9$ **(g)** $18, 9, 4\frac{1}{2}, 2\frac{1}{4}$
(h) $-100, -20, -4, -\frac{4}{5}$
4 $7, 12, 17, 22, 27, 32, 37, 42, 47, 52, 57, 62$
(a) 5 **(b)** 11 **(c)** 4 **(d)** 8 **(e)** 52 **(f)** 7th term
(g) Yes; 17th term. **(h)** No; they all end in 2 or 7.
(i) 47, the 9th term **(j)** 42, the 8th term
5 $\frac{3}{4}, 1\frac{1}{2}, 3, 6, 12, 24, 48, 96, 192, 384, 768, 1536$
(a) 6 **(b)** 10 **(c)** 3 **(d)** 10 **(e)** 384 **(f)** 9th term
(g) Yes; 8th term. **(h)** No **(i)** 384, the 10th term
(j) 48, the 7th term
6(a) $13, 14, 15, 16, 17$. Add 1. **(b)** $9, 14, 19, 24, 29$. Add 5. **(c)** $10, 5, 0, -5, -10$. Subtract 5.
(d) $6, 12, 24, 48, 96$. Multiply by 2.
(e) $-7, 7, -7, 7, -7$. Multiply by -1.
(f) $40, 20, 10, 5, 2\frac{1}{2}$. Divide by 2.
7(c) $100 = T_{33}$, 200 is not a member, $1000 = T_{333}$.
8(a) 16 is not a member, $35 = T_{20}$, $111 = T_{58}$.
(b) $44 = T_5$, 200 and 306 are not members.
(c) 40 is not a member, $72 = T_6$, $200 = T_{10}$.
(d) $8 = T_3$, 96 is not a member, $128 = T_7$.
9(c) 49 **(d)** $T_{20} = 204$
10(a) 52 **(b)** 73 **(c)** $T_{41} = 128$ **(d)** $T_{21} = 103$
11(a) $3, 5, 7, 9$ **(b)** $5, 17, 29, 41$ **(c)** $6, 3, 0, -3$
(d) $12, 2, -8, -18$ **(e)** $5, 10, 20, 40$
(f) $4, 20, 100, 500$ **(g)** $20, 10, 5, 2\frac{1}{2}$
(h) $1, -1, 1, -1$
12(a) $1, 0, -1, 0$, T_n where n is even. **(b)** $0, -1, 0, 1$, T_n where n is odd. **(c)** $-1, 1, -1, 1$. No terms are zero. **(d)** $0, 0, 0, 0$. All terms are zero.
13 $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$ The sum of two odd integers is even, and the sum of an even and an odd integer is odd.

Exercise **8B** (Page 201)

1(a) $14,\ 18,\ 22$ **(b)** $18,\ 23,\ 28$ **(c)** $5,\ -5,\ -15$ **(d)** $-7,\ -13,\ -19$ **(e)** $9,\ 10\frac{1}{2},\ 12$ **(f)** $6\frac{1}{2},\ 6,\ 5\frac{1}{2}$
2(a) AP: $a = 3,\ d = 4$ **(b)** AP: $a = 11,\ d = -4$ **(c)** AP: $a = 10,\ d = 7$ **(d)** not an AP **(e)** AP: $a = 50,\ d = -15$ **(f)** AP: $a = 23,\ d = 11$ **(g)** AP: $a = -12,\ d = 5$ **(h)** not an AP **(i)** not an AP **(j)** AP: $a = 8,\ d = -10$ **(k)** AP: $a = -17,\ d = 17$ **(l)** AP: $a = 10,\ d = -2\frac{1}{2}$
3(a) $3,\ 5,\ 7,\ 9$ **(b)** $7,\ 9,\ 11,\ 13$ **(c)** $7,\ 3,\ -1,\ -5$ **(d)** $17,\ 28,\ 39,\ 50$ **(e)** $30,\ 19,\ 8,\ -3$ **(f)** $-9,\ -5,\ -1,\ 3$ **(g)** $4\frac{1}{2},\ 4,\ 3\frac{1}{2},\ 3$ **(h)** $3\frac{1}{2},\ 1\frac{1}{2},\ -\frac{1}{2},\ -2\frac{1}{2}$ **(i)** $0{\cdot}9,\ 1{\cdot}6,\ 2{\cdot}3,\ 3$
4(a) 67 **(b)** -55 **(c)** $50\frac{1}{2}$
5(a) 29 **(b)** 51 **(c)** 29
6(a) $a = 6,\ d = 10$ **(b)** $86,\ 206,\ 996$ **(c)** $T_n = 10n - 4$
7(a) $a = -20,\ d = 11$ **(b)** $57,\ 310,\ 2169$ **(c)** $T_n = 11n - 31$
8(a) $a = 300,\ d = -40$ **(b)** $60,\ -1700,\ -39\,660$ **(c)** $T_n = 340 - 40n$
9(a) $d = 3,\ T_n = 5 + 3n$ **(b)** $d = -6,\ T_n = 27 - 6n$ **(c)** not an AP **(d)** $d = 4,\ T_n = 4n - 7$ **(e)** $d = 1\frac{1}{4}$, $T_n = \frac{1}{4}(2 + 5n)$ **(f)** $d = -17,\ T_n = 29 - 17n$ **(g)** $d = \sqrt{2},\ T_n = n\sqrt{2}$ **(h)** not an AP **(i)** $d = 3\frac{1}{2}$, $T_n = \frac{1}{2}(7n - 12) = \frac{7}{2}n - 6$
10(a) $T_n = 170 - 5n$ **(b)** 26 terms **(c)** $T_{35} = -5$
11(a) 11 terms **(b)** 34 terms **(c)** 16 terms **(d)** 13 terms **(e)** 9 terms **(f)** 667 terms
12(a) $T_n = 23 - 3n,\ T_8 = -1$ **(b)** $T_n = 55 - 5n$, $T_{12} = -5$ **(c)** $T_n = 74 - 7n,\ T_{11} = -3$ **(d)** $T_n = 85 - 3n,\ T_{29} = -2$ **(e)** $T_n = 353 - 8n$, $T_{45} = -7$ **(f)** $T_n = 25 - \frac{1}{2}n,\ T_{51} = -\frac{1}{2}$
13(a) $11,\ 15,\ 19,\ 23,\ a = 11,\ d = 4$ **(b)** $T_{50} + T_{25} = 314,\ T_{50} - T_{25} = 100$ **(d)** $815 = T_{202}$ **(e)** $T_{248} = 999,\ T_{249} = 1003$ **(f)** $T_{49} = 203,\ \ldots,\ T_{73} = 299$ lie between 200 and 300, making 25 terms.
14(a)(i) $T_n = 8n$ **(ii)** $T_{63} = 504,\ T_{106} = 848$ **(iii)** 44 terms **(b)** $T_{91} = 1001,\ T_{181} = 1991$, 91 terms **(c)** $T_{115} = 805,\ T_{285} = 1995$, 171 terms
15(a) $d = 3;\ 7,\ 10,\ 13,\ 16$ **(b)** $d = -18;\ 82,\ 64,\ 46,\ 28$ **(c)** $d = 8;\ T_{20} = 180$ **(d)** $d = -2;\ T_{100} = -166$
16(a) \$500, \$800, \$1100, \$1400, ... **(b)** $a = 500$, $d = 300$ **(c)** \$4700 **(d)** cost $= 200 + 300n$ **(e)** 32
17(a) $180,\ 200,\ 220,\ \ldots$ **(b)** $a = 180,\ d = 20$ **(c)** 400 km **(d)** length $= 160 + 20n$ **(e)** 19 months
18(a) $d = 4,\ x = 1$ **(b)** $d = 6x,\ x = \frac{1}{3}$
19(a) $d = \log_3 2,\ T_n = n\log_3 2$ **(b)** $d = -\log_a 3,\ T_n = \log_a 2 + (4 - n)\log_a 3$ **(c)** $d = x + 4y,\ T_n = nx + (4n - 7)y$ **(d)** $d = -4 + 7\sqrt{5},\ T_n = 9 - 4n + (7n - 13)\sqrt{5}$ **(e)** $d = -1{\cdot}88,\ T_n = 3{\cdot}24 - 1{\cdot}88n$ **(f)** $d = -\log_a x,\ T_n = \log_a 3 + (3 - n)\log_a x$

Exercise **8C** (Page 205)

1(a) $8,\ 16,\ 32$ **(b)** $3,\ 1,\ \frac{1}{3}$ **(c)** $-56,\ -112,\ -224$ **(d)** $-20,\ -4,\ -\frac{4}{5}$ **(e)** $-24,\ 48,\ -96$ **(f)** $200,\ -400,\ 800$ **(g)** $-5,\ 5,\ -5$ **(h)** $1,\ -\frac{1}{10},\ \frac{1}{100}$ **(i)** $40,\ 400,\ 4000$
2(a) GP: $a = 4,\ r = 2$ **(b)** GP: $a = 16,\ r = \frac{1}{2}$ **(c)** GP: $a = 7,\ r = 3$ **(d)** GP: $a = -4,\ r = 5$ **(e)** not a GP **(f)** GP: $a = -1000,\ r = \frac{1}{10}$ **(g)** GP: $a = -80,\ r = -\frac{1}{2}$ **(h)** GP: $a = 29,\ r = 1$ **(i)** not a GP **(j)** GP: $a = -14,\ r = -1$ **(k)** GP: $a = 6,\ r = \frac{1}{6}$ **(l)** GP: $a = -\frac{1}{3},\ r = -3$
3(a) $1,\ 3,\ 9,\ 27$ **(b)** $12,\ 24,\ 48,\ 96$ **(c)** $5,\ -10,\ 20,\ -40$ **(d)** $18,\ 6,\ 2,\ \frac{2}{3}$ **(e)** $18,\ -6,\ 2,\ -\frac{2}{3}$ **(f)** $50,\ 10,\ 2,\ \frac{2}{5}$ **(g)** $6,\ -3,\ 1\frac{1}{2},\ -\frac{3}{4}$ **(h)** $-13,\ -26,\ -52,\ -104$ **(i)** $-7,\ 7,\ -7,\ 7$
4(a) 40 **(b)** $\frac{3}{10}$ **(c)** -56 **(d)** -8 **(e)** -88 **(f)** 120
5(a) 3^{69} **(b)** 5×7^{69} **(c)** $8 \times (-3)^{69} = -8 \times 3^{69}$
6(a) $a = 7,\ r = 2$ **(b)** $T_6 = 224,\ T_{50} = 7 \times 2^{49}$ **(c)** $T_n = 7 \times 2^{n-1}$
7(a) $a = 10,\ r = -3$ **(b)** $T_6 = -2430$, $T_{25} = 10 \times (-3)^{24} = 10 \times 3^{24}$ **(c)** $T_n = 10 \times (-3)^{n-1}$
8(a) $a = -80,\ r = \frac{1}{2}$ **(b)** $T_{10} = -\frac{5}{32}$, $T_{100} = -80 \times (\frac{1}{2})^{99}$ **(c)** $T_n = -80 \times (\frac{1}{2})^{n-1}$
9(a) $T_n = 10 \times 2^{n-1},\ T_6 = 320$ **(b)** $T_n = 180 \times (\frac{1}{3})^{n-1},\ T_6 = \frac{20}{27}$ **(c)** not a GP **(d)** not a GP **(e)** $T_n = \frac{3}{4} \times 4^{n-1},\ T_6 = 768$ **(f)** $T_n = -48 \times (\frac{1}{2})^{n-1},\ T_6 = -1\frac{1}{2}$
10(a) $r = -1,\ T_n = (-1)^{n-1},\ T_6 = -1$ **(b)** $r = -2,\ T_n = -2 \times (-2)^{n-1} = (-2)^n,\ T_6 = 64$ **(c)** $r = -3,\ T_n = -8 \times (-3)^{n-1},\ T_6 = 1944$ **(d)** $r = -\frac{1}{2},\ T_n = 60 \times (-\frac{1}{2})^{n-1},\ T_6 = -\frac{15}{8}$ **(e)** $r = -\frac{1}{2},\ T_n = -1024 \times (-\frac{1}{2})^{n-1},\ T_6 = 32$ **(f)** $r = -6,\ T_n = \frac{1}{16} \times (-6)^{n-1},\ T_6 = -486$
11(a) $T_n = 2^{n-1}$, 7 terms **(b)** $T_n = -3^{n-1}$, 5 terms **(c)** $T_n = 8 \times 5^{n-1}$, 7 terms **(d)** $T_n = 7 \times 2^{n-1}$, 6 terms **(e)** $T_n = 2 \times 7^{n-1}$, 5 terms **(f)** $T_n = 5^{n-3}$, 7 terms

12(a) $r = 2$; 25, 50, 100, 200, 400 **(b)** $r = 2$; 3, 6, 12, 24, 48, 96 **(c)** Either $r = 3$, giving 1, 3, 9, 27, 81, or $r = -3$, giving 1, -3, 9, -27, 81.
13(a) $r = \frac{1}{9}$ or $-\frac{1}{9}$ **(b)** $r = 0{\cdot}1$ or $-0{\cdot}1$ **(c)** $r = -\frac{3}{2}$ **(d)** $r = \sqrt{2}$ or $-\sqrt{2}$
14(a) 50, 100, 200, 400, 800, 1600, $a = 50$, $r = 2$ **(b)** $6400 = T_8$ **(c)** $T_{50} \times T_{25} = 5^4 \times 2^{75}$, $T_{50} \div T_{25} = 2^{25}$ **(e)** The six terms $T_6 = 1600, \ldots, T_{11} = 51\,200$ lie between 1000 and 100 000.
15 The successive thicknesses form a GP with 101 terms, and with $a = 0{\cdot}1$ mm and $r = 2$. Hence thickness $= T_{101} = \frac{2^{100}}{10}$ mm $\doteqdot 1{\cdot}27 \times 10^{23}$ km $\doteqdot 1{\cdot}34 \times 10^{10}$ light years, which is almost exactly the present estimate of the distance to the Big Bang.
16(a) $r = \sqrt{2}$, $T_n = \sqrt{6} \times (\sqrt{2}\,)^{n-1}$ **(b)** $r = ax^2$, $T_n = a^n x^{2n-1}$ **(c)** $r = \frac{y}{x}$, $T_n = -x^{2-n} y^{n-2}$
17(a) $T_n = 2x^n$, $x = 1$ or -1
(b) $T_n = x^{6-2n}$, $x = \frac{1}{3}$ or $-\frac{1}{3}$
(c) $T_n = 2^{-16} \times 2^{4n-4} x = 2^{4n-20} x$, $x = 6$

Exercise 8D (Page 208)

1(a) 11 **(b)** 23 **(c)** -31 **(d)** -8 **(e)** 12 **(f)** 10
2(a) 6 or -6 **(b)** 12 or -12 **(c)** 30 or -30 **(d)** 14 or -14 **(e)** 5 **(f)** -16
3(a) 10; 8 or -8 **(b)** 25; 7 or -7 **(c)** $20\frac{1}{2}$; 20 or -20 **(d)** $-12\frac{1}{2}$; 10 or -10 **(e)** -30; 2 **(f)** 0; 6 **(g)** -3; 1 **(h)** 24; -3 **(i)** 40; 45 **(j)** 84; -16 **(k)** $-5\frac{3}{4}$; -36 **(l)** -21; 7
4(a) 7, 14, 21, 28, 35, 42 **(b)** 27, 18, 12, 8
(c) 40, $36\frac{1}{2}$, 33, $29\frac{1}{2}$, 26, $22\frac{1}{2}$, 19, $15\frac{1}{2}$, 12, $8\frac{1}{2}$, 5
(d) 1, 10, 100, 1000, 10 000, 100 000, 1 000 000 or 1, -10, 100, -1000, 10 000, $-100\,000$, 1 000 000
(e) 3, $14\frac{1}{4}$, $25\frac{1}{2}$, $36\frac{3}{4}$, 48
(f) 3, 6, 12, 24, 48 or 3, -6, 12, -24, 48
5(a) $d = 3$, $a = -9$ **(b)** $d = 4$, $a = -1$ **(c)** $d = -9$, $a = 60$ **(d)** $d = 3\frac{1}{2}$, $a = -4\frac{1}{2}$
6(a) $r = 2$, $a = 4$ **(b)** $r = 4$, $a = \frac{1}{16}$
(c) $r = 3$ and $a = \frac{1}{9}$, or $r = -3$ and $a = -\frac{1}{9}$
(d) $r = \sqrt{2}$ and $a = \frac{3}{2}$, or $r = -\sqrt{2}$ and $a = \frac{3}{2}$
7(a) $T_8 = 37$ **(b)** $T_2 = 59$ **(c)** $T_2 = \frac{3}{8}$
8(a) $n = 13$ **(b)** $n = 8$ **(c)** $n = 11$ **(d)** $n = 8$
9(b) $n = 19$ **(c)** $n = 29$ **(d)** $n = 66$ **(e)** 10 terms **(f)** 37 terms
10(a) $T_n = 98 \times (\frac{1}{7})^{n-1}$, 10 terms
(b) $T_n = 25 \times (\frac{1}{5})^{n-1} = (\frac{1}{5})^{n-3}$, 11 terms
(c) $T_n = (0{\cdot}9)^{n-1}$, 132 terms
11 152 sheets
12(a) $a = 28$, $d = -1$
(b) $a = \frac{1}{3}$ and $r = 3$, or $a = \frac{2}{3}$ and $r = -3$
(c) $T_6 = -2$
13(a) $x = 10$; 9, 17, 25 **(b)** $x = -2$; -2, -6, -10 **(c)** $x = 2$; -1, 5, 11 **(d)** $x = -4$; -14, -4, 6
14(a) $x = -\frac{1}{2}$; $-\frac{1}{2}$, $\frac{1}{2}$, $-\frac{1}{2}$
(b) $x = 1$; 1, 2, 4 or $x = 6$; -4, 2, -1
15(a)(i) $x = -48$ **(ii)** $x = 6$ **(b)(i)** $x = 0{\cdot}100\,01$ **(ii)** $x = 0{\cdot}002$ or $x = -0{\cdot}002$ **(c)(i)** $x = 0{\cdot}398$ **(ii)** $x = 20$ **(d)(i)** They can't form an AP. **(ii)** $x = 9$ **(e)(i)** $x = 2$ **(ii)** $x = 4$ or $x = 0$ **(f)(i)** $x = \sqrt{5}$ **(ii)** $x = 2$ or $x = -2$ **(g)(i)** $x = \frac{3}{2}\sqrt{2}$ **(ii)** $x = 2$ or $x = -2$ **(h)(i)** $x = 40$ **(ii)** 2^5 or -2^5 **(i)(i)** $x = 0$ **(ii)** They can't form a GP.
16(a) $a = 6\frac{1}{4}$ and $b = 2\frac{1}{2}$, or $a = 4$ and $b = -2$
(b) $a = 1$, $b = 0$
17(a) $T_n = 2^{8-3n}$
18(b) $\frac{T_8}{T_1} = (\frac{1}{2})^{\frac{7}{12}} \doteqdot 0{\cdot}6674 \doteqdot \frac{2}{3}$
(c) $\frac{T_5}{T_1} = (\frac{1}{2})^{\frac{4}{12}} \doteqdot 0{\cdot}7937 \doteqdot \frac{4}{5}$
(d) $\frac{T_6}{T_1} = (\frac{1}{2})^{\frac{5}{12}} \doteqdot 0{\cdot}7491 \doteqdot \frac{3}{4}$, $\frac{T_4}{T_1} = (\frac{1}{2})^{\frac{3}{12}} \doteqdot 0{\cdot}8409 \doteqdot \frac{5}{6}$
(e) $\frac{T_3}{T_1} = (\frac{1}{2})^{\frac{2}{12}} \doteqdot 0{\cdot}8908 \doteqdot \frac{8}{9}$, $\frac{T_2}{T_1} = (\frac{1}{2})^{\frac{1}{12}} \doteqdot 0{\cdot}9439 \doteqdot \frac{17}{18}$

Exercise 8E (Page 212)

1(a) 24 **(b)** 80 **(c)** 0 **(d)** $3\frac{3}{4}$
2(a) 450 **(b)** 24 **(c)** -54 **(d)** 15·6
3(a) 10, 30, 60, 100, 150 **(b)** 1, -2, 7, -20, 61 **(c)** 1, 5, 14, 30, 55 **(d)** 3, $7\frac{1}{2}$, $13\frac{1}{2}$, 21, 30
4(a) -2, 3, -3 **(b)** 120, 121, $121\frac{1}{3}$ **(c)** 60, 50, 30 **(d)** 0·1111, 0·111 11, 0·111 111
5(a) S_n: 2, 7, 15, 26, 40, 57, 77
(b) S_n: 40, 78, 114, 148, 180, 210, 238
(c) S_n: 2, -2, 2, -2, 2, -2, 2
(d) S_n: 7, 0, 7, 0, 7, 0, 7
6(a) T_n: 1, 3, 5, 7, 9, 11, 13
(b) T_n: 2, 4, 8, 16, 32, 64, 128
(c) T_n: -3, -5, -7, -9, -11, -13, -15
(d) T_n: 8, -8, 8, -8, 8, -8, 8
7(a) T_n: 1, 1, 1, 2, 3, 5, 8, 13
(b) T_n: 3, 1, 3, 4, 7, 11, 18, 29
8(a) 42 **(b)** 75 **(c)** 15 **(d)** 174 **(e)** 100 **(f)** 63 **(g)** 117 **(h)** -1 **(i)** 0 **(j)** 404 **(k)** 7 **(l)** -7

10(a) $\displaystyle\sum_{n=1}^{40} n^3$ **(b)** $\displaystyle\sum_{n=1}^{40} \frac{1}{n}$ **(c)** $\displaystyle\sum_{n=1}^{20} (n+2)$ **(d)** $\displaystyle\sum_{n=1}^{12} 2^n$
(e) $\displaystyle\sum_{n=1}^{10} (-1)^n n$ **(f)** $\displaystyle\sum_{n=1}^{10} (-1)^{n+1} n$ or $\displaystyle\sum_{n=1}^{10} (-1)^{n-1} n$

Exercise 8F (Page 216)

1(a) $n = 100$, 5050 **(b)** $n = 50$, 2500
(c) $n = 50$, 2550 **(d)** $n = 100$, 15 150
(e) $n = 50$, 7500 **(f)** $n = 9000$, 49 504 500
2(a) 180 **(b)** 78 **(c)** -153 **(d)** -222
3(a) $a = 2$, $d = 4$, 882 **(b)** $a = 3$, $d = 7$, 1533
(c) $a = -6$, $d = 5$, 924 **(d)** $a = 10$, $d = -5$, -840
(e) $a = -7$, $d = -3$, -777 **(f)** $a = 1\frac{1}{2}$, $d = 2$, $451\frac{1}{2}$
4(a) 222 **(b)** -630 **(c)** 78 400 **(d)** 0 **(e)** 65 **(f)** 30
5(a) 101 terms, 10 100 **(b)** 13 terms, 650
(c) 11 terms, 275 **(d)** 100 terms, 15 250
(e) 11 terms, 319 **(f)** 10 terms, $61\frac{2}{3}$
6(a) 500 terms, 250 500 **(b)** 2001 terms, 4 002 000
(c) 3160 **(d)** 1440
7(a) $S_n = \frac{1}{2}n(5 + 5n)$ **(b)** $S_n = \frac{1}{2}n(17 + 3n)$
(c) $S_n = n(1 + 2n)$ **(d)** $\frac{1}{2}n(5n - 23)$
(e) $S_n = \frac{1}{4}n(21 - n)$ **(f)** $\frac{1}{2}n(2 + n\sqrt{2} - 3\sqrt{2})$
8(a) $\frac{1}{2}n(n + 1)$ **(b)** n^2 **(c)** $\frac{3}{2}n(n + 1)$ **(d)** $100n^2$
9(a) 450 legs. No creatures have the mean number of 5 legs. **(b)** 16 860 years **(c)** \$352 000
10(a) $a = 598$, $\ell = 200$, 79 800
(b) $a = 90$, $\ell = -90$, 0 **(c)** $a = -47$, $\ell = 70$, 460
(d) $a = 53$, $\ell = 153$, 2163
11(a) $\ell = 22$ **(b)** $a = -7{\cdot}1$ **(c)** $d = 11$ **(d)** $a = -3$
12(b)(i) 16 terms **(ii)** more than 16 terms
(c) 5 terms or 11 terms **(d)** $n = 18$ or $n = -2$, but n must be a positive integer.
(e) $n = 4, 5, 6, \ldots, 12$ **(f)** Solving $S_n > 256$ gives $(n - 8)^2 < 0$, which has no solutions.
13(a) $S_n = n(43 - n)$, 43 terms
(b) $S_n = \frac{3}{2}n(41 - n)$, 41 terms
(c) $S_n = 3n(n + 14)$, 3 terms
(d) $\frac{1}{4}n(n + 9)$, 6 terms
14(a) 20 rows, 29 logs on bottom row
(b) $S_n = 5n^2$, 7 seconds
(c) 11 trips, deposits are 1 km apart.
15(a) $d = -2$, $a = 11$, $S_{10} = 20$
(b) $a = 9$, $d = -2$, $T_2 = 7$
(c) $d = -3$, $a = 28\frac{1}{2}$, $T_4 = 19\frac{1}{2}$
16(a) 10 terms, $55 \log_a 2$ **(b)** 11 terms, 0
(c) 6 terms, $3(4 \log_b 3 - \log_b 2)$
(d) $15(\log_x 2 - \log_x 3)$

Exercise 8G (Page 220)

1 2801 kits, cats, sacks, wives and man
2(a) 1093 **(b)** 547
3(a) 1023, $2^n - 1$
(b) 242, $3^n - 1$
(c) $-11\,111$, $-\frac{1}{9}\Big(10^n - 1\Big)$ **(d)** -781, $-\frac{1}{4}\Big(5^n - 1\Big)$
(e) -341, $\frac{1}{3}\Big(1 - (-2)^n\Big)$ **(f)** 122, $\frac{1}{2}\Big(1 - (-3)^n\Big)$
(g) -9091, $-\frac{1}{11}\Big(1 - (-10)^n\Big)$
(h) -521, $-\frac{1}{6}\Big(1 - (-5)^n\Big)$
4(a) $\frac{1023}{64}$, $16\Big(1 - (\frac{1}{2})^n\Big)$ **(b)** $\frac{364}{27}$, $\frac{27}{2}\Big(1 - (\frac{1}{3})^n\Big)$
(c) $\frac{605}{9}$, $\frac{135}{2}\Big(1 - (\frac{1}{3})^n\Big)$ **(d)** $\frac{211}{24}$, $\frac{4}{3}\Big((\frac{3}{2})^n - 1\Big)$
(e) $\frac{341}{64}$, $\frac{16}{3}\Big(1 - (-\frac{1}{2})^n\Big)$ **(f)** $\frac{182}{27}$, $\frac{27}{4}\Big(1 - (-\frac{1}{3})^n\Big)$
(g) $-\frac{305}{9}$, $-\frac{135}{4}\Big(1 - (-\frac{1}{3})^n\Big)$ **(h)** $\frac{55}{24}$, $\frac{4}{15}\Big(1 - (-\frac{3}{2})^n\Big)$
5(a) $5\Big((1{\cdot}2)^n - 1\Big)$, 25·96 **(b)** $20\Big(1 - (0{\cdot}95)^n\Big)$, 8·025
(c) $100\Big((1{\cdot}01)^n - 1\Big)$, 10·46
(d) $100\Big(1 - (0{\cdot}99)^n\Big)$, 9·562
6(a)(i) 2^{63} **(ii)** $2^{64} - 1$ **(b)** 615 km^3
7(a) $S_n = \Big((\sqrt{2}\,)^n - 1\Big)\Big(\sqrt{2} + 1\Big)$,
$S_{10} = 31(\sqrt{2} + 1)$
(b) $S_n = \frac{1}{2}\Big(1 - (-\sqrt{5}\,)^n\Big)\Big(\sqrt{5} - 1\Big)$,
$S_{10} = -1562(\sqrt{5} - 1)$
8(a) $a = 6$, $r = 2$, 762 **(b)** $a = 9$, $r = 3$, 3276
(c) $a = 12$, $r = \frac{1}{2}$, $\frac{765}{32}$
9(a) $\frac{1}{8} + \frac{3}{4} + \frac{9}{2} + 27 + 162 = 194\frac{3}{8}$
or $\frac{1}{8} - \frac{3}{4} + \frac{9}{2} - 27 + 162 = 138\frac{7}{8}$
(b) $15\frac{3}{4}$
(c) 1562·496 **(d)** 7 **(e)** 640
10(a)(i) 0·01172 tonnes **(ii)** 11·99 tonnes
(b) $4{\cdot}9 \times 10^{-3}$ g
(c)(i) $S_n = 10P(1{\cdot}1^{10} - 1)$ **(ii)** \$56.47
11(a) 6 terms **(b)** 8 terms **(c)** 5 terms **(d)** 7 terms
12(b) $n = 8$ **(c)** 14 terms **(d)** $S_{14} = 114\,681$
13(a) 41 powers of 3 **(b)** 42 terms

Exercise 8H (Page 224)

1(a) 18, 24, 26, $26\frac{2}{3}$, $26\frac{8}{9}$, $26\frac{26}{27}$ **(b)** $S_\infty = 27$
(c) $S_\infty - S_6 = 27 - 26\frac{26}{27} = \frac{1}{27}$
2(a) 24, 12, 18, 15, $16\frac{1}{2}$, $15\frac{3}{4}$ **(b)** $S_\infty = 16$
(c) $S_\infty - S_6 = 16 - 15\frac{3}{4} = \frac{1}{4}$
3(a) $a = 1$, $S_\infty = 2$ **(b)** $a = 8$, $S_\infty = 16$
(c) $a = -4$, $S_\infty = -8$
4(a) $a = 1$, $S_\infty = \frac{3}{4}$ **(b)** $a = 36$, $S_\infty = 27$
(c) $a = -60$, $S_\infty = -45$

5(a) $r = \frac{1}{4}$, $S_\infty = 80$ **(b)** $r = -\frac{1}{2}$, $S_\infty = 40$
(c) $r = -\frac{1}{5}$, $S_\infty = 50$
6(a) $r = -\frac{1}{2}$, $S_\infty = \frac{2}{3}$ **(b)** $r = \frac{1}{3}$, $S_\infty = \frac{3}{2}$
(c) $r = -\frac{2}{3}$, $S_\infty = \frac{3}{5}$ **(d)** $r = \frac{3}{5}$, $S_\infty = 2\frac{1}{2}$
(e) $r = -\frac{3}{2}$, no limiting sum **(f)** $r = \frac{1}{3}$, $S_\infty = 18$
(g) $r = \frac{1}{10}$, $S_\infty = 1111\frac{1}{9}$
(h) $r = -\frac{1}{10}$, $S_\infty = 909\frac{1}{11}$
(i) $r = -1$, no limiting sum **(j)** $r = \frac{9}{10}$, $S_\infty = 1000$
(k) $r = -\frac{1}{5}$, $S_\infty = -\frac{5}{3}$ **(l)** $r = \frac{1}{5}$, $S_\infty = -\frac{5}{6}$
7(a) The successive down-and-up distances form a GP with $a = 12$ and $r = \frac{1}{2}$. **(b)** $S_\infty = 24$ metres
8(a) T_n: 10, 10, 10, 10, 10, 10. S_n: 10, 20, 30, 40, 50, 60. $S_n \to \infty$ as $n \to \infty$. **(b)** T_n: 10, −10, 10, −10, 10, −10. S_n: 10, 0, 10, 0, 10, 0. S_n oscillates between 10 and 0 as $n \to \infty$. **(c)** T_n: 10, 20, 40, 80, 160, 320. S_n: 10, 30, 70, 150, 310, 630. $S_n \to \infty$ as $n \to \infty$. **(d)** T_n: 10, −20, 40, −80, 160, −320. S_n: 10, −10, 30, −50, 110, −210. S_n oscillates between larger and larger positive and negative numbers as $n \to \infty$.
9(a) $S_\infty - S_4 = 160 - 150 = 10$
(b) $S_\infty - S_4 = 111\frac{1}{9} - 111\frac{1}{10} = \frac{1}{90}$
(c) $S_\infty - S_4 = 55\frac{5}{9} - 32\frac{4}{5} = 22\frac{34}{45}$
10(a) $a = 2000$ and $r = \frac{1}{5}$ **(b)** $S_\infty = 2500$
(c) $S_\infty - S_4 = 4$
11(a) $S_\infty = 10\,000$ **(b)** $S_\infty - S_{10} \doteqdot 3487$
12(a) $S_\infty = \frac{5}{1-x}$, $x = \frac{1}{2}$ **(b)** $S_\infty = \frac{5}{1-x}$, $x = -\frac{2}{3}$
(c) $S_\infty = \frac{5}{1+x}$, $x = -\frac{2}{3}$ **(d)** $S_\infty = \frac{3x}{2}$, $x = \frac{4}{3}$
(e) $S_\infty = \frac{3x}{4}$, $x = \frac{8}{3}$ **(f)** $S_\infty = 3x$, $x = \frac{2}{3}$
13(a) $-1 < x < 1$, $\frac{7}{1-x}$ **(b)** $-\frac{1}{3} < x < \frac{1}{3}$, $\frac{2x}{1-3x}$
(c) $0 < x < 2$, $\frac{1}{2-x}$ **(d)** $-2 < x < 0$, $-\frac{1}{x}$
14(a) $r = 1{\cdot}01$, no limiting sum
(b) $r = -0{\cdot}99$, $S_\infty = \frac{100}{199}$
(c) $r = (1{\cdot}01)^{-1}$, $S_\infty = 101$
(d) $r = -\frac{1}{6}$, $S_\infty = \frac{108}{175}$
15(a) $r = \frac{1}{4}$, $S_\infty = \frac{64}{3}\sqrt{5}$ **(b)** $r = -\frac{1}{3}$, $S_\infty = 81\sqrt{7}$
(c) $\frac{7}{6}(7+\sqrt{7})$ **(d)** $4(2-\sqrt{2})$ **(e)** $5(5-2\sqrt{5})$
(f) $r = \frac{1}{3}\sqrt{10} > 1$, so there is no limiting sum.
(g) $\frac{1}{3}\sqrt{3}$ **(h)** $\frac{1}{2}(\sqrt{3}+1)$
16(a) $a = \frac{1}{3}$, $r = \frac{1}{3}$, $S_\infty = \frac{1}{2}$
(b) $a = \frac{7}{2}$, $r = \frac{1}{2}$, $S_\infty = 7$
(c) $a = -24$, $r = -\frac{3}{5}$, $S_\infty = -15$
17(a) $r = \frac{4}{5}$ **(b)** $18 + 6 + 2 + \cdots$ or $9 + 6 + 4 + \cdots$
(c) $r = \frac{5}{6}$ **(d)(i)** $r = -\frac{1}{2} + \frac{1}{2}\sqrt{5}$ ($r = -\frac{1}{2} - \sqrt{5} < -1$, so it is not a possible solution.)
(ii) $r = \frac{1}{2}$ **(iii)** $r = \frac{1}{2}\sqrt{2}$ or $-\frac{1}{2}\sqrt{2}$
18(a) $-\sqrt{2} < x < \sqrt{2}$ and $x \neq 0$, $S_\infty = \dfrac{1}{2-x^2}$
(b) $x \neq 0$, $S_\infty = \dfrac{1+x^2}{x^2}$

Exercise **8I** (Page 227)

1(a) $0{\cdot}3 + 0{\cdot}03 + 0{\cdot}003 + \cdots = \frac{1}{3}$
(b) $0{\cdot}1 + 0{\cdot}01 + 0{\cdot}001 + \cdots = \frac{1}{9}$
(c) $0{\cdot}7 + 0{\cdot}07 + 0{\cdot}007 + \cdots = \frac{7}{9}$
(d) $0{\cdot}6 + 0{\cdot}06 + 0{\cdot}006 + \cdots = \frac{2}{3}$
2(a) $0{\cdot}27 + 0{\cdot}0027 + 0{\cdot}000027 + \cdots = \frac{3}{11}$ **(b)** $\frac{81}{99}$
(c) $\frac{1}{11}$ **(d)** $\frac{4}{33}$ **(e)** $\frac{26}{33}$ **(f)** $\frac{1}{37}$ **(g)** $\frac{5}{37}$ **(h)** $\frac{5}{27}$
4(a) $12 + (0{\cdot}4 + 0{\cdot}04 + \cdots) = 12\frac{4}{9}$ **(b)** $7\frac{9}{11}$
(c) $8{\cdot}4 + (0{\cdot}06 + 0{\cdot}006 + \cdots) = 8\frac{7}{15}$
(d) $0{\cdot}2 + (0{\cdot}036 + 0{\cdot}00036 + \cdots) = \frac{13}{55}$
5(a) $0{\cdot}\dot{9} = 0{\cdot}9 + 0{\cdot}09 + 0{\cdot}009 + \cdots = \frac{0{\cdot}9}{1-0{\cdot}1} = 1$
(b) $2{\cdot}7\dot{9} = 2{\cdot}7 + (0{\cdot}09 + 0{\cdot}009 + 0{\cdot}0009 + \cdots)$
$= 2{\cdot}7 + \frac{0{\cdot}09}{1-0{\cdot}1} = 2{\cdot}7 + 0{\cdot}1 = 2{\cdot}8$
6(a) $\frac{29}{303}$ **(b)** $\frac{25}{101}$ **(c)** $\frac{3}{13}$ **(d)** $\frac{3}{7}$
(e) $0{\cdot}25 + (0{\cdot}0057 + 0{\cdot}000057 + \cdots) = \frac{211}{825}$ **(f)** $1\frac{14}{135}$
(g) $\frac{1}{3690}$ **(h)** $7\frac{27}{35}$

Review Exercise **8J** (Page 228)

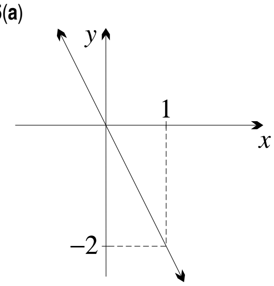
1 14, 5, −4, −13, −22, −31, −40, −49 **(a)** 6 **(b)** 4
(c) −31 **(d)** T_8 **(e)** No **(f)** $T_{11} = -40$
2(a) 52, −62, −542, −5 999 942
(b) 20 no, $10 = T_8$, $-56 = T_{19}$, −100 no
(c) $T_{44} = -206$ **(d)** $T_{109} = -596$
3(a) −5, 5, −5, 5, −5, 5, −5, 5, **(b)** −5, 0
(c) Take the opposite. **(d)** 5, −5, −5
4(a) AP, $d = 7$ **(b)** AP, $d = -121$ **(c)** neither
(d) GP, $r = 3$ **(e)** neither **(f)** GP, $r = -\frac{1}{2}$
5(a) $a = 23$, $d = 12$ **(b)** $T_{20} = 251$, $T_{600} = 7211$
(d) $143 = T_{11}$, 173 is not a term. **(e)** $T_{83} = 2007$, $T_{165} = 1991$ **(f)** 83 (Count both T_{83} and T_{165}.)
6(a) $a = 20$, $d = 16$ **(b)** $T_n = 4 + 16n$
(c) 12 cases, \$4 change **(d)** 18
7(a) $a = 50$, $r = 2$ **(b)** $T_n = 50 \times 2^{n-1}$
(c) $T_8 = 6400$, $T_{12} = 102\,400$
(d) $1600 = T_6$, 4800 is not a term.
(e) 320 000 **(f)** 18 terms
8(a) $a = 486$, $r = \frac{1}{3}$
(b) 486, 162, 54, 18, 6, 2 (no fractions)
(c) 4 **(d)** $S_6 = 728$ **(e)** 729
9(a) 75 **(b)** 45 or −45
10(a) 11 111 **(b)** −16 400 **(c)** 1025
11(a) $n = 45$, $S_{45} = 4995$ **(b)** $n = 101$, $S_{101} = 5050$
(c) $n = 77$, $S_{77} = 2387$

12(a) 189 **(b)** -1092 **(c)** $-157\frac{1}{2}$
13(a) 300 **(b)** $r = -\frac{3}{2} < -1$, so there is no limiting sum. **(c)** $-303\frac{3}{4}$
14(a) $-3 < x < -1$ **(b)** $S_\infty = -\frac{2+x}{1+x}$
15(a) $\frac{13}{33}$ **(b)** $\frac{52}{111}$ **(c)** $12\frac{335}{1100}$
16(a) $d = 5,\ 511$ **(b)** -1450 **(c)** $r = -2,\ -24$ **(d)** $d = -5$ **(e)** $n = 2$ or $n = 8$ **(f)** $r = -\frac{1}{3}$ **(g)** 16

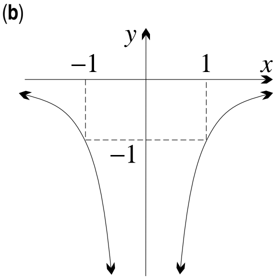
Chapter Nine

Exercise **9A** (Page 232)

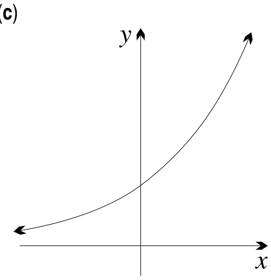
1 The values of $f'(x)$ should be about -4, -3, -2, -1, 0, 1, 2, 3, 4. The graph of $y = f'(x)$ should approximate a line of gradient 2 through the origin; its exact equation is $f'(x) = 2x$.
2 The values of $f'(x)$ should be about $1\frac{1}{2}$, 0, $-0{\cdot}9$, $-1{\cdot}2$, $-0{\cdot}9$, 0, $1\frac{1}{2}$. The graph of $y = f'(x)$ is a parabola crossing the x-axis at $x = -2$ and $x = 2$.
3(a) 2 **(b)** 7 **(c)** -5 **(d)** -3 **(e)** $\frac{1}{2}$ **(f)** 0
4(a) 10 **(b)** $\frac{2}{3}$ **(c)** -1
5(a)



(b)



(c)



Exercise **9B** (Page 235)

1(b) $10h + 5h^2$
2(b) $10xh + 5h^2$
3(d) At A, $f'(1) = -2$.
(e) At B, $f'(3) = 2$; at C, $f'(2) = 0$.
4(a) $2xh + h^2$ **(b)** $f'(x) = 2x$
(c) $f'(0) = 0$ **(d)** $f'(3) = 6$
5(a) $2x + h + 4,\ 2x + 4$ **(b)** $f'(0) = 4,\ f'(-2) = 0$
6(a) $2x + h - 2,\ f'(x) = 2x - 2$
(b) $f'(0) = -2,\ f'(2) = 2$
7(a) $2x + h + 6,\ f'(x) = 2x + 6$
(b) $f'(0) = 6,\ f'(-3) = 0$
8(a) $4 + h,\ f'(2) = 4$ **(b)** $2h + 3,\ f'(0) = 3$
(c) $-6 + h,\ f'(-1) = -6$
9(a) $2x + h - 6,\ f'(x) = 2x - 6$ **(b)** It is a concave-up parabola with x-intercepts $x = 0$ and $x = 6$.
(c) $f'(0) = -6,\ f'(6) = 6$ **(d)** $f'(3) = 0$

10(a) $8-2x-h$, $f'(x)=8-2x$ **(b)** It is a concave-down parabola with x-intercepts $x=0$ and $x=8$. **(c)** $f'(0)=8$, $f'(8)=-8$ **(d)** $f'(4)=0$
11(a) $3x^2$ **(b)** $4x^3$
13(c)(i) $0<|h|<0{\cdot}2$ **(ii)** $0<|h|<0{\cdot}02$
(iii) $0<|h|<0{\cdot}0002$ **(iv)** $0<|h|<2\times10^{-11}$

Exercise **9C** (Page 240)

1(a) $7x^6$ **(b)** $5x^4$ **(c)** $45x^4$ **(d)** $6x$ **(e)** $2x^5$ **(f)** $4x^7$ **(g)** 0 **(h)** 0 **(i)** 0
2(a) $5,\ 5,\ 5$ **(b)** $-1,\ -1,\ -1$ **(c)** $2x+5,\ 5,\ 7$ **(d)** $6x-5,\ -5,\ 1$ **(e)** $4x^3-10x,\ 0,\ -6$ **(f)** $-3-15x^2,\ -3,\ -18$ **(g)** $4x^3+3x^2+2x+1,\ 1,\ 10$ **(h)** $x^3+x^2+x,\ 0,\ 3$ **(i)** $2x^5-2x^3+2x,\ 0,\ 2$
3(a) $4-2x$ **(b)** $3x^2+1$ **(c)** $6x-16x^3$ **(d)** $2x+2$ **(e)** $8x$ **(f)** $4x^3+12x$ **(g)** $2x-14$ **(h)** $3x^2-10x+3$ **(i)** $18x-30$
4(a) $2x+1$ **(b)** $f'(0)=1$ **(c)** $45°$
5(a) $-1+2x$ **(c)** $71°34'$
6(a) $2x+8$ **(b)** $x=-4,\ (-4,-9)$ **(c)** $x=2,\ (2,27)$
7(a) $-4x$ **(b)** $x=0,\ (0,3)$ **(c)** $x=5,\ (5,-47)$
8 $f'(x)=2x-3$ **(a)** $3,\ 71°34'$ **(b)** $1,\ 45°$ **(c)** $0,\ 0°$ **(d)** $-1,\ 135°$ **(e)** $-3,\ 108°26'$
9(a) $2x-2,\ (1,6)$ **(b)** $2x+4,\ (-2,-14)$ **(c)** $2x-10,\ (5,-10)$
10 $f'(x)=2x-5$ **(a)** $(4,-3)$ **(b)** $(0,1)$ **(c)** $(3,-5)$ **(d)** $(2,-5)$
11(a) $f'(x)=3x^2-3,\ (1,0),\ (-1,4)$
(b) $f'(x)=4x^3-36x,\ (0,0),\ (3,-81),\ (-3,-81)$
(c) $f'(x)=3x^2,\ (5,131),\ (-5,-119)$
12 The tangent has gradient $2a-6$, the normal has gradient $\dfrac{1}{6-2a}$. **(a)(i)** 3 **(ii)** 4 **(iii)** $3\frac14$ **(b)** $2\frac12$ **(c)(i)** $3\frac13$ **(ii)** $2\frac14$

Exercise **9D** (Page 244)

1(a) $\frac{dy}{dx}=2x+7,\ 27$ **(b)** $\frac{dy}{dx}=3x^2+6x+6,\ 366$ **(c)** $\frac{dy}{dx}=4x^3+2x+8,\ 4028$ **(d)** $\frac{dy}{dx}=x^2-\frac12x+1,\ 96$ **(e)** $\frac{dy}{dx}=4,\ 4$ **(f)** $\frac{dy}{dx}=0,\ 0$
2(a) $2x$ **(b)** $6x-5x^4$ **(c)** $2x-3$
3(a) $5x^4+3x^2+1$ **(b)** $1,\ -1$ **(c)** $9,\ -\frac19$
4(a) $3x^2-2,\ 1,\ 10$ **(b)** $45°,\ 84°17'$
5(a) $-2+2x,\ (1,2)$ **(b)** $4x^3+36x,\ (0,0)$
6 $\frac{dy}{dx}=2x+1,\ x=3,\ (3,12)$
7 $\frac{dy}{dx}=3x^2,\ x=2$ or $-2,\ (2,7),\ (-2,-9)$
8(a) $\frac{dy}{dx}=2x-3,\ \frac{dy}{dx}=5$ **(b)** $y=5x-16$
(c) $-\frac15,\ y=-\frac15x+4\frac45$
9 $\frac{dy}{dx}=2x-8$ **(a)** $y=-6x+14,\ x-6y+47=0$
(b) $y=4x-21,\ x+4y-18=0$
(c) $y=-8x+15,\ x-8y+120=0$
(d) $y=-1,\ x=4$
10(a) $2x-6,\ y=-6x,\ y=\frac16x$
(b) $3x^2-4,\ y=8x-16,\ x+8y-2=0$
(c) $2x-4x^3,\ y=2x+2,\ x+2y+1=0$
(d) $3x^2-3,\ y=0,\ x=1$
11(a) $y=4x-4,\ y=-\frac14x+4\frac12$
(b) $A=(0,-4),\ B=(0,4\frac12)$ **(c)** $AB=8\frac12,\ 8\frac12\ \mathrm{u}^2$
12(a) $y=-2x+10,\ x-2y+15=0$
(b) $A=(5,0),\ B=(-15,0)$ **(c)** $AB=20,\ 80\,\mathrm{u}^2$
13(b) $y=2x+5,\ y=-2x+5$ **(c)** $(0,5)$
14(a) $\frac{dy}{dx}=-2x,\ A=(-1,3)$ and $B=(1,3)$
(b) tangent at A: $y=2x+5$, tangent at B: $y=-2x+5$. They meet at $(0,5)$.
(c) normal at A: $y=-\frac12x+2\frac12$, normal at B: $y=\frac12x+2\frac12$. They meet at $(0,2\frac12)$.
15 $y=3x-2,\ x+3y=4,\ P=(0,-2),\ Q=(0,1\frac13),\ |\triangle QUP|=1\frac23$ square units
16(a) $2ax+b$ **(b)** $12ax^3+12bx^2$
17(a) $b=7,\ c=0$ **(b)** $b=-2,\ c=-3$ **(c)** $b=-10,\ c=25$ **(d)** $b=-1,\ c=-2$ **(e)** $b=-9,\ c=17$ **(f)** $b=-5\frac23,\ c=7$
18(a) all real x **(b)** $x<0$ **(c)** $x<-1$ or $x>1$ **(d)** $-1<x<0$ or $x>1$

Exercise **9E** (Page 247)

1(a) $-x^{-2}$ **(b)** $-5x^{-6}$ **(c)** $-3x^{-2}$ **(d)** $-10x^{-3}$ **(e)** $4x^{-4}$ **(f)** $-4x^{-3}-4x^{-9}$
2(a) $-\frac{1}{x^2}$ **(b)** $-\frac{2}{x^3}$ **(c)** $-\frac{4}{x^3}$ **(d)** $-\frac{3}{x^2}$
3(a) $4x^3-2x,\ 2$ **(b)** $2x^5-2x^3+2x,\ 2$ **(c)** $\frac12x^4-\frac12x^2+\frac12,\ \frac12$ **(d)** $2x,\ 2$ **(e)** $6x-5x^4,\ 1$ **(f)** $4x-5,\ -1$ **(g)** $-4x^{-2},\ -4$ **(h)** $-2x^{-4},\ -2$ **(i)** $-x^{-2}+2x^{-3},\ 1$
4(a) $y=3x^3-5x,\ \frac{dy}{dx}=9x^2-5$
(b) $y=x^2-4,\ \frac{dy}{dx}=2x$
(c) $y=\frac53x^3+\frac43x^2,\ \frac{dy}{dx}=5x^2+\frac83x$
(d) $y=3x-x^{-1},\ \frac{dy}{dx}=3+x^{-2}$
(e) $y=x^{-3}+7x^{-2},\ \frac{dy}{dx}=-3x^{-4}-14x^{-3}$
(f) $y=3x^3-5x+x^{-1},\ \frac{dy}{dx}=9x^2-5-x^{-2}$
5(a) $-\frac{6}{x^7}+\frac{8}{x^9}$ **(b)** $-\frac{1}{3x^2}$ **(c)** $-\frac{15}{x^4}$ **(d)** $-\frac{4}{5x^5}$ **(e)** $\frac{7}{x^2}$ **(f)** $-\frac{7}{2x^2}$ **(g)** $\frac{7}{3x^2}$ **(h)** $\frac{3}{x^6}$
6(a) $f'(x)=-\frac{1}{x^2},\ f'(3)=-\frac19,\ f'(\frac13)=-9$
(b) $(1,1),\ (-1,-1)$ **(c)** $(\frac12,2),\ (-\frac12,-2)$
(d) No; the derivative $-\frac{1}{x^2}$ can never be zero.
(e) Yes, all of them; the derivative $-\frac{1}{x^2}$ is negative for all points on the curve.

7(a) $f'(x) = \frac{3}{x^2}$, $f'(2) = \frac{3}{4}$, $f'(6) = \frac{1}{12}$
(b) $(1, -3)$ and $(-1, 3)$
8(a) $f'(x) = -\frac{12}{x^2}$, $f'(2) = -3$, $f'(6) = -\frac{1}{3}$
(b) At $M(2,6)$, tangent: $y = -3x + 12$, normal: $x - 3y + 16 = 0$. At $N(6,2)$, tangent: $y = -\frac{1}{3}x + 4$, normal: $y = 3x - 16$. **(c)** $(1, 12)$ and $(-1, -12)$
9 $-\frac{a}{x^2} + \frac{b}{cx^3}$
11(b) $a = 6$ and $y = 27x$, or $a = -6$ and $y = 3x$

Exercise 9F (Page 249)

1(a) $\frac{1}{2}x^{-\frac{1}{2}}$ **(b)** $-\frac{1}{2}x^{-1\frac{1}{2}}$ **(c)** $\frac{3}{2}x^{\frac{1}{2}}$ **(d)** $4x^{-\frac{1}{3}}$
(e) $-4x^{-1\frac{1}{3}}$ **(f)** $x^{-\frac{3}{4}} - 2x^{-\frac{5}{4}}$ **(g)** $\frac{49}{3}x^{1\frac{1}{3}}$
(h) $-\frac{10}{3}x^{-1\frac{2}{3}}$ **(i)** $6x^{-1 \cdot 6}$
2(a) $\frac{1}{2}x^{-\frac{1}{2}}$ **(b)** $\frac{1}{3}x^{-\frac{2}{3}}$ **(c)** $\frac{1}{4}x^{-\frac{3}{4}}$ **(d)** $5x^{-\frac{1}{2}}$
3(a) $y = x\sqrt{x} = x^1 \times x^{\frac{1}{2}} = x^{1\frac{1}{2}} = x^{\frac{3}{2}}$, $\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}}$
(b) $y = x^2\sqrt{x} = x^2 \times x^{\frac{1}{2}} = x^{2\frac{1}{2}} = x^{\frac{5}{2}}$, $\frac{dy}{dx} = \frac{5}{2}x^{1\frac{1}{2}}$
(c) $y = x^{-\frac{1}{2}}$, $\frac{dy}{dx} = -\frac{1}{2}x^{-1\frac{1}{2}}$
(d) $y = \frac{1}{x^1 \times x^{\frac{1}{2}}} = x^{-1\frac{1}{2}} = x^{-\frac{3}{2}}$, $\frac{dy}{dx} = -\frac{3}{2}x^{-2\frac{1}{2}}$
4(a) $12x^{-\frac{1}{2}}$ **(b)** $8x^{-\frac{2}{3}}$ **(c)** $\frac{2}{3}x^{-\frac{1}{3}}$ **(d)** $20x^{-\frac{1}{3}}$
(e) $4x^{-\frac{1}{2}}$ **(f)** $\frac{5}{2}x^{-\frac{1}{2}}$ **(g)** $3\sqrt{x}$ **(h)** $18\sqrt{x}$ **(i)** $10x^{\frac{3}{2}}$
(j) $-\frac{1}{2}x^{-1\frac{1}{2}}$ **(k)** $-3x^{-\frac{3}{2}}$ **(l)** $-\frac{15}{2}x^{-2\frac{1}{2}}$
5(a) $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$, $\frac{1}{2}$ & $\frac{1}{4}$ **(b)** $y = \frac{1}{2}x + \frac{1}{2}$, $y = \frac{1}{4}x + 1$
(c) $y = -2x + 3$, $y = -4x + 18$
6(a) $\frac{dy}{dx} = -2x^{-1\frac{1}{2}} = -\frac{2}{x\sqrt{x}}$ **(b)** $y = -\frac{1}{4}x + 3$, $y = 4x - 14$ **(c)** $\frac{1}{\sqrt{x}}$ is undefined when $x \le 0$; $-\frac{2}{x\sqrt{x}} < 0$ for all $x > 0$.
7(a) $\frac{1}{\sqrt{x}}$, $\frac{1}{2}$, -2 **(b)** $y = \frac{1}{2}x + 2$, $y = -2x + 12$
(c) $A(-4, 0)$, $B(6, 0)$ **(d)** $AB = 10$, 20 square units
8(a) $(1, 1)$ and $(-1, -1)$ **(b)** $(1, \frac{1}{2})$ **(c)** $(\frac{1}{4}, -\frac{1}{2})$
(d) $(0, 0)$, $(1, -1\frac{1}{4})$ $(-1, \frac{3}{4})$
11(a) At P, $\frac{dy}{dx} = 2a - 10$.
(b) At P, $y = a^2 - 10a + 9$.
(c) $a = 3$ and $y = -4x$, or $a = -3$ and $y = -16x$

Exercise 9G (Page 252)

1(a) $12(3x+7)^3$ **(b)** $30(5x-9)^5$ **(c)** $-28(5-4x)^6$
(d) $-4(1-x)^3$ **(e)** $24x(x^2+1)^{11}$ **(f)** $14x(x^2-2)^6$
(g) $-6x(5-x^2)^2$ **(h)** $42x(3x^2+7)^6$
(i) $-16x(5-2x^2)^3$ **(j)** $16x^3(x^4+1)^3$
(k) $45x^2(3x^3-7)^4$ **(l)** $-400x^4(5-8x^5)^9$
2(a) $-7(7x+2)^{-2}$ **(b)** $-6(x-1)^{-3}$
(c) $-12x^2(x^3-12)^{-5}$ **(d)** $-30x(5x^2-2)^{-4}$
(e) $-64x(7-x^2)^3$ **(f)** $-18(3x^2+1)(x^3+x+1)^5$
(g) $-4(1+2x+3x^2)(1-x-x^2-x^3)^3$
(h) $-5(3x^2-2x)(x^3-x^2)^{-6}$
(i) $-9(2x+3)(x^2+3x+1)^{-10}$
3(a) $25(5x-7)^4$ **(b)** $49(7x+3)^6$ **(c)** $180(5x+3)^3$
(d) $-21(4-3x)^6$ **(e)** $-22(3-x)$ **(f)** $-28(4x-5)^{-8}$
(g) $-30(3x+7)^{-6}$ **(h)** $12(10-3x)^{-5}$
(i) $84(5-7x)^{-5}$
4 $2x - 6$
5 $24x - 12$
6(a) $\dfrac{-2}{(2x+7)^2}$ **(b)** $\dfrac{1}{(2-x)^2}$ **(c)** $\dfrac{-5}{(3+5x)^2}$
(d) $\dfrac{21}{(4-3x)^2}$ **(e)** $\dfrac{-60}{(3x-1)^6}$ **(f)** $\dfrac{15}{(x+1)^4}$
7(a) $20(5x-4)^3$ **(b)** $y = 1$, $\frac{dy}{dx} = 20$
(c) $y = 20x - 19$, $x + 20y = 21$
8(a) $y = 24x - 16$ **(b)** $3x + y = 4$ **(c)** $x + 2y = 2$
9(a) $4(x-5)^3$, $(5, 0)$ **(b)** $6x(x^2-1)^2$, $(0, -1)$, $(1, 0)$, $(-1, 0)$
(c) $10(x+1)(2x+x^2)^4$, $(0, 0)$, $(-2, 0)$, $(-1, -1)$
(d) $\dfrac{-5}{(5x+2)^2}$, none
(e) $6(x-5)^5$, $(5, 4)$
(f) $\dfrac{-2x}{(1+x^2)^2}$, $(0, 1)$
10(a) $2\frac{1}{2}$ and 1 **(b)** 2 and $1\frac{1}{2}$
11(a) $\dfrac{5}{2\sqrt{5x+4}}$ **(b)** $\dfrac{-1}{\sqrt{3-2x}}$ **(c)** $\dfrac{7x}{\sqrt{x^2+1}}$
(d) $\dfrac{-x}{\sqrt{9-x^2}}$ **(e)** $\dfrac{x}{\sqrt{a^2-x^2}}$ **(f)** $\dfrac{1}{2\sqrt{x+4}}$
(g) $\dfrac{-3}{2\sqrt{4-3x}}$ **(h)** $\dfrac{m}{2\sqrt{mx+b}}$ **(i)** $\frac{1}{2}(5-x)^{-1\frac{1}{2}}$
12(a) $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{2x+5}}$, tangent: $y = \frac{1}{3}x + 2\frac{1}{3}$, normal: $y = -3x + 9$ **(b)** $\dfrac{dy}{dx} = \dfrac{x}{\sqrt{x^2-3}}$, tangent: $y = 2x - 3$, normal: $y = -\frac{1}{2}x + 2$
13(a) $-\dfrac{1}{\sqrt{3-2x}}$, none **(b)** $\dfrac{x-1}{\sqrt{x^2-2x+5}}$, $(1, 2)$
(c) $\dfrac{x-1}{\sqrt{x^2-2x}}$, none ($x = 1$ is outside the domain.)
14(a) $\dfrac{dy}{dx} = \dfrac{-x}{\sqrt{169-x^2}}$ **(c)** The tangent at P has gradient $-\frac{12}{5}$, the radius OP has gradient $\frac{5}{12}$.
15(a) $\dfrac{11(\sqrt{x}-3)^{10}}{2\sqrt{x}}$ **(b)** $\dfrac{-3}{4\sqrt{4-\frac{1}{2}x}}$
(c) $\dfrac{3\sqrt{2}}{(1-x\sqrt{2})^2}$ **(d)** $\frac{1}{2}(5-x)^{-1\frac{1}{2}}$
(e) $\frac{1}{2}a^2(1+ax)^{-1\frac{1}{2}}$ **(f)** $\frac{1}{2}b(c-x)^{-1\frac{1}{2}}$
(g) $-16\left(1 - \dfrac{1}{x^2}\right)\left(x + \dfrac{1}{x}\right)^3$
(h) $6\left(\dfrac{1}{2\sqrt{x}} - \dfrac{1}{2x\sqrt{x}}\right)\left(\sqrt{x} + \dfrac{1}{\sqrt{x}}\right)^5$

16(a) $\frac{dy}{dx} = 3(x-a)^2$, $a=4$ or $a=8$
(b) $\dfrac{dy}{dx} = \dfrac{-1}{(x+a)^2}$, $a=-5$ or $a=-7$
17(a) $a=8$, $b=1$ (b) $a=\frac{1}{16}$, $b=12$
18(a) $x+y(b-4)^2 = 2b-4$ (b)(i) $x+4y=0$
(ii) $x+y=6$

Exercise 9H (Page 255)

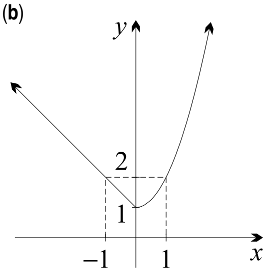
1 $2x^2(2x-3) = 4x^3-6x^2$
2 $4x-9$
3 $4x^3$
4(a) $u'=4x^3$, $v'=10(2x-1)^4$
(d) $x=0$, $x=\frac{1}{2}$, $x=\frac{2}{9}$
5(a) $(3x+5)^2(12x+5)$ (b) $x(x-1)^2(5x-2)$
(c) $2x^3(1-5x)^5(2-25x)$
6 $y=x$, $y=-x$
7(a) $y'=x^2(1-x)^4(3-8x)$
8(a) $(x-1)^3(5x-1)$, 1, $\frac{1}{5}$
(b) $(x+5)^4(6x+5)$, -5, $-\frac{5}{6}$
(c) $2(4-3x)^4(2-9x)$, $\frac{4}{3}$, $\frac{2}{9}$
(d) $3(3-2x)^4(1-4x)$, $1\frac{1}{2}$, $\frac{1}{4}$
(e) $x^2(x+1)^3(7x+3)$, 0, -1, $-\frac{3}{7}$
(f) $3x^2(3x-2)^3(7x-2)$, 0, $\frac{2}{3}$, $\frac{2}{7}$
(g) $x^4(1-x)^6(5-12x)$, 0, 1, $\frac{5}{12}$
(h) $(x-2)^2(4x-5)$, 2, $\frac{5}{4}$
(i) $(x+5)^5(7x+17)$, -5, $-\frac{17}{7}$
9(b) $y=2x-1$, $y=-\frac{1}{2}x+1\frac{1}{2}$
10 $y=8x+8$, $x+8y+1=0$
11(a) $10x(x^2+1)^4$, $(x^2+1)^4(11x^2+1)$
(b) $-8x(1-x^2)^3$, $2x^2(1-x^2)^3(3-11x^2)$
(c) $3(2x+1)(x^2+x+1)^2$,
$-2(x^2+x+1)^2(7x^2+4x+1)$
(d) $(4-9x^4)^3(4-45x^4)$
12(a) $10x^3(x^2-10)^2(x^2-4)$, $(0,0)$
(b) $(\sqrt{10},0)$, $(-\sqrt{10},0)$, $(2,-3456)$, $(-2,-3456)$
13(a) $\dfrac{3(3x+2)}{\sqrt{x+1}}$, $-\frac{2}{3}$ (b) $\dfrac{4(3x-1)}{\sqrt{1-2x}}$, $\frac{1}{3}$
(c) $\dfrac{10x(5x-2)}{\sqrt{2x-1}}$, 0 and $\frac{2}{5}$
14(a) $(x+1)^2(x+2)^3(7x+10)$, -1, -2, $-\frac{10}{7}$
(b) $6(2x-3)^3(2x+3)^4(6x-1)$, $1\frac{1}{2}$, $-1\frac{1}{2}$, $\frac{1}{6}$
(c) $\dfrac{1-2x^2}{\sqrt{1-x^2}}$, $\sqrt{\frac{1}{2}}$ and $-\sqrt{\frac{1}{2}}$
15(a) $y'=2a(x-3)$ (b) $y'(1)=-4a$, $y'(5)=4a$
(c) $y=-4ax+4a$, $y=4ax-20a$, $M=(3,-8a)$
(d) $V=(3,-4a)$

Exercise 9I (Page 258)

1(a) $u'=2$, $v'=3$
2(a) $\dfrac{1}{(x+1)^2}$ (b) $\dfrac{4}{(x+2)^2}$ (c) $\dfrac{1}{(1-3x)^2}$
(d) $\dfrac{-2}{(x-1)^2}$ (e) $\dfrac{-4}{(x-2)^2}$ (f) $\dfrac{4}{(x+2)^2}$
(g) $\dfrac{-5}{(2x-3)^2}$ (h) $\dfrac{-40}{(5+4x)^2}$
3(a) $\dfrac{x(x+2)}{(x+1)^2}$, $x=0$, $x=-2$ (b) $\dfrac{3+x^2}{(3-x^2)^2}$, none
(c) $\dfrac{x(2-x)}{(1-x)^2}$, $x=0$, $x=2$ (d) $\dfrac{1+x^2}{(1-x^2)^2}$, none
(e) $\dfrac{4x}{(x^2+1)^2}$, $x=0$ (f) $\dfrac{10x}{(x^2-4)^2}$, $x=0$
4 $\dfrac{-3}{(3x-2)^2}$
5(b) 5, $78^\circ 41'$ (c) $y=5x-12$, $x+5y+8=0$
6(b) $\frac{4}{3}$, $53^\circ 8'$ (c) $4x-3y=4$, $3x+4y=28$
7(a) $y=x$ (c) $A(-1,0)$, $B(0,\frac{1}{4})$
(d) $\frac{1}{8}$ square units (e) $(\frac{1}{3},\frac{1}{3})$
8(a) $\dfrac{c^2+2c}{(c+1)^2}$, $c=0$ or -2
(b) $\dfrac{12k}{(9-k)^2}=1$, $k=3$ or 27
9(a) $12(3x-7)^3$ (b) $\dfrac{x^2+2}{x^2}$ (c) $8x$ (d) $\dfrac{-2x}{x^2-9)^2}$
(e) $4(1-x)(4-x)^2$ (f) $\dfrac{-6}{(3+x)^2}$ (g) $20x^3(x^4-1)^4$
(h) $\dfrac{1}{2(2-x)^{\frac{3}{2}}}$ (i) $6x^2(x^3+5)$ (j) $\dfrac{3x^2+x-1}{4x\sqrt{x}}$
(k) $\frac{2}{3}x(5x^3-2)$ (l) $\dfrac{5}{(x+5)^2}$ (m) $\frac{1}{2}\sqrt{x}(3+5x)$
(n) $\dfrac{2(x-1)(x+1)(x^2+1)}{x^3}$ (o) $x^2(x-1)^7(11x-3)$
(p) $\dfrac{(x+1)(x-1)}{x^2}$
10(b) The denominator is positive, being a square, so the sign of y' is the sign of $\alpha-\beta$.

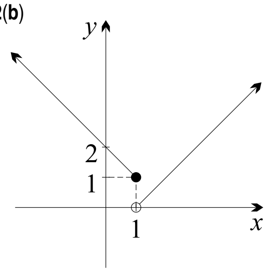
Exercise 9J (Page 261)

1(b)



(c) Yes (d) domain: all real x, range: $y \geq 1$

2(b)

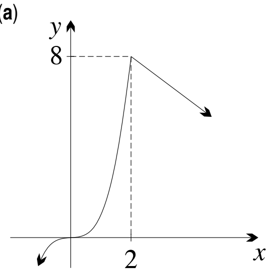


(c) No **(d)** domain: all real x, range: $y > 0$

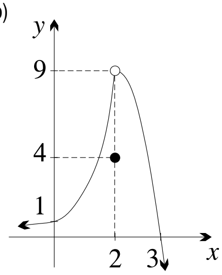
3(a) The denominator is zero when $x = 5$.
(b) $\dfrac{(x-5)(x+5)}{x-5} = x + 5$, for $x \neq 5$ **(c)** 10

4(a) 4 **(b)** -4 **(c)** 1 **(d)** -6 **(e)** -2 **(f)** 0

5(a)



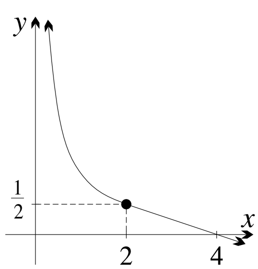
(b)



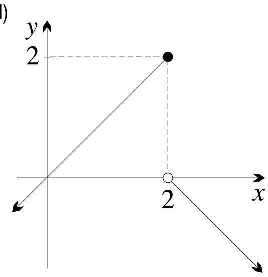
(a) $\lim\limits_{x\to2^-} f(x) = \lim\limits_{x\to2^+} f(x) = f(2) = 8$,
continuous at $x = 2$,
domain: all real x, range: $y \leq 8$

(b) $\lim\limits_{x\to2^-} f(x) = 9$, $\lim\limits_{x\to2^+} f(x) = 9$, $f(2) = 4$,
not continuous at $x = 2$,
domain: all real x, range: $y < 9$

(c)



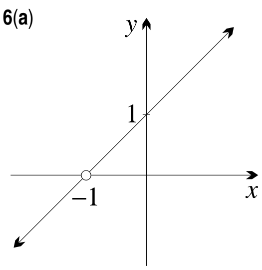
(d)



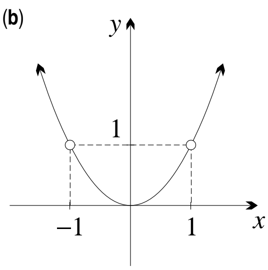
$\lim\limits_{x\to2^-} f(x) = \lim\limits_{x\to2^+} f(x) = \frac{1}{2}$, $f(2) = \frac{1}{2}$,
continuous at $x = 2$,
domain: $x > 0$, range: all real y

$\lim\limits_{x\to2^-} f(x) = f(2) = 2$, $\lim\limits_{x\to2^+} f(x) = 0$, not continuous at $x = 2$, domain: all real x, range: $y \leq 2$

6(a)



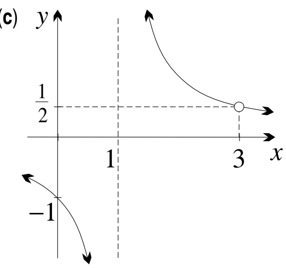
(b)



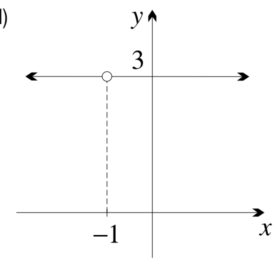
(a) $y = x + 1$ where $x \neq -1$,
domain: $x \neq -1$, range: $y \neq 0$

(b) $y = x^2$ where $x \neq -1$ or 1,
domain: $x \neq -1$ or 1, range: $y \geq 0$, $y \neq 1$

(c)



(d)



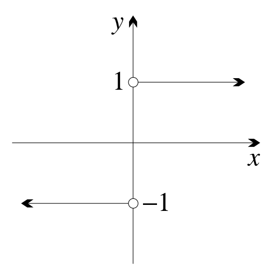
(c) $y = \dfrac{1}{x-1}$ where $x \neq 3$,
domain: $x \neq 1$ or 3, range: $y \neq 0$ or $\frac{1}{2}$

(d) $y = 3$ where $x \neq -1$,
domain: $x \neq -1$, range: $y = 3$

7(a)(i) $2x$ **(ii)** $2x + 5$ **(iii)** $4 - 2x$ **(b)(i)** $f'(x) = 2x$
(ii) $f'(x) = 2x + 5$ **(iii)** $f'(x) = 4 - 2x$

9(a) zeroes: 0, discontinuities: 3
(b) zeroes: 0, discontinuities: 7 and -1
(c) zeroes: 1 and -1, discontinuities: 0, 3 and -3

10(a)

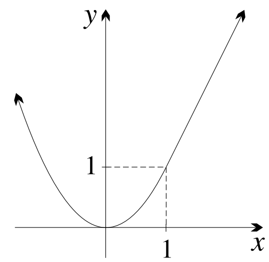


(c) $y = \begin{cases} 1, & \text{for } x > 0, \\ -1, & \text{for } x < 0, \\ \text{undefined}, & \text{for } x = 0. \end{cases}$
domain: $x \neq 0$, range: $y = 1$ or -1

Exercise **9K** (Page 264)

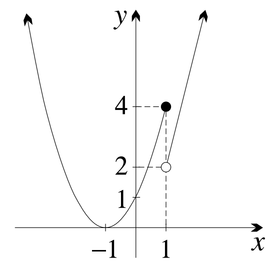
1(a) differentiable and continuous at $x = 0$, neither at $x = 2$
(b) differentiable and continuous at $x = 2$, neither at $x = 0$
(c) differentiable and continuous at $x = 0$, continuous but not differentiable at $x = 2$
(d) differentiable and continuous at $x = 2$, continuous but not differentiable at $x = 0$

2(a)



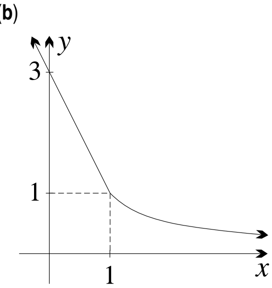
(d) The tangent at $x = 1$ has gradient 3, and so $f'(1) = 2$.

3(a)



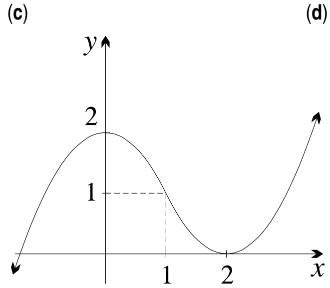
not differentiable at $x = 1$

(b)



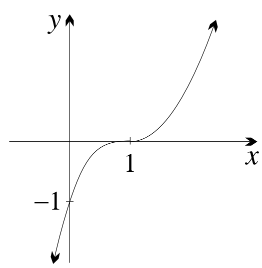
continuous but not differentiable at $x = 1$

(c)



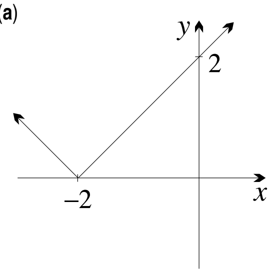
differentiable at $x = 1$, $f'(1) = -2$

(d)



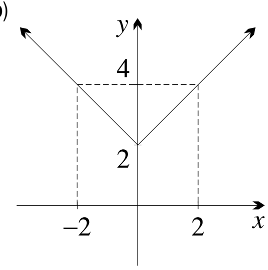
differentiable at $x = 1$, $f'(1) = 0$

4(a)



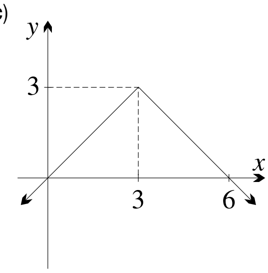
continuous but not differentiable at $x = -2$

(b)



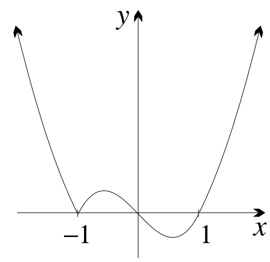
continuous but not differentiable at $x = 0$

(c)



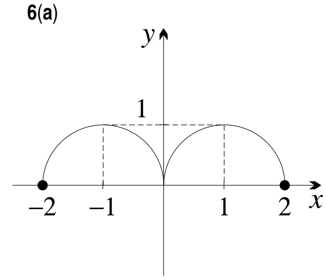
continuous but not differentiable at $x = 3$

5

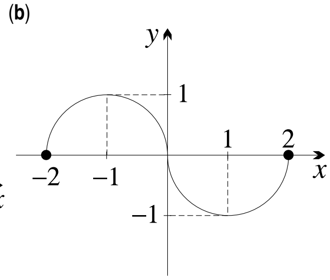


continuous but not differentiable at $x = -1$

6(a)



(b)



Review Exercise **9L** (Page 265)

1(a) $2x+5$ **(b)** $-2x$ **(c)** $6x-2$

2(a) $3x^2-4x+3$ **(b)** $6x^5-16x^3$
(c) $9x^2-30x^4$ **(d)** $2x+1$ **(e)** $-12x+7$
(f) $-6x^{-3}+2x^{-2}$ **(g)** $12x^2+12x^{-4}$
(h) $\frac{3}{2}x^{-\frac{1}{2}}-\frac{3}{2}x^{-1\frac{1}{2}}$ **(i)** $x^{-2}-2x^{-3}$

3(a) $-\dfrac{3}{x^2}$ **(b)** $-\dfrac{1}{3x^3}$ **(c)** $\dfrac{7}{2\sqrt{x}}$ **(d)** $\dfrac{6}{\sqrt{x}}$ **(e)** $-\dfrac{9}{2}\sqrt{x}$
(f) $-\dfrac{3}{x\sqrt{x}}$

4(a) $6x-2$ **(b)** $x-\frac{1}{2}$ **(c)** $10x+\dfrac{7}{x^2}$ **(d)** $-\dfrac{2}{x^2}-\dfrac{2}{x^3}$
(e) $\dfrac{2}{\sqrt{x}}$ **(f)** $3\sqrt{x}+\dfrac{3}{2\sqrt{x}}$

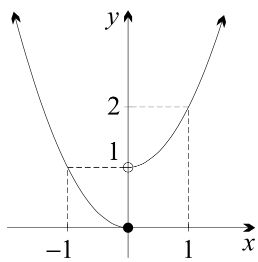
5(a) $9(3x+7)^2$ **(b)** $-4(5-2x)$ **(c)** $-\dfrac{5}{(5x-1)^2}$
(d) $\dfrac{14}{(2-7x)^3}$ **(e)** $\dfrac{5}{2\sqrt{5x+1}}$ **(f)** $\dfrac{1}{2(1-x)^{\frac{3}{2}}}$

6(a) $42x(7x^2-1)^2$ **(b)** $-15x^2(1+x^3)^{-6}$
(c) $8(1-2x)(1+x-x^2)^7$ **(d)** $-6x(x^2-1)^4$
(e) $-\dfrac{x}{\sqrt{9-x^2}}$ **(f)** $\dfrac{x}{(9-x^2)^{\frac{3}{2}}}$

7(a) $x^8(x+1)^6(16x+9)$ **(b)** $\dfrac{x(2-x)}{(1-x)^2}$
(c) $2x(4x^2+1)^3(20x^2+1)$ **(d)** $\dfrac{12}{(2x+3)^2}$
(e) $(9x-1)(x+1)^4(x-1)^3$ **(f)** $\dfrac{(x-5)(x+1)}{(x-2)^2}$

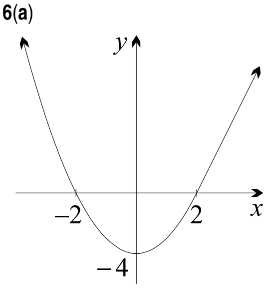
8 $\frac{dy}{dx}=2x+3$
(a) $3,\ 71^\circ 34'$ **(b)** $1,\ 45^\circ$ **(c)** $-1,\ 135^\circ$

9(a) tangent: $y = -3x$, normal: $3y = x$
(b) tangent: $y = -2$, normal: $x = 1$
(c) $(1, -2)$ and $(-1, 2)$ **(d)** $(2, 2)$ and $(-2, -2)$
10(a) $y = -x - 4$, $y = x - 8$ **(b)** $A(-4, 0)$, $B(8, 0)$
(c) $AB = 12$, $|\triangle ABP| = 36$ square units
11 The tangent is $y = x$.
12(a) $(1, -6\frac{2}{3})$, $(-1, -7\frac{1}{3})$ **(b)** $(-1, \frac{2}{3})$
13 At $(1, -3)$ the tangent is ℓ: $x + y + 2 = 0$, at $(-1, 3)$ the tangent is $x + y - 2 = 0$.
14(a) -2 **(b)** $\frac{1}{2}$ **(c)** $\frac{1}{2}$
15(a)



(b) 0, 1, 0; not continuous at $x = 0$
(c) domain: all real x, range: $y \geq 0$

16(a)

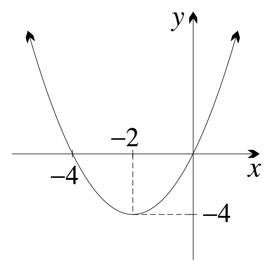


(b) 0, 0, 0; yes.
(c) 4, 4; differentiable at $x = 2$, with $f'(2) = 4$
(d) domain: all real x, range: $y \geq 4$

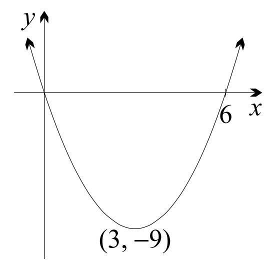
Chapter Ten

Exercise 10A (Page 272)

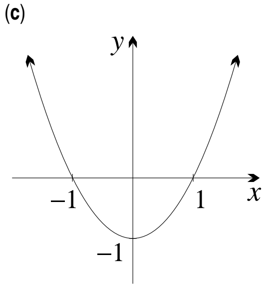
1(a)



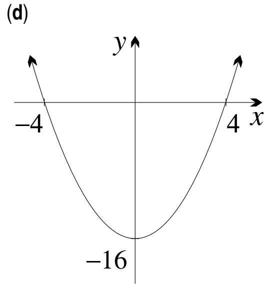
(b)



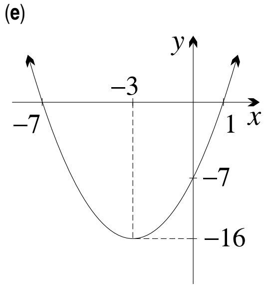
(c)



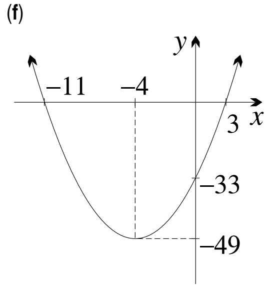
(d)



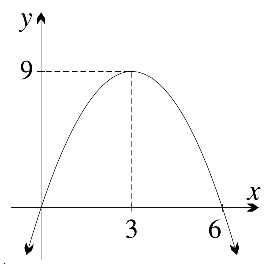
(e)

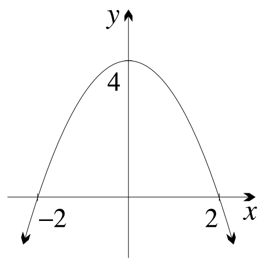


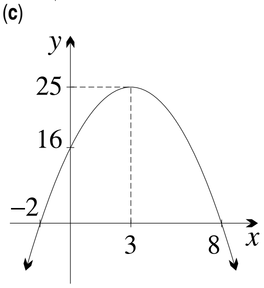
(f)

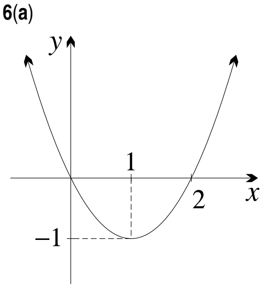


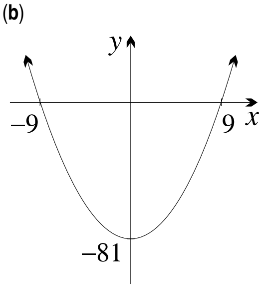
2(a) $-4 < x < 0$
(b) $x < 0$ or $x > 6$
(c) $-1 \leq x \leq 1$
(d) $x \leq -4$ or $x \geq 4$
(e) $-7 < x < 1$
(f) $x \leq -11$ or $x \geq 3$
3(a) $y = (x - 4)(x - 6)$
(b) $y = (x - 3)(x - 8)$
(c) $y = (x + 3)(x - 5)$
(d) $y = (x + 6)(x + 1)$
4(a) $y = (x - 3)(x - 5)$
(b) $y = x(x + 4)$
(c) $y = (x + 1)(x - 3)$
(d) $y = (x + 2)(x + 5)$

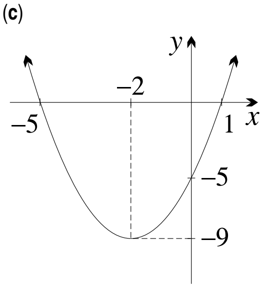
5(a)


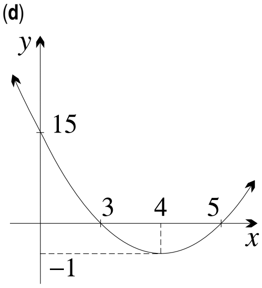
(b)


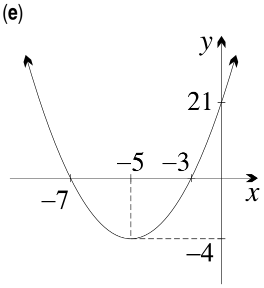
(c)


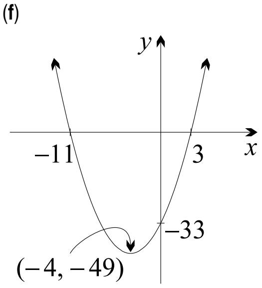
6(a)


(b)


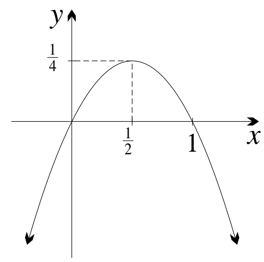
(c)


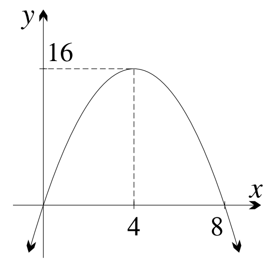
(d)


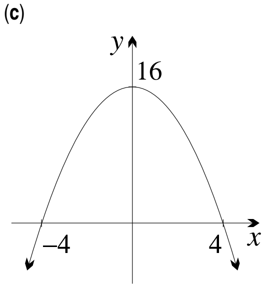
(e)


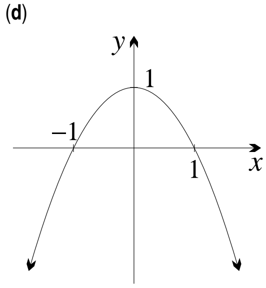
(f)


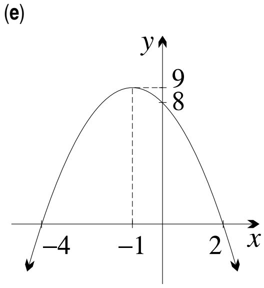
7(a) $x > 1$ or $x < -5$ **(b)** $x \geq 5$ or $x \leq 3$
(c) $-7 \leq x \leq -3$

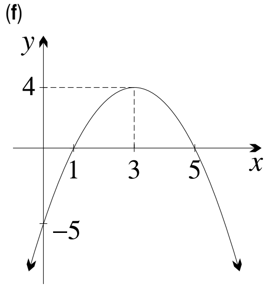
8(a)


(b)


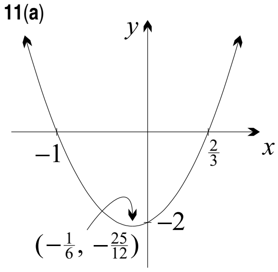
(c)


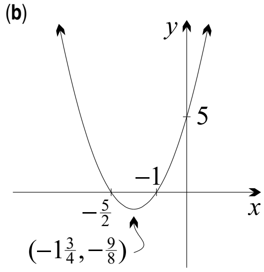
(d)


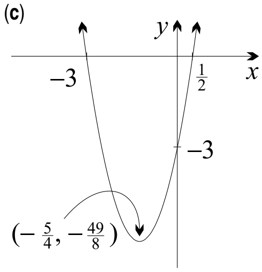
(e)


(f)


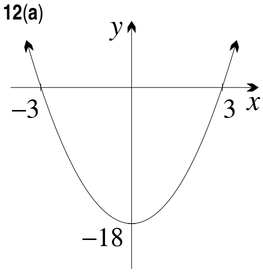
9(a) $-1 < x < 1$ **(b)** $-4 < x < 2$
(c) $x < 1$ or $x > 5$
10(a) $0 < x < 1$ **(b)** $x \leq 7$ or $x \geq 0$ **(c)** $2 \leq x \leq 3$
(d) $x < 0$ or $x > 9$ **(e)** $-5 < x < 4$ **(f)** $x = 5$

11(a)


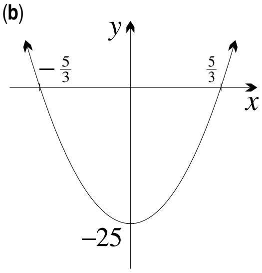
(b)


(c)


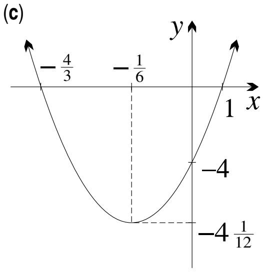
12(a)



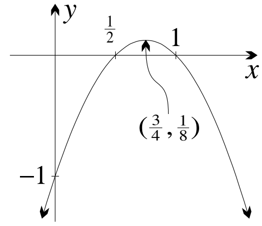
(b)



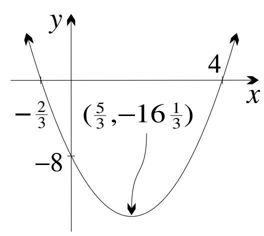
(c)



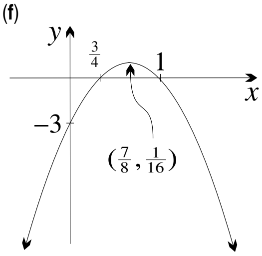
(d)



(e)



(f)

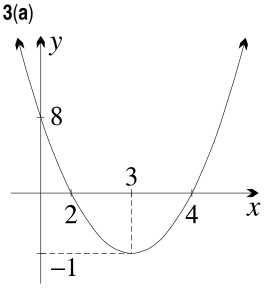


13(a) $y = 3(x-2)(x-8)$ **(b)** $y = -\frac{3}{16}(x-2)(x-8)$
(c) $y = -(x-2)(x-8)$ **(d)** $y = -\frac{5}{2}(x-2)(x-8)$
(e) $y = \frac{4}{3}(x-2)(x-8)$ **(f)** $y = -\frac{20}{7}(x-2)(x-8)$
14(a) $y = (x+1)(x-2)$ **(b)** $y = -(x+3)(x-2)$
(c) $y = 3(x+2)(x-4)$ **(d)** $y = -\frac{1}{2}(x-2)(x+2)$

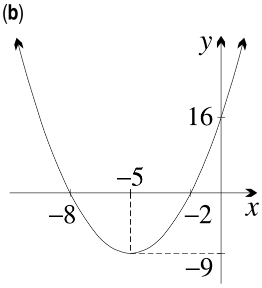
Exercise **10B** (Page 277)

1(a) $y = (x-2)^2 + 1$ **(b)** $y = (x+3)^2 + 2$
(c) $y = (x-1)^2 + 7$
2(a) Shift $y = x^2$ up 1 unit. **(b)** Shift $y = x^2$ down 3 units. **(c)** Shift $y = x^2$ right 4 units.
(d) Shift $y = x^2$ left 2 units.
(e) Shift $y = x^2$ right 1 unit and up 4 units.
(f) Shift $y = x^2$ right 3 units and down 2 units.
(g) Shift $y = x^2$ left 7 units and up 2 units.
(h) Shift $y = x^2$ left 4 units and down 1 unit.

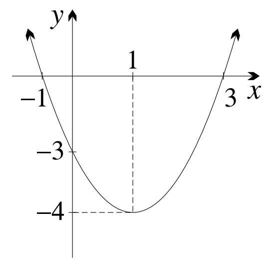
3(a)



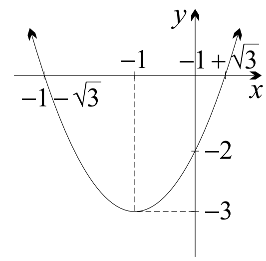
(b)



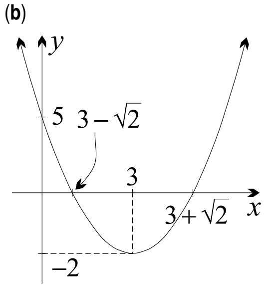
(c)



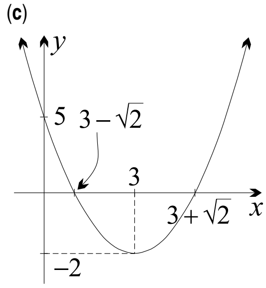
4(a)



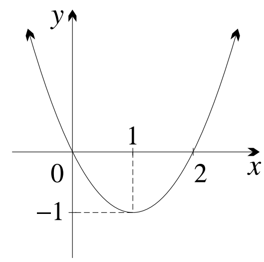
(b)



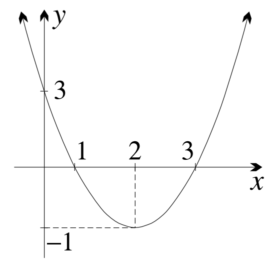
(c)



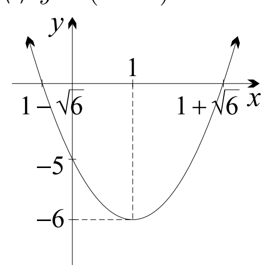
5(a) $y = (x-1)^2 - 1$



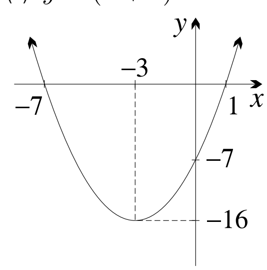
(b) $y = (x-2)^2 - 1$



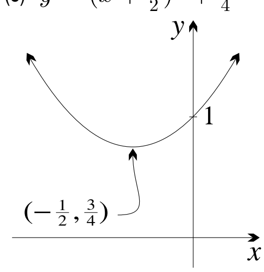
(c) $y = (x-1)^2 - 6$



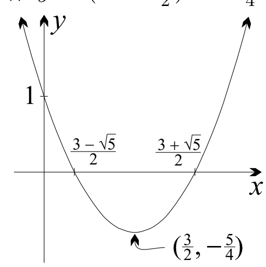
(d) $y = (x+3)^2 - 16$



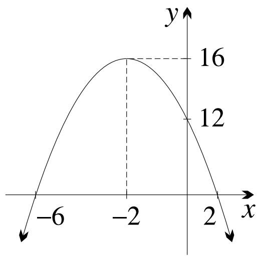
(e) $y = (x+\frac{1}{2})^2 + \frac{3}{4}$



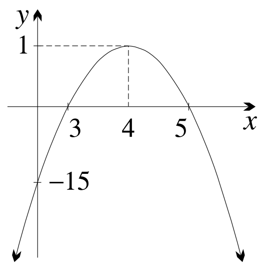
(f) $y = (x-1\frac{1}{2})^2 - 1\frac{1}{4}$



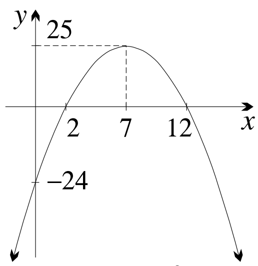
6(a)



(b)



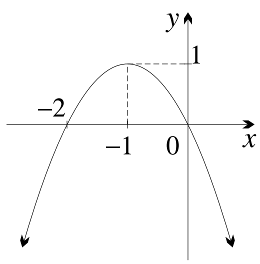
(c)



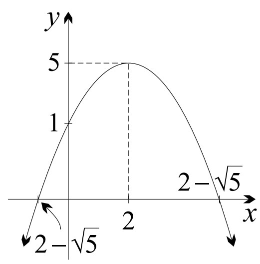
7(a) $y = (x-1)^2 + 2$ **(b)** $y = (x+2)^2 - 3$
(c) $y = -(x-2)^2 - 1$ **(d)** $y = -(x-3)^2 + 5$

8(a) $y = (x-2)^2 + 5$
(b) $y = x^2 - 3$
(c) $y = (x+1)^2 + 7$
(d) $y = (x-3)^2 - 11$

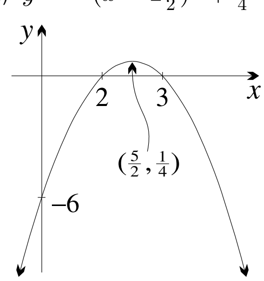
9(a) $y = -(x+1)^2 + 1$



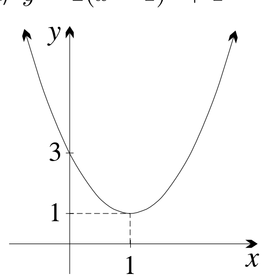
(b) $y = -(x-2)^2 + 5$



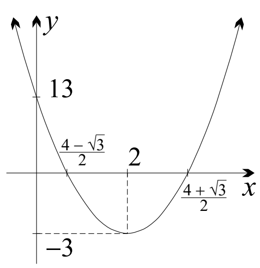
(c) $y = -(x - 2\frac{1}{2})^2 + \frac{1}{4}$



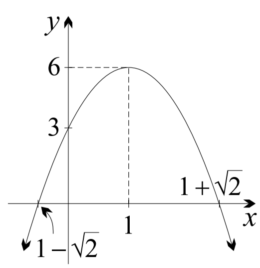
(d) $y = 2(x-1)^2 + 1$



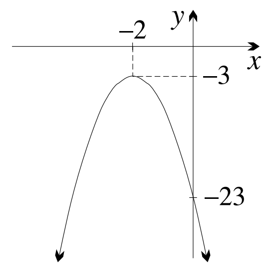
(e) $y = 4(x-2)^2 - 3$



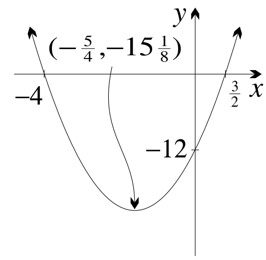
(f) $y = -3(x-1)^2 + 6$



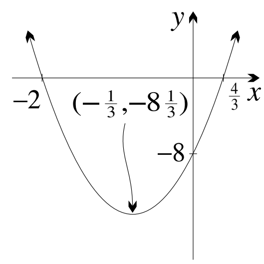
(g) $y = -5(x+2)^2 - 3$



(h) $y = 2(x + \frac{5}{4})^2 - 15\frac{1}{8}$



(i) $y = 3(x + \frac{1}{3})^2 - 8\frac{1}{3}$



10 Let $h = -4$ and $k = 2$ in the formula $y = a(x-h)^2 + k$.
(a) $y = (x+4)^2 + 2$ **(b)** $y = 3(x+4)^2 + 2$
(c) $y = -\frac{2}{49}(x+4)^2 + 2$ **(d)** $y = \frac{7}{8}(x+4)^2 + 2$
(e) $y = -\frac{1}{8}(x+4)^2 + 2$ **(f)** $y = \frac{18}{25}(x+4)^2 + 2$

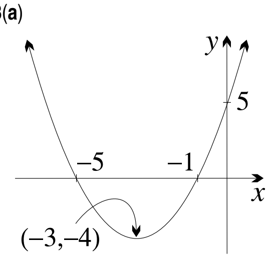
11 The vertex is $\left(-\dfrac{b}{2a}, \dfrac{4ac-b^2}{4a}\right)$, the zeroes are $x = \dfrac{-b+\sqrt{b^2-4ac}}{2a}$ & $x = \dfrac{-b-\sqrt{b^2-4ac}}{2a}$, the y-intercept is c.

Exercise **10C** (Page 279)

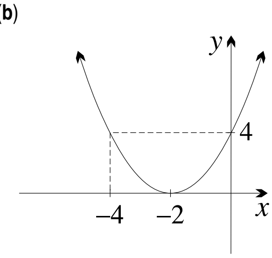
1(a) -1 or -2 **(b)** -3 **(c)** 3 or 6 **(d)** -3 or $\frac{1}{2}$
(e) $\frac{1}{2}$ or 2 **(f)** $\frac{2}{3}$ or 1

2(a) $-1+\sqrt{3}$ or $-1-\sqrt{3}$, $0{\cdot}7321$ or $-2{\cdot}732$
(b) $2+\sqrt{3}$ or $2-\sqrt{3}$, $3{\cdot}732$ or $0{\cdot}2679$
(c) $\frac{1}{2}(-3+\sqrt{17})$ or $\frac{1}{2}(-3-\sqrt{17})$, $0{\cdot}5616$ or $-3{\cdot}562$
(d) $-1+\sqrt{5}$ or $-1-\sqrt{5}$, $1{\cdot}236$ or $-3{\cdot}236$
(e) $\frac{1}{3}(1+\sqrt{7})$ or $\frac{1}{3}(1-\sqrt{7})$, $1{\cdot}215$ or $-0{\cdot}5486$
(f) $\frac{1}{2}(-2+\sqrt{6})$ or $\frac{1}{2}(-2-\sqrt{6})$, $0{\cdot}2247$ or $-2{\cdot}225$

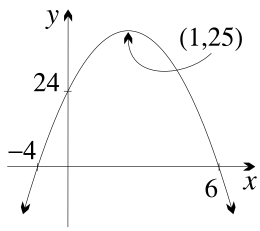
3(a)



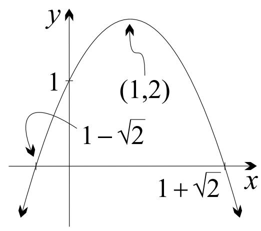
(b)



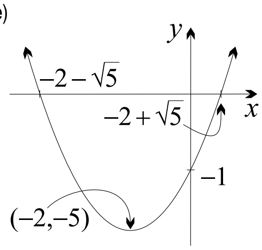
(c)



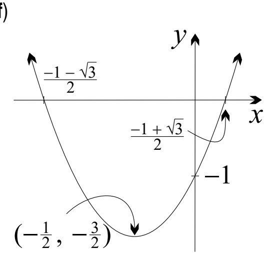
(d)



(e)



(f)



4(a) $-5 < x < -1$ **(b)** $x \neq -2$
(c) $x \leq -4$ or $x \geq 6$ **(d)** $1 - \sqrt{2} \leq x \leq 1 + \sqrt{2}$
5(a) $(x+5)(x-4)$ **(b)** $(x+1)(x+4)$ **(c)** $(x-2)(x-7)$
6(a) $(0, 3)$ and $(5, 8)$. The line and parabola intersect twice.
(b) $(-1, -9)$. The line meets the parabola once, and so it is a tangent to the parabola.
(c) The line and the parabola do not intersect.
(d) $(1, 2)$ and $(2, 1)$. The line and the parabola intersect twice.
7(a) 16, twice **(b)** 5, twice **(c)** 0, once
(d) -31, no times

Exercise **10D** (Page 280)

1(a) $u = 4$ or 9 **(b)** 3, -3, 2 or -2
2(a) $u = 25$ or 4 **(b)** 2, -2, 5 or -5
3(a) $u = 1$ or 8 **(b)** 1 or 2
4(a) $u = 27$ or 1 **(b)** 1 or 3
5(a) $u = 6$ or 12 **(b)** 3, -3, 4 or -2
6(a) $u = -12$ or 4 **(b)** $2 + 2\sqrt{2}$ or $2 - 2\sqrt{2}$
7(a) $u = \frac{4}{3}$ or 2 **(b)** $\sqrt{2}$, $-\sqrt{2}$, $\frac{2}{3}\sqrt{3}$ or $-\frac{2}{3}\sqrt{3}$
8(a) $u = \frac{1}{16}$ or 16 **(b)** $\frac{1}{4}$, $-\frac{1}{4}$, 4 or -4
9(a) 2, -2, $\frac{1}{3}\sqrt{3}$, $-\frac{1}{3}\sqrt{3}$ **(b)** -1,3 **(c)** 1, -3
(d) 1,-2
10(a) 1 or 2 **(b)** 1 or 0 **(c)** 2 or 3 **(d)** -1 or 2
11(a) $(1, 3)$ and $(\frac{9}{5}, \frac{13}{5})$ **(b)** $(2, -1)$
(c) $(-2, 3)$ and $(\frac{100}{13}, -\frac{45}{13})$
12(a) 30°, 90°or 150° **(b)** 45°or 225°
(c) 120°, 180°or 240° **(d)** 135°or 315°
13(a) $\frac{1}{2}$, 2, $\frac{1}{2}(-3 + \sqrt{5})$ or $\frac{1}{2}(-3 - \sqrt{5})$
(b) $\frac{1}{2}(-3 + \sqrt{29})$ or $\frac{1}{2}(-3 - \sqrt{29})$

Exercise **10E** (Page 283)

1(a) minimum of -4 when $x = 3$
(b) minimum of 4 when $x = 1$
(c) minimum of -2 when $x = 3$
(d) minimum of -9 when $x = 5$
(e) minimum of 7 when $x = 0$
(f) minimum of -4 when $x = -2$
2(a) maximum of -1 when $x = -2$
(b) maximum of 8 when $x = 1$
(c) maximum of 6 when $x = -3$
(d) maximum of 9 when $x = 0$
(e) maximum of 4 when $x = 2$
(f) maximum of 9 when $x = -1$
3(a) minimum of -4 when $x = 5$
(b) minimum of -9 when $x = -5$
(c) maximum of 9 when $x = 4$
4(b) 4 when $x = 2$
5(b) 225, when $x = 15$
6(b) 18 when $x = 3$
7(a) 1 second **(b)** 105 metres
8(a) 2 machines **(b)** \$7000
9(a) minimum of $-\frac{1}{4}$ when $x = \frac{3}{2}$
(b) minimum of $-\frac{1}{4}$ when $x = 2\frac{1}{2}$
(c) maximum of $\frac{5}{4}$ when $x = \frac{3}{2}$
(d) maximum of $\frac{9}{8}$ when $x = \frac{1}{4}$
(e) maximum of $-\frac{5}{4}$ when $x = \frac{1}{2}$
(f) minimum of $\frac{11}{4}$ when $x = \frac{1}{4}$
10(c) 24 metres long and 12 metres wide
11(b) $8x - x^2$ **(c)** $16\,\text{m}^2$
12 $100\,\text{m}^2$
13(c) $x = 15$ **(d)** $300\,\text{m}^2$
14(c) $x = \frac{800}{3}$ and $y = 200$
15(c) $15\frac{5}{8}\,\text{cm}^2$
16(b) 23
17(b) \$40.50
18(c) $\frac{1280}{41}$ cm and $\frac{2000}{41}$ cm

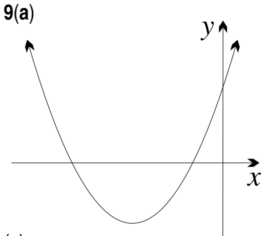
Exercise **10F** (Page 287)

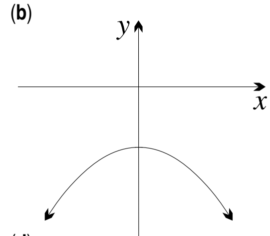
1(a) irrational, unequal
(b) unreal (that is, no roots)
(c) rational, equal (that is, a single rational root)
(d) rational, unequal
(e) unreal (that is, no roots)
(f) rational, unequal
2(a) $\Delta = 4$, two rational roots
(b) $\Delta = 0$, one rational root
(c) $\Delta = 32$, two irrational roots

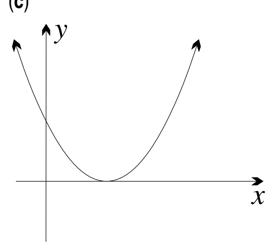
(d) $\Delta = -7$, no roots
(e) $\Delta = 24$, two irrational roots
(f) $\Delta = 0$, one rational root
(g) $\Delta = -31$, no roots
(h) $\Delta = 361 = 19^2$, two rational roots
(i) $\Delta = 36$, two rational roots
3(a) $\Delta = 100 - 4g,\ g = 25$ **(b)** $\Delta = 16 - 4g,\ g = 4$
(c) $\Delta = 1 - 8g,\ g = \frac{1}{8}$ **(d)** $\Delta = 44 - 4g,\ g = 11$
4(a) $\Delta = 4 - 4k,\ k \le 1$ **(b)** $\Delta = 64 - 8k,\ k \le 8$
(c) $\Delta = 4 - 12k,\ k \le \frac{1}{3}$ **(d)** $\Delta = 33 - 16k,\ k \le \frac{33}{16}$
5(a) $\Delta = 36 - 4\ell,\ \ell > 9$ **(b)** $\Delta = 100 - 4\ell,\ \ell > 25$
(c) $\Delta = 1 - 8\ell,\ \ell > \frac{1}{8}$ **(d)** $\Delta = 48 - 12\ell,\ \ell > 4$
6(a) $\Delta = g^2 - 4g,\ g = 4$ (If $g = 0$, then $y = 1$ for all x, and so y is never zero.)
(b) $\Delta = 49 - 4g^2,\ g = \frac{7}{2}$ or $-\frac{7}{2}$
(c) $\Delta = 16(g^2 - 6g - 7),\ g = -1$ or 7
(d) $\Delta = 4(g^2 + 2g - 8),\ g = -4$ or 2
7(a) $\Delta = k^2 - 16,\ k \le -4$ or $k \ge 4$
(b) $\Delta = 9k^2 - 36,\ k \le -2$ or $k \ge 2$
(c) $\Delta = k^2 + 12k + 20,\ k \le -10$ or $k \ge -2$
(d) $\Delta = k^2 - 12k,\ k \le 0$ or $k \ge 12$
8(a) $\Delta = \ell^2 - 16,\ -4 < \ell < 4$
(b) $\Delta = 36 - 4\ell^2,\ \ell < -3$ or $\ell > 3$
(c) $\Delta = \ell^2 + 2\ell - 15,\ -5 < \ell < 3$
(d) $\Delta = 4(4\ell^2 + \ell + 4)$, no values
(e) $\Delta = 20\ell^2 - 20\ell,\ 0 < \ell < 1$
(f) $\Delta = 4(\ell + 6),\ \ell < -6$
9(b) $\Delta = 28$. Since $\Delta > 0$, the quadratic equation has two roots.
10(a) They intersect twice.
(b) They do not intersect.
(c) They intersect once.
(d) They intersect twice.
11(a) $m < -\frac{1}{4}$ **(b)** $m > -\frac{9}{4}$ **(c)** $m = \frac{39}{8}$
(d) $-1 < m < 2$
12(b) $b = -4$
13(a) $\Delta = (m + 4)^2$ **(b)** $\Delta = (m - 2)^2$
(c) $\Delta = (2m - n)^2$ **(d)** $\Delta = (4m - 1)^2$
14(a) $\Delta = \lambda^2 + 4$ **(b)** $\Delta = 4\lambda^2 + 48$ **(c)** $\Delta = \lambda^2 + 16$
(d) $\Delta = (\lambda - 1)^2 + 8$
15(b) $b^2 = ac$

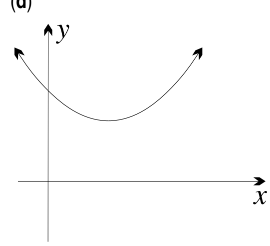
Exercise **10G** (Page 291)

1(a) $\Delta = -19 < 0,\ a = 1 > 0$ **(b)** $\Delta = 5 > 0$
(c) $\Delta = -11 < 0,\ a = -1 < 0$ **(d)** $\Delta = 13 > 0$
2(a) positive definite **(b)** indefinite
(c) negative definite **(d)** indefinite
(e) indefinite **(f)** positive definite
3(a)(i) $k > 4$ **(ii)** $k \le 4$
(b)(i) $k > \frac{25}{32}$ **(ii)** $k \le \frac{25}{32}$
4(a)(i) $m < -9$ **(ii)** $m \ge -9$
(b)(i) $m > \frac{9}{8}$ **(ii)** $m \le \frac{9}{8}$
5(a)(i) $-8 < k < 8$ **(ii)** $k \le -8$ or $k \ge 8$
(b)(i) $0 < k < 24$ **(ii)** $k \le 0$ or $k \ge 24$
(c)(i) $2 < k < 5$ **(ii)** $k \le 2$ or $k \ge 5$
6(a)(i) $-4 < m < 4$ **(ii)** $m \le -4$ or $m \ge 4$
(b)(i) $-8 < m < 12$ **(ii)** $m \le -8$ or $m \ge 12$
(c)(i) $0 < m < 2$ **(ii)** $m \le 0$ or $m \ge 2$
8(a) -4 and 4
(b) 2 (When $\ell = 0$, it is not a quadratic.)
(c) 1 (When $\ell = -\frac{3}{10}$, the expression becomes $-\frac{1}{10}(5x - 4)^2$, which is a multiple of a perfect square, but is not itself a perfect square.)
(d) $-\frac{2}{9}$ and 2

9(a)


(b)


(c)


(d)


10(a) $a > 0$ and $b^2 < 3ac$
(b) $a < 0$ and $b^2 < 3ac$ **(c)** $b^2 \ge 3ac$

Exercise **10H** (Page 294)

1(a) $x^2 - 11x + 28 = 0$ **(b)** $x^2 + 3x - 10 = 0$
(c) $x^2 + 17x + 72 = 0$ **(d)** $x^2 - 12x = 0$
2(a) $\alpha + \beta = -7,\ \alpha\beta = 10$, the roots are -2 and -5.
3(a) $2, 5$ **(b)** $-1, -6$ **(c)** $-1, 0$ **(d)** $-\frac{3}{2}, -\frac{1}{2}$ **(e)** $2, \frac{1}{3}$
(f) $3, 6$
4(a) $3, 2$ **(b)** $2, 5$ **(c)** $\frac{1}{3}, -\frac{4}{3}$ **(d)** $-1, \frac{4}{5}$ **(e)** $\frac{2}{3}$ $-\frac{4}{3}$
(f) $-5, -1$
5(a) 4 **(b)** -5 **(c)** 8 **(d)** $-\frac{4}{5}$

6(a) 7 **(b)** 12 **(c)** 1008 **(d)** $\frac{7}{6}$
7(a) 3 **(b)** 2 **(c)** 21 **(d)** 6 **(e)** 20 **(f)** $\frac{3}{2}$ **(g)** 5
(h) $\frac{5}{2}$
8(a) $\frac{5}{2}$ **(b)** $\frac{1}{2}$ **(c)** -1 **(d)** 5 **(e)** $\frac{5}{8}$ **(f)** 10 **(g)** $\frac{21}{4}$
(h) 21
9(b)(i) $3,\ 1,\ 5$ **(ii)** $-5,\ -7,\ 53$
(c) $\sqrt{5},\ \sqrt{53}$
10(a) $x^2-4x+3=0$ **(b)** $x^2-4x-12=0$
(c) $x^2+5x+4=0$ **(d)** $4x^2-8x+3=0$
(e) $x^2-4x+1=0$ **(f)** $x^2+2x-4=0$
11(a) -1 **(b)** 1 **(c)** $\frac{1}{10}$ **(d)** -3
12(a) $\frac{5}{2}$ **(b)** -4 **(c)** $\frac{7}{2}$ **(d)** $\frac{1}{3}$
13(a) 8 **(b)** $\frac{1}{3}\left(2+2\sqrt{10}\right)$ or $\frac{1}{3}\left(2-2\sqrt{10}\right)$
14(b) $\frac{305}{27}$
15(b) 6 **(c)** 3 **(d)** 27
16(b) $-\frac{12}{5}$ **(c)** $\left(-\frac{6}{5}, \frac{3}{5}\right)$

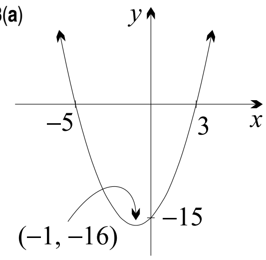
Exercise **10I** (Page 297)

1 $a=2$ and $b=3$
2 $a=1$ and $b=2$
3(a) $c=1$ **(b)** $a=2$ **(c)** $b=3$
4(a) $c=3$ **(b)** $a=1$ **(c)** $b=3$
5(a) $a=1, b=7, c=12$ **(b)** $2(x+1)^2-(x+1)-7$
(c) $a=2$, $b=16$ and $c=35$
9(a) $(x+1)^2-2(x+1)+1$ **(b)** $(n-4)^2+8(n-4)+16$
(c) $(x+2)^2-4(x+2)+4$
10 $a=3$, $b=-3$ and $c=1$
11(b) 961 **(d)** n^2

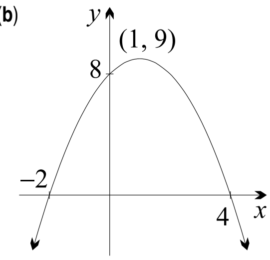
Review Exercise **10J** (Page 298)

1(a) 2 or 3 **(b)** -3 or 7 **(c)** $x=-\frac{3}{2}$ or 5
2(a) $1+\sqrt{2}$ or $1-\sqrt{2}$ **(b)** $\frac{1}{2}(7+\sqrt{5})$ or $\frac{1}{2}(7-\sqrt{5})$
(c) $\frac{1}{2}(2+\sqrt{2})$ or $\frac{1}{2}(2-\sqrt{2})$
3(a) $y=(x+3)^2-2$ **(b)** $y=(x-4)^2-13$
(c) $y=-(x+2)^2+9$
4(a) Shift $y=x^2$ up by 5 units.
(b) Shift $y=x^2$ down by 1 unit.
(c) Shift $y=x^2$ right by 3 units.
(d) Shift $y=x^2$ left by 4 units and up by 7 units.
5(a) $y=(x-2)(x-5)$ **(b)** $y=(x-3)(x+1)$
(c) $y=(x+4)(x-7)$ **(d)** $y=(x+5)(x+2)$
6(a) $y=(x-1)^2$ **(b)** $y=x^2-2$
(c) $y=(x+1)^2+5$ **(d)** $y=(x-4)^2-9$
7(a) $y=(x-1)(x-5)$ **(b)** $y=-x(x-3)$
(c) $y=(x-1)^2+2$ **(d)** $y=-(x-4)^2+1$

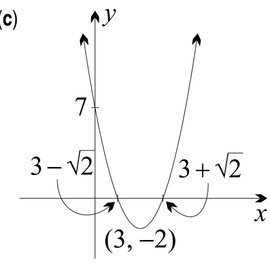
8(a)



(b)



(c)



9(a) $x>2$ or $x<-2$ **(b)** $-4\le x\le 3$
(c) $x\ge 5$ or $x\le -2$
10(a) $(-1,1)$ and $(4,16)$. The line and parabola intersect twice.
(b) $(-2,-4)$. The line meets the parabola once, and so it is a tangent to the parabola.
11(a) $1,\ -1,\ 3$ or -3 **(b)** 2 or -1 **(c)** 0 or 2
(d) $-3,\ -1,\ 2$ or 4
12(a) minimum of 4 when $x=1$
(b) maximum of 1 when $x=2$
(c) minimum of 3 when $x=2$
(d) minimum of $-\frac{23}{4}$ when $x=-\frac{3}{2}$
13(b) $r_1=r_2=5$
14(a) $\Delta=25$, two rational roots
(b) $\Delta=0$, one rational root
(c) $\Delta=29$, two irrational roots
15(a) $k\le 9$ **(b)** $k=9$ **(c)** $k>9$
16(a) $m\ge 2$ or $m\le -2$ **(b)** $m=2$ or $m=-2$
(c) $-2<m<2$
17(a) positive definite **(b)** negative definite
(c) indefinite
18 $g>\frac{2}{3}$
19 $\ell=6$ or $\ell=-3$
20(a) 3 **(b)** -2 **(c)** $-\frac{3}{2}$ **(d)** 13
21(a) $\frac{4}{3}$ **(b)** $-\frac{1}{3}$ **(c)** $\frac{4}{27}$ **(d)** $\frac{22}{9}$
22 $x^2-4x+1=0$
23 $a=3$, $b=1$ and $c=5$

Chapter Eleven

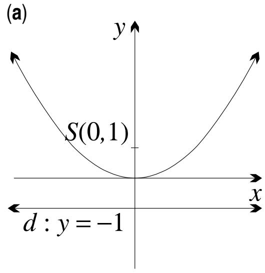
Exercise 11A (Page 303)

1(a) $y=-2$ **(b)** $x=-1$ **(c)** $y=2$ **(d)** $x^2+y^2=9$ **(e)** $(x+3)^2+(y-1)^2=9$ **(f)** $y=x+5$ **(g)** $y=3x$ or $y=-3x$
2(a) $x^2+y^2=4$ **(b)** $(x-2)^2+(y-3)^2=1$ **(c)** $x^2+(y+4)^2=9$ **(d)** $(x+5)^2+(y+2)^2=\frac{1}{4}$
3(a) $C(0,0)$, $r=1$ **(b)** $C(-1,0)$, $r=2$ **(c)** $C(2,-3)$, $r=\sqrt{5}$ **(d)** $C(0,4)$, $r=8$
4(a) $C(0,1)$, $r=2$ **(b)** $C(-3,0)$, $r=1$ **(c)** $C(2,-3)$, $r=4$ **(d)** $C(4,-7)$, $r=10$
5(a) $(x-3)^2+(y-1)^2$ **(b)** $(x-3)^2+(y-1)^2=16$
6(a) $6x-4y+15=0$ **(b)(i)** $(-\frac{1}{2},3)$, $-\frac{2}{3}$ **(ii)** $6x-4y+15=0$ **(iii)** The locus of the point P that moves so that it is equidistant from R and S is the perpendicular bisector of RS.
7(a) $\dfrac{y}{x-4}$, $\dfrac{y}{x+2}$ **(c)** $(x-1)^2+y^2=9$, which is a circle with centre $(1,0)$ and radius 3.
8(a) $2x+y-1=0$, the perpendicular bisector of AB **(b)** $x^2+y^2+2x-6y+5=0$, circle with centre $(-1,3)$ and radius $\sqrt{5}$ **(c)** $x^2-2x-8y+17=0$, parabola
9(a) $x^2+y^2=4$ **(b)** $3x^2+3y^2-28x+18y+39=0$
10(a) $(x+1)^2+(y-3)^2$ **(b)** y^2 **(c)** $(x+1)^2=6(y-\frac{3}{2})$, vertex: $(-1,\frac{3}{2})$
11(b) $C(2,1)$, radius $\sqrt{5}$
12 $3x^2-y^2+12x+10y-25=0$
13(a) P may satisfy $x-y+12=0$ or $7x+7y-60=0$. **(b)** The gradients are 1 and -1, so the lines are perpendicular.
14(a) two concentric circles with centre the origin and radii 1 and 3 **(b)** The inner locus is a square with centre the origin and side length 2 units. The outer locus is a square of side length 6 whose corners have been rounded to quadrants of a circle.
15 the single point $(3,5)$

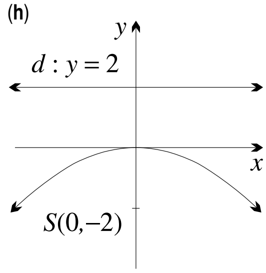
Exercise 11B (Page 308)

1(iii) Only parts (a), (h), (m) and (t) are sketched below. The details of all the parabolas follow these sketches.

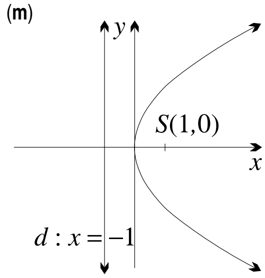
(a)



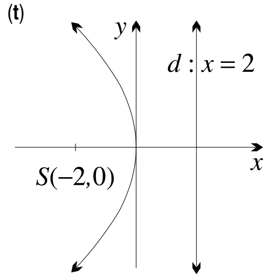
(h)



(m)



(t)



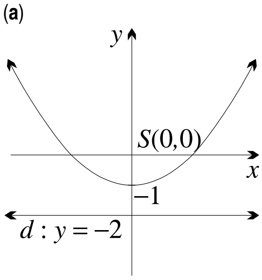
(a) $V(0,0)$, $S(0,1)$, directrix: $y=-1$
(b) $V(0,0)$, $S(0,2)$, directrix: $y=-2$
(c) $V(0,0)$, $S(0,5)$, directrix: $y=-5$
(d) $V(0,0)$, $S(0,\frac{1}{4})$, directrix: $y=-\frac{1}{4}$
(e) $V(0,0)$, $S(0,\frac{1}{2})$, directrix: $y=-\frac{1}{2}$
(f) $V(0,0)$, $S(0,\frac{3}{2})$, directrix: $y=-\frac{3}{2}$
(g) $V(0,0)$, $S(0,-1)$, directrix: $y=1$
(h) $V(0,0)$, $S(0,-2)$, directrix: $y=2$
(i) $V(0,0)$, $S(0,-3)$, directrix: $y=3$
(j) $V(0,0)$, $S(0,-\frac{1}{4})$, directrix: $y=\frac{1}{4}$
(k) $V(0,0)$, $S(0,-\frac{1}{2})$, directrix: $y=\frac{1}{2}$
(l) $V(0,0)$, $S(0,-0{\cdot}1)$, directrix: $y=0{\cdot}1$
(m) $V(0,0)$, $S(1,0)$, directrix: $x=-1$
(n) $V(0,0)$, $S(3,0)$, directrix: $x=-3$
(o) $V(0,0)$, $S(4,0)$, directrix: $x=-4$
(p) $V(0,0)$, $S(\frac{1}{4},0)$, directrix: $x=-\frac{1}{4}$
(q) $V(0,0)$, $S(\frac{3}{2},0)$, directrix: $x=-\frac{3}{2}$
(r) $V(0,0)$, $S(\frac{5}{2},0)$, directrix: $x=-\frac{5}{2}$
(s) $V(0,0)$, $S(-1,0)$, directrix: $x=1$
(t) $V(0,0)$, $S(-2,0)$, directrix: $x=2$
(u) $V(0,0)$, $S(-3,0)$, directrix: $x=3$
(v) $V(0,0)$, $S(-\frac{1}{4},0)$, directrix: $x=\frac{1}{4}$
(w) $V(0,0)$, $S(-\frac{1}{2},0)$, directrix: $x=\frac{1}{2}$
(x) $V(0,0)$, $S(-0{\cdot}3,0)$, directrix: $x=0{\cdot}3$
2 Details rather than sketches are given:
(a) $V(0,0)$, $S(0,4)$, directrix: $y=-4$
(b) $V(0,0)$, $S(0,-1)$, directrix: $y=1$
(c) $V(0,0)$, $S(\frac{1}{2},0)$, directrix: $x=-\frac{1}{2}$
(d) $V(0,0)$, $S(-\frac{5}{2},0)$, directrix: $x=\frac{5}{2}$
3(a) $x^2=20y$ **(b)** $x^2=-12y$ **(c)** $x^2=8y$ **(d)** $x^2=-2y$ **(e)** $x^2=4y$ **(f)** $x^2=-y$
4(a) $y^2=2x$ **(b)** $y^2=-4x$ **(c)** $y^2=16x$

(d) $y^2 = -8x$ **(e)** $y^2 = 12x$ **(f)** $y^2 = -6x$
5(a) $x^2 = 16y$ **(b)** $x^2 = \frac{1}{2}y$
(c) $y^2 = 2x$ **(d)** $y^2 = -x$
6(a) $x^2 = 8y$ or $x^2 = -8y$
(b) $x^2 = 12y$, $x^2 = -12y$, $y^2 = 12x$ or $y^2 = -12x$
(c) $x^2 = y$ or $y^2 = x$ **(d)** $y^2 = 2x$ or $y^2 = -2x$
7(b) $x^2 = 12y$
8(a) $x^2 = 20y$ **(b)** $y^2 = 8x$
9(a) $x^2 = -4ay$ **(b)** $y^2 = 4ax$

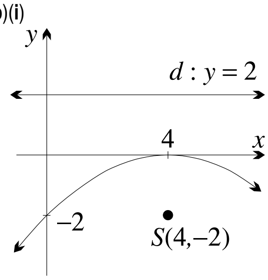
Exercise **11C** (Page 310)

1 Only graphs of (a)(i), (b)(i), (c)(v) and (d)(v) have been sketched. The details of all graphs are given afterwards.

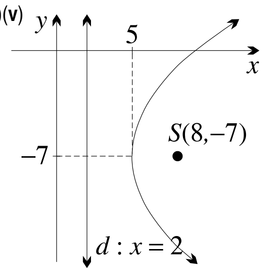
(a)



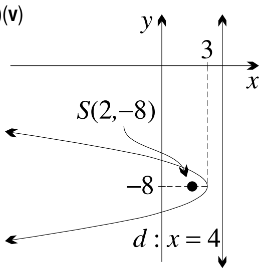
(b)(i)



(c)(v)



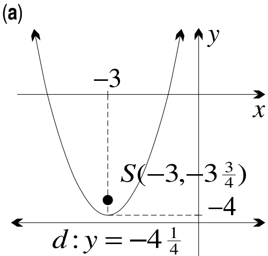
(d)(v)



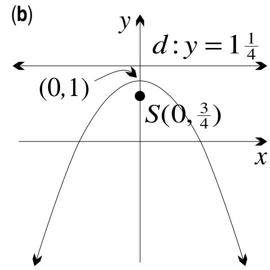
(a)(i) $V(0,-1)$, $S(0,0)$, directrix: $y = -2$
(ii) $V(-2,0)$, $S(-2,1)$, directrix: $y = -1$
(iii) $V(3,-5)$, $S(3,-3)$, directrix: $y = -7$
(iv) $V(-1,2)$, $S(-1,5)$, directrix: $y = -1$
(v) $V(0,4)$, $S(0,\frac{9}{2})$, directrix: $y = \frac{7}{2}$
(vi) $V(1,-3)$, $S(1,-\frac{3}{2})$, directrix: $y = -\frac{9}{2}$
(b)(i) $V(4,0)$, $S(4,-2)$, directrix: $y = 2$
(ii) $V(0,1)$, $S(0,-2)$, directrix: $y = 4$
(iii) $V(-5,3)$, $S(-5,2)$, directrix: $y = 4$
(iv) $V(1,-2)$, $S(1,-4)$, directrix: $y = 0$
(v) $V(0,-3)$, $S(0,-3\frac{1}{2})$, directrix: $y = -2\frac{1}{2}$
(vi) $V(-3,-1)$, $S(-3,-\frac{7}{2})$, directrix: $y = \frac{3}{2}$
(c)(i) $V(-1,0)$, $S(0,0)$, directrix: $x = -2$
(ii) $V(0,2)$, $S(1,2)$, directrix: $x = -1$
(iii) $V(-2,0)$, $S(-\frac{1}{2},0)$, directrix: $x = -3\frac{1}{2}$
(iv) $V(0,1)$, $S(4,1)$, directrix: $x = -4$
(v) $V(5,-7)$, $S(8,-7)$, directrix: $x = 2$
(vi) $V(-1,3)$, $S(1,3)$, directrix: $x = -3$
(d)(i) $V(3,0)$, $S(2,0)$, directrix: $x = 4$
(ii) $V(0,1)$, $S(-2,1)$, directrix: $x = 2$
(iii) $V(-6,0)$, $S(-8\frac{1}{2},0)$, directrix: $x = -3\frac{1}{2}$
(iv) $V(0,3)$, $S(-\frac{1}{2},3)$, directrix: $x = \frac{1}{2}$
(v) $V(3,-8)$, $S(2,-8)$, directrix: $x = 4$
(vi) $(-1,3)$, $S(-4,3)$, directrix: $x = 2$
2(a) $(x+2)^2 = 8(y-4)$ **(b)** $(y-1)^2 = 16(x-1)$
(c) $(x-2)^2 = -12(y-2)$ **(d)** $y^2 = -4(x-1)$
(e) $(x+5)^2 = 8(y-2)$ **(f)** $(y+2)^2 = 16(x+7)$
(g) $(x-8)^2 = -12(y+7)$ **(h)** $(y+3)^2 = -8(x+1)$
(i) $(x-6)^2 = 12(y+3)$
3(a) $(x-2)^2 = 8(y+1)$ **(b)** $y^2 = 4(x-1)$
(c) $(x+3)^2 = -8(y-4)$ **(d)** $(y-5)^2 = -12(x-2)$
(e) $(x-3)^2 = 8(y-1)$ **(f)** $(y-2)^2 = 12(x+4)$
(g) $x^2 = -8(y+\frac{3}{2})$ **(h)** $(y+4)^2 = -12(x+1)$
(i) $(x+7)^2 = 2(y+5)$
4(a) $x^2 = 8(y-2)$ **(b)** $y^2 = 12(x-3)$
(c) $x^2 = -4(y+1)$ **(d)** $y^2 = -8(x+2)$
(e) $(x-1)^2 = 8(y-5)$
(f) $(y+2)^2 = 4(x-2)$ **(g)** $(x+1)^2 = -2(y-\frac{9}{2})$
(h) $(y-\frac{1}{2})^2 = -4(x-4)$ **(i)** $(x-5)^2 = 10(y+\frac{13}{2})$

5 Only graphs (a) and (b) have been sketched.

(a)



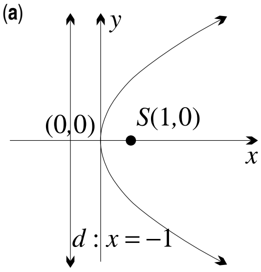
(b)



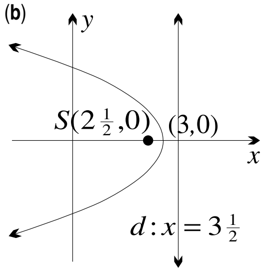
(a) $y+4 = (x+3)^2$, vertex: $(-3,-4)$, focus: $(-3,-3\frac{3}{4})$, directrix: $y = -4\frac{1}{4}$
(b) $x^2 = -(y-1)$, vertex: $(0,1)$, focus: $(0,\frac{3}{4})$, directrix: $y = \frac{5}{4}$
(c) $(x-6)^2 = 6(y+6)$, vertex: $(6,-6)$, focus: $(6,-4\frac{1}{2})$, directrix: $y = -7\frac{1}{2}$
(d) $x^2 = 4(y+\frac{1}{2})$, vertex: $(0,-\frac{1}{2})$, focus: $(0,\frac{1}{2})$, directrix: $y = -\frac{3}{2}$
(e) $(x+3)^2 = y+25$, vertex: $(-3,-25)$, focus: $(-3,-24\frac{3}{4})$, directrix: $y = -25\frac{1}{4}$
(f) $(x+4)^2 = 8(y-3)$, vertex: $(-4,3)$, focus: $(-4,5)$, directrix: $y = 1$
(g) $(x-3)^2 = -2(y+1\frac{1}{2})$, vertex: $(3,-1\frac{1}{2})$, focus: $(3,-2)$, directrix: $y = -1$
(h) $(x-4)^2 = -12(y-1)$, vertex: $(4,1)$, focus: $(4,-2)$, directrix: $y = 4$

6 Only graphs (a) and (b) have been sketched.

(a)



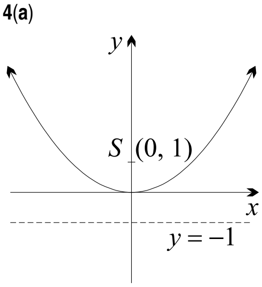
(b)



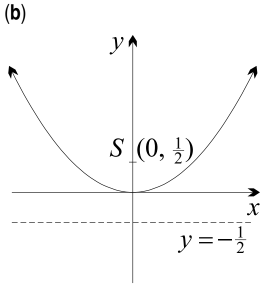
(a) $y^2 = 4x$ vertex: $(0,0)$, focus: $(1,0)$, directrix: $x = -1$
(b) $y^2 = -2(x-3)$ vertex: $(3,0)$, focus: $(2\frac{1}{2},0)$, directrix: $x = 3\frac{1}{2}$
(c) $y^2 = 6(x-3)$, vertex: $(3,0)$, focus: $(4\frac{1}{2},0)$, directrix: $x = 1\frac{1}{2}$
(d) $(y-1)^2 = 4(x-1)$, vertex: $(1,1)$, focus: $(2,1)$, directrix: $x = 0$
(e) $(y-2)^2 = 8(x+\frac{1}{2})$, vertex: $(-\frac{1}{2},2)$, focus: $(1\frac{1}{2},2)$, directrix: $x = -2\frac{1}{2}$
(f) $(y-3)^2 = 2(x+1)$, vertex: $(-1,3)$, focus: $(-\frac{1}{2},3)$, directrix: $x = -1\frac{1}{2}$
(g) $(y+2)^2 = -6(x-5)$, vertex: $(5,-2)$, focus: $(3\frac{1}{2},-2)$, directrix: $x = 6\frac{1}{2}$
(h) $(y-5)^2 = 12(x+1)$, vertex: $(-1,5)$, focus: $(2,5)$, directrix: $x = -4$
7(b) $(x-3)^2 = 8(y-1)$
8(a) $(x+7)^2 = 12(y+5)$ **(b)** $(y-2)^2 = 4(x+1)$
9(a) $(x-1)^2 = 4(y-4)$ **(b)** $(y+2)^2 = 2(x+3)$
10(a) $(x-3)^2 = 8(y+1)$ or $(x-3)^2 = -8(y+1)$ or $(y+1)^2 = 8(x-3)$ or $(y+1)^2 = -8(x-3)$
(b) $(y+1)^2 = 8(x+1)$ or $(y+1)^2 = -8(x-3)$
(c) $(x+2)^2 = 4(y-3)$ or $(x+2)^2 = -4(y-5)$
(d) $(y-2)^2 = 6(x-3)$ or $(y-2)^2 = -6(x-3)$
(e) $(x-6)^2 = -20(y-2)$ or $(y+3)^2 = 20(x-1)$

Review Exercise **11D** (Page 312)

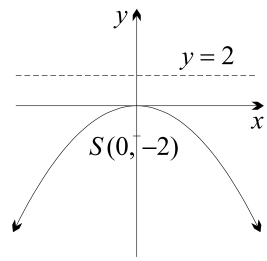
1(a) $y = 3$ **(b)** $x = 1$ **(c)** $x^2 + y^2 = 16$
(d) $(x-1)^2 + (y+2)^2 = 4$
2(a) $C(0,0)$, $r = 3$ **(b)** $C(1,-3)$, $r = 2$
(c) $C(2,-4)$, $r = 5$
3(a) $6x - 8y + 37 = 0$ **(b)** $x^2 + y^2 - x - 10y + 19 = 0$
4(a)



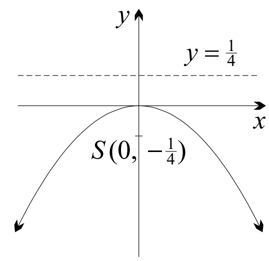
(b)



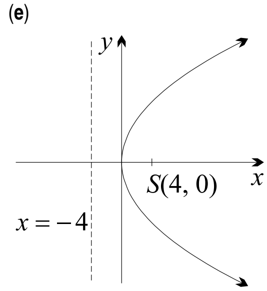
(c)



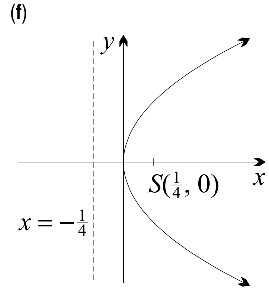
(d)



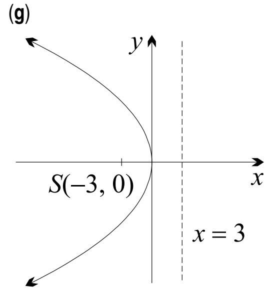
(e)



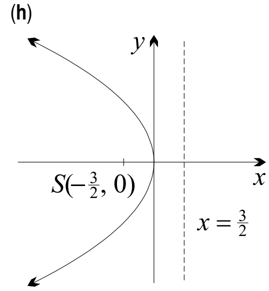
(f)



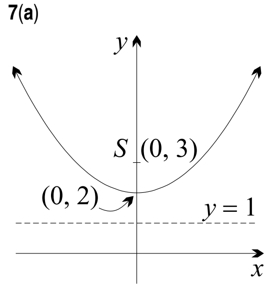
(g)



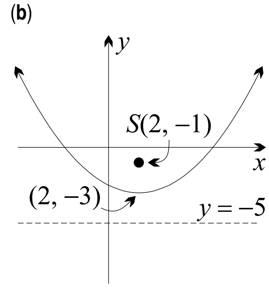
(h)



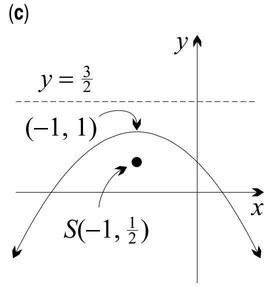
5(a) $x^2 = 16y$ **(b)** $x^2 = -8y$ **(c)** $x^2 = 4y$
6(a) $y^2 = 20x$ **(b)** $y^2 = -8x$ **(c)** $y^2 = -12x$
7(a)



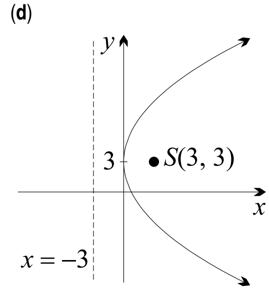
(b)



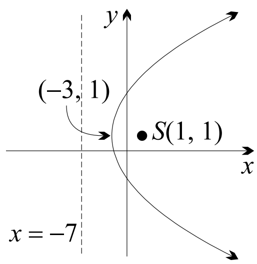
(c)



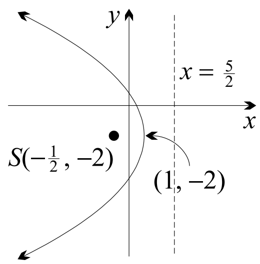
(d)



(e)

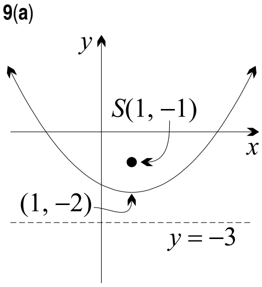


(f)

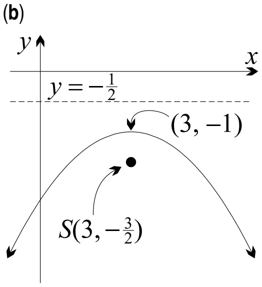


8(a) $(x-1)^2 = 8(y-4)$ **(b)** $(y-3)^2 = 4(x+2)$
(c) $x^2 = -12(y-2)$

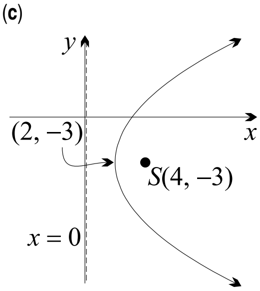
9(a)



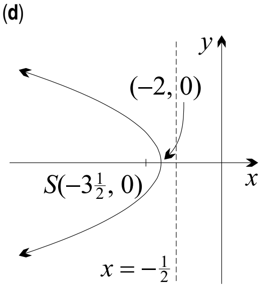
(b)



(c)



(d)

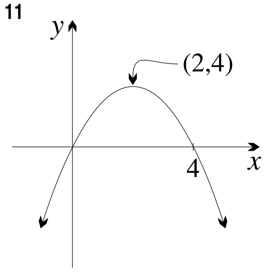


Chapter Twelve

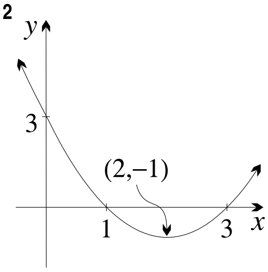
Exercise 12A (Page 315)

1(a) A, G and I **(b)** C and E **(c)** B, D, F and H
2(a) increasing **(b)** stationary **(c)** decreasing
(d) decreasing **(e)** increasing **(f)** stationary
3(a) $2x-6$ **(b)(i)** decreasing **(ii)** decreasing
(iii) stationary **(iv)** increasing **(v)** decreasing
4(a) $3x^2-12x+9$ **(b)(i)** increasing **(ii)** stationary
(iii) decreasing **(iv)** stationary **(v)** increasing
5(a) $x=1$ **(b)** $x=2$ **(c)** $x=-3$ **(d)** $x=4$
(e) $x=0$ or $x=2$ **(f)** $x=2$ or $x=-2$
6(a) The derivative is always negative.
(b) The derivative is always positive.
(c) $f'(x)=3x^2$, which is positive except at $x=0$.
(d) $f'(x)=2x$, which is positive if $x>0$ and negative if $x<0$. At $x=0$ the function is stationary.
7(a) increasing **(b)** decreasing **(c)** stationary
8(a) stationary **(b)** increasing **(c)** decreasing
9(a) increasing **(b)** decreasing **(c)** increasing
10(a) increasing **(b)** decreasing **(c)** increasing
11(a) $4-2x$ **(b)(i)** $x<2$ **(ii)** $x>2$ **(iii)** $x=2$
12(a) $2x-4$ **(b)(i)** $x>2$ **(ii)** $x<2$ **(iii)** $x=2$

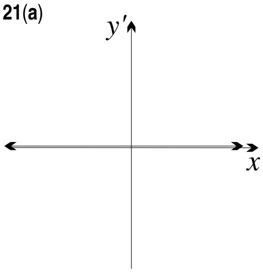
11

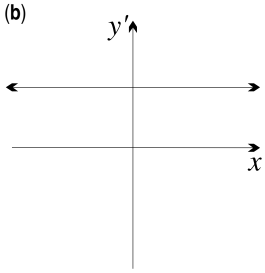


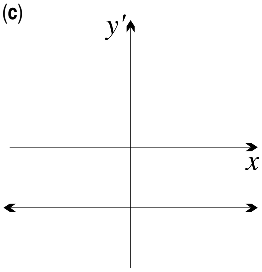
12

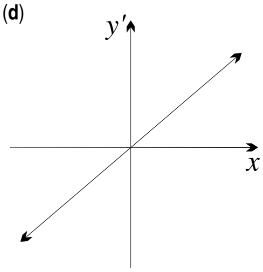


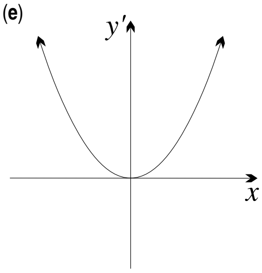
14(a) $3x^2+4x+1$ **(b)** $-\frac{1}{3}$ and -1 **(d)** $-1<x<-\frac{1}{3}$
15(a) $x>2$ **(b)** $x<-3$ **(c)** $x>1$ or $x<-1$
(d) $x<0$ or $x>2$
16(a) III **(b)** I **(c)** IV **(d)** II
17(a) $\dfrac{1}{x^2}$
(b) The function is not continuous at $x=0$.
18(a) $-\dfrac{6}{(x-3)^2}$ **(b)** $f'(x)$ is negative for $x\neq 3$.
19(a) $\dfrac{x^2(x^2+3)}{(x^2+1)^2}$ **(b)** $f'(x)$ is positive for $x\neq 0$.
20(a) x^2+2x+5 **(b)** $f'(x)>0$ for all values of x.
(c) $f(-3)=-8$, $f(0)=7$, $f(x)$ is increasing for all x. Hence the curve crosses the x-axis exactly once between $x=-3$ and $x=0$ and nowhere else.

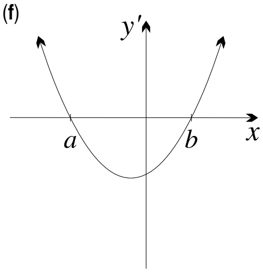
21(a)


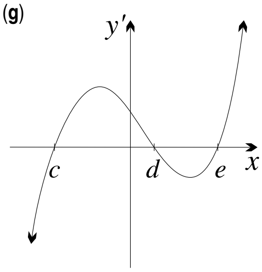
(b)


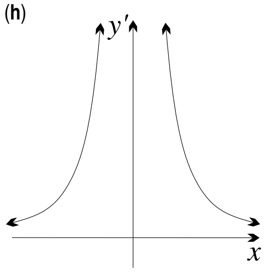
(c)


(d)


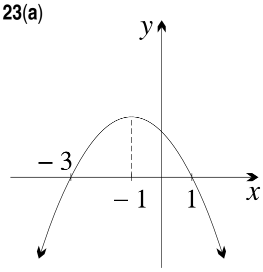
(e)


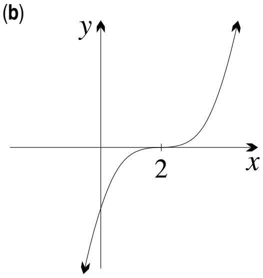
(f)


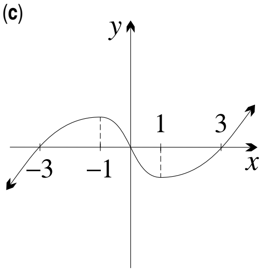
(g)


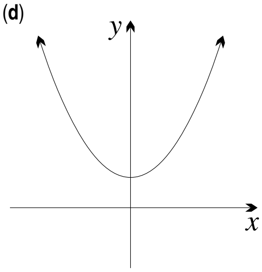
(h)


22(a) $-3x^2 + 4x - 5$
(b) As $\Delta < 0$, $f'(x)$ is negative definite. **(c)** 1

23(a)


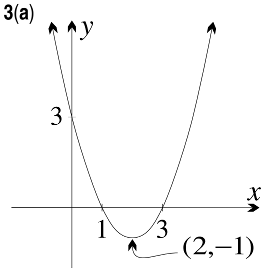
(b)


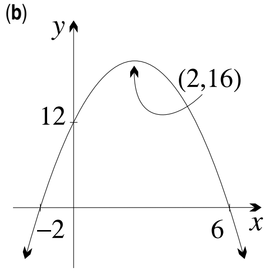
(c)


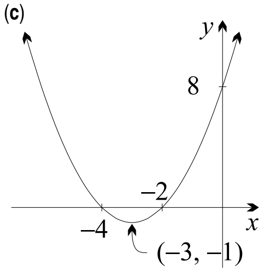
(d)


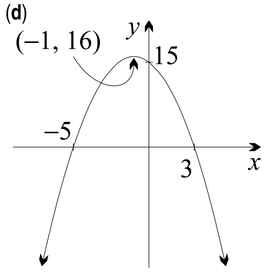
Exercise **12B** (Page 320)

1(a) $x = 3$ **(b)** $x = -2$ **(c)** $x = 1$ or -1
2(a) $(2, 3)$ **(b)** $(4, 0)$ **(c)** $(1, -2)$ **(d)** $(1, 0)$
(e) $(0, 0)$ and $(2, -4)$ **(f)** $(1, -2)$

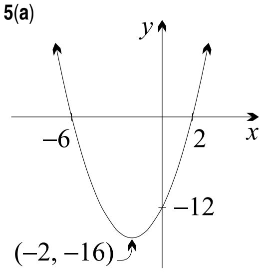
3(a)


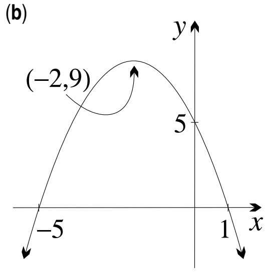
(b)


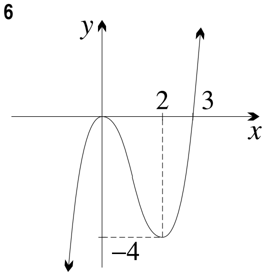
(c)


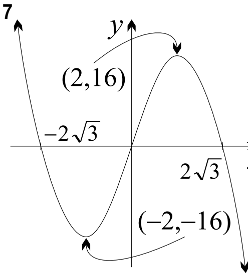
(d)


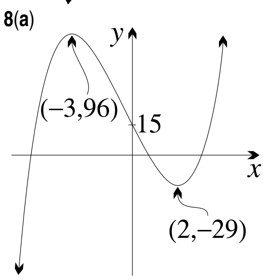
4(a) minimum **(b)** maximum **(c)** minimum
(d) stationary point of inflexion

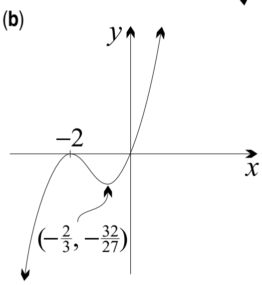
5(a)


(b)


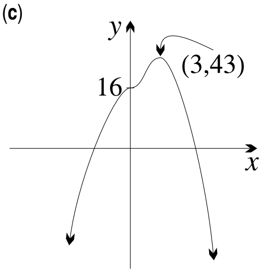
6


7


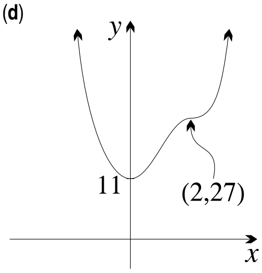
8(a)


(b)


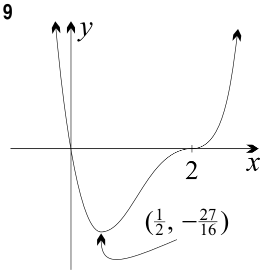
(c)



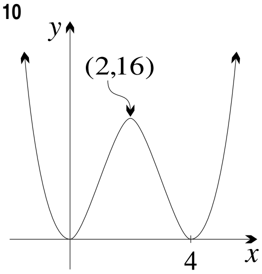
(d)



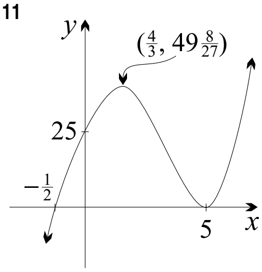
9



10

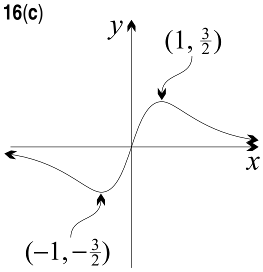


11



12(a) $a = -8$ **(b)** $a = 2$
13(a) $a = 2$ and $c = 3$
(b) $b = -3$ and $c = -24$
14(b) $a = b = -1$, $c = 6$
15(c) $a = -1$

16(c)



(d)(i) no roots **(ii)** 1 root
(iii) 2 roots **(iv)** 1 root
17(b) $9a - 3b + 3c = -27$
(c) $a = 2$, $b = 3$, $c = -12$
(d) $d = 7$

Exercise **12C** (Page 323)

1(a) $3x^2$, $6x$, 6 **(b)** $10x^9$, $90x^8$, $720x^7$
(c) $7x^6$, $42x^5$, $210x^4$ **(d)** $2x$, 2, 0
(e) $8x^3$, $24x^2$, $48x$ **(f)** $15x^4$, $60x^3$, $180x^2$
(g) -3, 0, 0 **(h)** $2x - 3$, 2, 0
(i) $12x^2 - 2x$, $24x - 2$, 24
(j) $20x^4 + 6x^2$, $80x^3 + 12x$, $240x^2 + 12$
2(a) $2x + 3$, 2 **(b)** $3x^2 - 8x$, $6x - 8$ **(c)** $2x - 1$, 2
(d) $6x - 13$, 6 **(e)** $30x^4 - 36x^3$, $120x^3 - 108x^2$
(f) $32x^7 + 40x^4$, $224x^6 + 160x^3$
3(a) $0{\cdot}3x^{-0{\cdot}7}$, $-0{\cdot}21x^{-1{\cdot}7}$, $0{\cdot}357x^{-2{\cdot}7}$
(b) $-\dfrac{1}{x^2}$, $\dfrac{2}{x^3}$, $-\dfrac{6}{x^4}$ **(c)** $-\dfrac{2}{x^3}$, $\dfrac{6}{x^4}$, $-\dfrac{24}{x^5}$
(d) $-\dfrac{15}{x^4}$, $\dfrac{60}{x^5}$, $-\dfrac{300}{x^6}$
(e) $2x - \dfrac{1}{x^2}$, $2 + \dfrac{2}{x^3}$, $-\dfrac{6}{x^4}$

4(a) $-\dfrac{3}{x^4}$, $\dfrac{12}{x^5}$ **(b)** $-\dfrac{4}{x^5}$, $\dfrac{20}{x^6}$ **(c)** $-\dfrac{6}{x^3}$, $\dfrac{18}{x^4}$
(d) $-\dfrac{6}{x^4}$, $\dfrac{24}{x^5}$
5(a) $2(x + 1)$, 2 **(b)** $9(3x - 5)^2$, $54(3x - 5)$
(c) $8(4x - 1)$, 32 **(d)** $-11(8 - x)^{10}$, $110(8 - x)^9$
6(a) $\dfrac{-1}{(x + 2)^2}$, $\dfrac{2}{(x + 2)^3}$ **(b)** $\dfrac{2}{(3 - x)^3}$, $\dfrac{6}{(3 - x)^4}$
(c) $\dfrac{-15}{(5x + 4)^4}$, $\dfrac{300}{(5x + 4)^5}$
(d) $\dfrac{12}{(4 - 3x)^3}$, $\dfrac{108}{(4 - 3x)^4}$
7(a) $\dfrac{1}{2\sqrt{x}}$, $\dfrac{-1}{4x\sqrt{x}}$ **(b)** $\frac{1}{3}x^{-\frac{2}{3}}$, $-\frac{2}{9}x^{-\frac{5}{3}}$
(c) $\frac{3}{2}\sqrt{x}$, $\dfrac{3}{4\sqrt{x}}$ **(d)** $-\frac{1}{2}x^{-\frac{3}{2}}$, $\frac{3}{4}x^{-\frac{5}{2}}$
(e) $\dfrac{1}{2\sqrt{x + 2}}$, $\dfrac{-1}{4(x + 2)^{\frac{3}{2}}}$ **(f)** $\dfrac{-2}{\sqrt{1 - 4x}}$, $\dfrac{-4}{(1 - 4x)^{\frac{3}{2}}}$
8(a) $f'(x) = 3x^2 + 6x + 5$, $f''(x) = 6x + 6$ **(b)(i)** 5
(ii) 14 **(iii)** 6 **(iv)** 12
9(a)(i) 15 **(ii)** 12 **(iii)** 6 **(iv)** 0
(b)(i) -8 **(ii)** 48 **(iii)** -192 **(iv)** 384
10(a) $\dfrac{1}{(x + 1)^2}$, $\dfrac{-2}{(x + 1)^3}$ **(b)** $\dfrac{7}{(2x + 5)^2}$, $\dfrac{-28}{(2x + 5)^3}$
11 $(x - 1)^3(5x - 1)$, $4(x - 1)^2(5x - 2)$
12(a) $1, -1$ **(b)** $-\frac{1}{3}$
13(a) nx^{n-1}, $n(n - 1)x^{n-2}$, $n(n - 1)(n - 2)x^{n-3}$
(b) $n(n - 1)(n - 2)\ldots 1$, 0

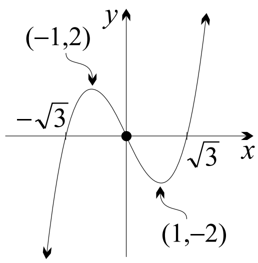
Exercise **12D** (Page 327)

1

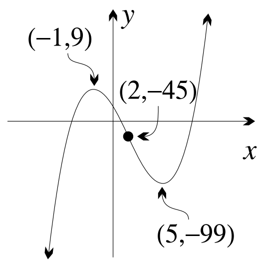
Point	A	B	C	D	E	F	G	H	I
y'	0	+	0	−	0	−	0	+	0
y''	+	0	−	0	0	0	+	0	0

2(a) concave down **(b)** concave up **(c)** concave up
(d) concave down
3(a) minimum **(b)** maximum **(c)** minimum
(d) minimum
4(a) $y'' = 2$, so $y'' > 0$ for all values of x.
(b) $y'' = -6$, so $y'' < 0$ for all values of x.
5(a) $y'' = 6x - 6$ **(b)(i)** $x > 1$ **(ii)** $x < 1$
6(a) $y'' = 6x - 2$ **(b)(i)** $x > \frac{1}{3}$ **(ii)** $x < \frac{1}{3}$

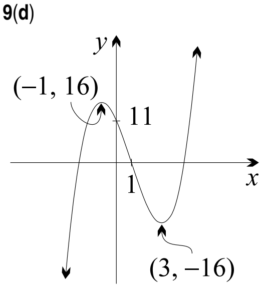
7(e)



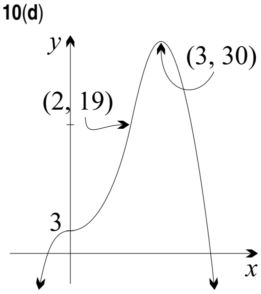
8(d)



9(d)

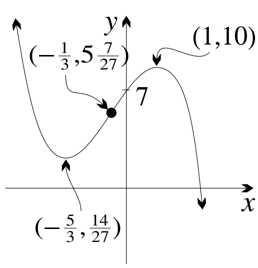


10(d)



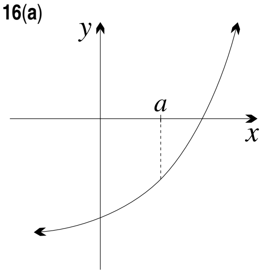
11(a) $x > 2$ or $x < -1$ **(b)** $-1 < x < 2$ **(c)** $x > \frac{1}{2}$
(d) $x < \frac{1}{2}$
12(a) $y' = 3x^2 + 6x - 72$, $y'' = 6x + 6$
(d) $75x + y - 13 = 0$
13(b) $f''(x) = g''(x) = 0$, no
(c) $f(x)$ has a stationary point of inflexion, $g(x)$ has a minimum turning point.
14(a) $y'' = 6x - 2a$, $a = 6$
(b) $y'' = 6x + 4a$, $a > 1\frac{1}{2}$
(c) $y'' = 12x^2+6ax+2b$, $a = -5$, $b = 6$ **(d)** $a > -\frac{2}{3}$

15

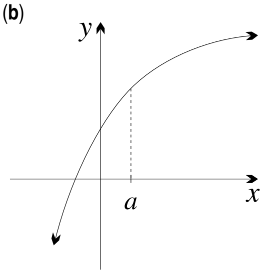


(e) $\frac{16}{3}$

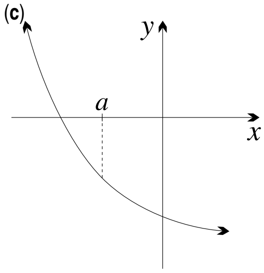
16(a)



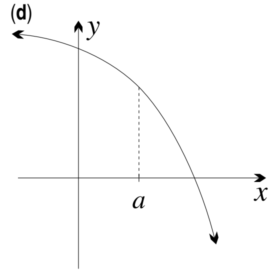
(b)



(c)



(d)

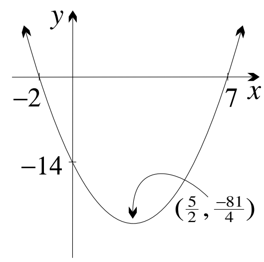


17 $a = 2$, $b = -3$, $c = 0$ and $d = 5$

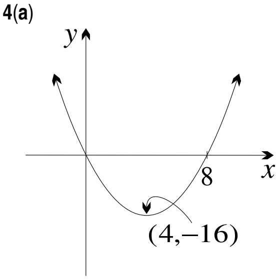
Exercise 12E (Page 331)

1(a) $(6, 0)$ **(b)** $(4, 32)$ **(c)** $(2, 16)$
2(a) $x = -1$ or $x = 2$ **(b)** $x = 0$ **(c)** $-1 < x < 2$
(d) $x < 0$

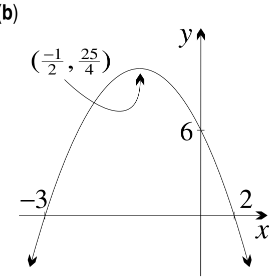
3



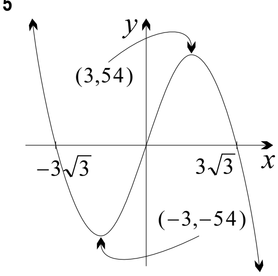
4(a)



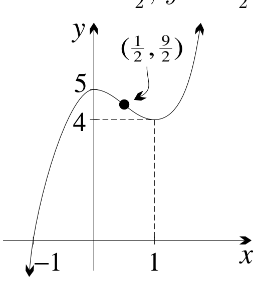
(b)



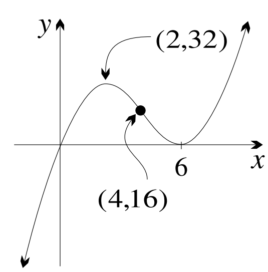
5



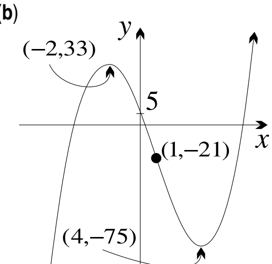
(a) Show that $f(-x) = -f(x)$. Point symmetry in the origin. **(e)** When $x = 0$, $y' = 27$.
6 When $x = \frac{1}{2}$, $y' = 1\frac{1}{2}$.



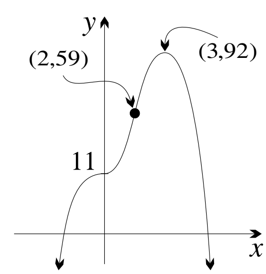
7(a)



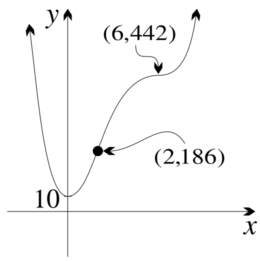
(b)



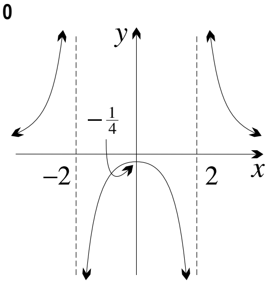
8



9

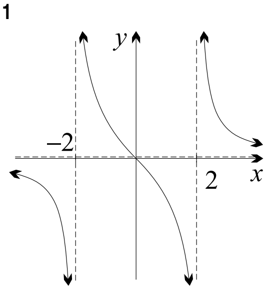


10



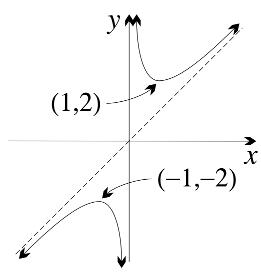
(c) line symmetry in the y-axis **(d)** domain: $x \neq 2$ and $x \neq -2$, asymptotes: $x = 2$ and $x = -2$ **(e)** $y = 0$
(g) $y > 0$ or $y \leq -\frac{1}{4}$

11



(c) gradient $= -\frac{1}{4}$
(d) domain: $x \neq 2$ and $x \neq -2$, asymptotes: $x = 2$ and $x = -2$ **(e)** $y = 0$
(f) point symmetry in the origin **(i)** all real y

12



(c) point symmetry in the origin
(d) domain: $x \neq 0$, asymptote: $x = 0$
(g) $y \geq 2$ or $y \leq -2$

Exercise **12F** (Page 334)

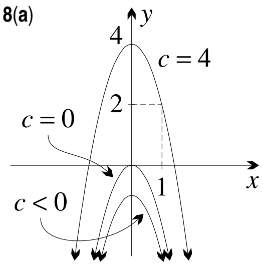
1(a) A relative maximum, B relative minimum
(b) C absolute maximum, D relative minimum, E relative maximum, F absolute minimum
(c) G absolute maximum, H horizontal point of inflexion **(d)** I horizontal point of inflexion, J absolute minimum
2(a) $0,\ 4$ **(b)** $2,\ 5$ **(c)** $0,\ 4$ **(d)** $0,\ 5$ **(e)** $0,\ 2\sqrt{2}$
(f) $-1,\ -\frac{1}{4}$ **(g)** $-1,\ 2$
3(a) $-1,\ 8$ **(b)** $-49,\ 5$ **(c)** $0,\ 4$ **(d)** $0,\ 9$
4(a) global minimum -5, global maximum 20
(b) global minimum -5, local maximum 11, global maximum 139
(c) global minimum 4, global maximum 11

Exercise **12G** (Page 337)

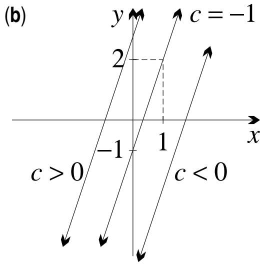
1(b) $3\frac{1}{8}$ metres
2(a) $11x - 2x^2$ **(b)** $\frac{11}{4}$ **(c)** $\frac{121}{8}$
3(a) $2x^2 - 16x + 64$ **(b)** 4 **(c)** 32
4(b) 5 **(c)** $25\,\text{cm}^2$
5(c) 10 **(d)** $200\,\text{m}^2$
6 after 2 hours and 40 minutes
7(d) $24\,\text{cm}$
8(b) $x = 30\,\text{m}$ and $y = 20\,\text{m}$
9(a) 20 and 20 **(b)** 20 and 20
10(c) Each of the 6 rectangles has dimensions $\frac{3}{4} \times \frac{2}{3}$.
11(a) $\dfrac{x}{4},\ \dfrac{10-x}{4}$ **(c)** 5 **(d)** $\frac{25}{8}\,\text{m}^2$
12(a) $R = x(47 - \frac{1}{3}x)$ **(b)** $-\frac{8}{15}x^2 + 32x - 10$ **(c)** 30
13(b) $\frac{5}{2}$ **(c)** $\frac{7}{2}$ units
14(c) $\frac{10}{3}$
15(c) $\frac{10}{3\pi}$ **(d)** $\dfrac{1000}{27\pi}\,\text{m}^3$
16(c) $20\sqrt{10}\,\pi\,\text{cm}^3$
17(c) $4 \times 4 \times 2$
18(a) $S = 16x + 4y$ **(c)** $27 \times 9 \times 18$
19(b) width $16\sqrt{3}\,\text{cm}$, depth $16\sqrt{6}\,\text{cm}$
20(d) $8\,\text{cm}$
21(d) $2(\sqrt{10} + 1) \times 4(\sqrt{10} + 1)$
22(b) $80\,\text{km/h}$ **(c)** \$400

Exercise **12H** (Page 344)

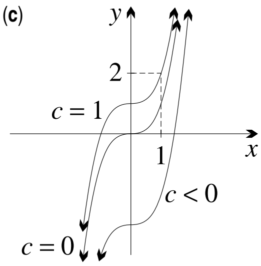
1(a) $\frac{1}{7}x^7 + C$ **(b)** $\frac{1}{4}x^4 + C$ **(c)** $\frac{1}{11}x^{11} + C$ **(d)** $\frac{3}{2}x^2 + C$
(e) $5x + C$ **(f)** $\frac{1}{2}x^{10} + C$ **(g)** $3x^7 + C$ **(h)** C
2(a) $\frac{1}{3}x^3 + \frac{1}{5}x^5 + C$ **(b)** $x^4 - x^5 + C$ **(c)** $\frac{2}{3}x^3 + \frac{5}{8}x^8 + C$
(d) $\frac{1}{3}x^3 - \frac{1}{2}x^2 + x + c$ **(e)** $3x - 2x^2 + 2x^8 + C$
(f) $x^3 - x^4 - x^5 + C$
3(a) $\frac{1}{3}x^3 - \frac{3}{2}x^2 + C$ **(b)** $\frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x + C$
(c) $x^3 + \frac{11}{2}x^2 - 4x + C$ **(d)** $\frac{5}{6}x^6 - x^4 + C$
(e) $x^8 + \frac{1}{2}x^4 + C$ **(f)** $\frac{1}{2}x^2 + \frac{1}{4}x^4 - 3x - x^3 + C$
4(a)(i) $y = x^2 + 3x + 3$ **(ii)** $y = x^2 + 3x + 4$
(b)(i) $y = 3x^3 + 4x + 1$ **(ii)** $y = 3x^3 + 4x - 2$
(c)(i) $y = x^3 - 2x^2 + 7x$ **(ii)** $y = x^3 - 2x^2 + 7x - 7$
5(a) $-\dfrac{1}{x} + C$ **(b)** $-\dfrac{1}{2x^2} + C$ **(c)** $\dfrac{1}{x^2} + C$
(d) $\dfrac{1}{x^3} + C$ **(e)** $-\dfrac{1}{x} + \dfrac{1}{2x^2} + C$
6(a) $\frac{2}{3}x^{\frac{3}{2}} + C$ **(b)** $2\sqrt{x} + C$ **(c)** $\frac{3}{4}x^{\frac{4}{3}} + C$ **(d)** $4\sqrt{x} + C$
(e) $\frac{5}{8}x^{\frac{8}{5}} + C$
7(a) $y = \frac{2}{3}x^{\frac{3}{2}} + 1$
(b) $y = \frac{2}{3}x^{\frac{3}{2}} - 16$

8(a)


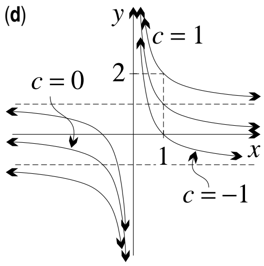
$y = -2x^2 + C$,
$y = 4 - 2x^2$

(b)


$y = 3x + C$,
$y = 3x - 1$

(c)


$y = x^3 + C$,
$y = x^3 + 1$

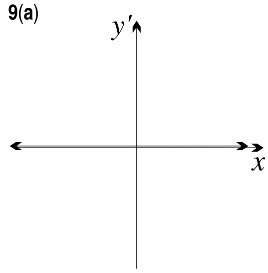
(d)


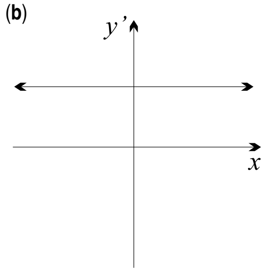
$y = \frac{1}{x} + C$,
$y = \frac{1}{x} + 1$

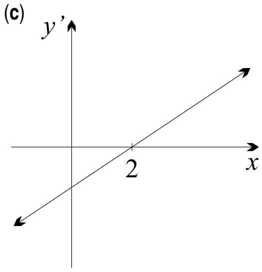
9(a) $\frac{1}{4}(x+1)^4 + C$ **(b)** $\frac{1}{6}(x-2)^6 + C$
(c) $\frac{1}{3}(x+5)^3 + C$ **(d)** $\frac{1}{10}(2x+3)^5 + C$
(e) $\frac{1}{21}(3x-4)^7 + C$ **(f)** $\frac{1}{20}(5x-1)^4 + C$
(g) $-\frac{1}{4}(1-x)^4 + C$ **(h)** $-\frac{1}{28}(1-7x)^4 + C$
(i) $\dfrac{-1}{3(x-2)^3} + C$ **(j)** $\dfrac{1}{9(1-x)^9} + C$
10(a) $\frac{2}{3}(x+1)^{\frac{3}{2}} + C$ **(b)** $\frac{2}{3}(x-5)^{\frac{3}{2}} + C$
(c) $-\frac{2}{3}(1-x)^{\frac{3}{2}} + C$ **(d)** $\frac{1}{3}(2x-7)^{\frac{3}{2}} + C$
(e) $\frac{2}{9}(3x-4)^{\frac{3}{2}} + C$
11(a) $y = \frac{1}{5}(x-1)^5$ **(b)** $\frac{1}{8}(2x+1)^4 - \frac{9}{8}$
(c) $y = \frac{1}{3}(2x+1)^{\frac{3}{2}}$
12(a) $y = \frac{3}{5}x^5 - \frac{1}{4}x^4 + x$ **(b)** $y = -\frac{1}{4}x^4 + x^3 + 2x - 2$
(c) $y = -\frac{1}{20}(2-5x)^4 + \frac{21}{20}$
13 30
14 The rule would give the primitive of x^{-1} as $\dfrac{x^0}{0}$, which is meaningless. This problem will be solved in Chapters Two & Three of the Year 12 volume.
15 $y = x^3 + 2x^2 - 5x + 6$
17 $y = -x^3 + 4x^2 + 3$

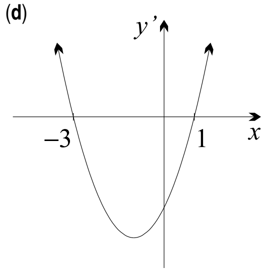
Review Exercise 12I (Page 346)

1(a) C and H **(b)** A and F **(c)** B, D, E and G
(d) A, B, G and H **(e)** D **(f)** C, E and F
2(a) $f'(x) = 3x^2 - 2x - 1$ **(b)(i)** decreasing
(ii) stationary **(iii)** increasing **(iv)** increasing
3(a) $2x - 4$ **(b)(i)** $x > 2$ **(ii)** $x < 2$ **(iii)** $x = 2$
4(a) $y' = 3x^2$, increasing **(b)** $y' = 2x-1$, increasing
(c) $y' = 5(x-1)^4$, stationary
(d) $y' = -\dfrac{4}{(x-3)^2}$, decreasing
5(a) $7x^6$, $42x^5$ **(b)** $3x^2 - 8x$, $6x - 8$
(c) $5(x-2)^4$, $20(x-2)^3$ **(d)** $-\dfrac{1}{x^2}$, $\dfrac{2}{x^3}$
6(a) concave up **(b)** concave down
7(a) $12x - 6$ **(b)(i)** $x > \frac{1}{2}$ **(ii)** $x < \frac{1}{2}$
8(a) $x < 1$ or $x > 3$ **(b)** $1 < x < 3$ **(c)** $x > 2$
(d) $x < 2$

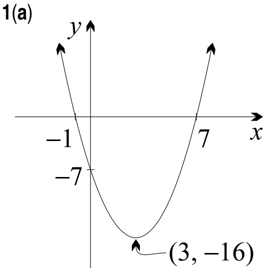
9(a)


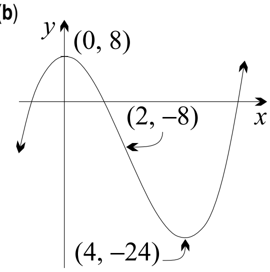
(b)


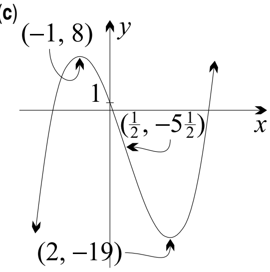
(c)


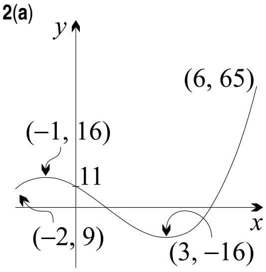
(d)


10(a) $P(-1, 3)$, $Q(\frac{1}{3}, \frac{49}{27})$ **(b)** $x > -\frac{1}{3}$
(c) $\frac{49}{27} < k < 3$

11(a)


(b)


(c)


12(a)


(b) 65 and -16
13(a) $a = -2$ **(b)** $a = 3$ and $b = 6$
14(b) -16
15(a) 175 **(b)** 0 **(c)** 256
16(b) $\frac{1600}{27}\ \text{cm}^3$
17(b) $r = 8\,\text{m}$

18(a) $\frac{1}{8}x^8 + C$ **(b)** $x^2 + C$ **(c)** $4x + C$ **(d)** $2x^5 + C$
(e) $4x^2 + x^3 - x^4 + C$
19(a) $x^3 - 3x^2 + C$ **(b)** $\frac{1}{3}x^3 - 2x^2 - 5x + C$
(c) $\frac{4}{3}x^3 - 6x^2 + 9x + C$
20(a) $\frac{1}{6}(x+1)^6 + C$ **(b)** $\frac{1}{8}(x-4)^8 + C$
(c) $\frac{1}{8}(2x-1)^4 + C$
21(a) $-\frac{1}{x} + C$ **(b)** $\frac{2}{3}x^{\frac{3}{2}} + C$
22 $f(x) = x^3 - 2x^2 + x + 3$
23 25

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